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Riemann-Lebesgue properties of Green's functions with applications to inverse scattering *

Richard Ford

Abstract

Saitō's method has been applied successfully for measuring potentials with compact support in three dimensions. Also potentials have been reconstructed in the sense of distributions using a weak version of the method. Saitō's method does not depend on the decay of the boundary value of the resolvent operator, but instead on certain Reimann-Lebesgue type properties of convolutions of the kernel of the unperturbed resolvent. In this paper these properties are extended from three to higher dimensions. We also provide an important application to inverse scattering by extending reconstruction results to measure potentials with unbounded support.

1 Introduction

The field of inverse potential scattering has matured over the last decade resulting in the emergence of a variety of effective techniques. Early work by Faddeev [7] uses a high energy limit of the scattering amplitude which he shows converges to the Fourier Transform of the potential for a certain class of potentials. A different method that also utilizes the high energy data from the scattering amplitude has been developed by Saitō and applied to general short-range potentials in $\mathbb{R}^3$ [17], [18] and subsequently to $\mathbb{R}^n$, $n \geq 2$ [19]. Newton's method [14] exploits the reciprocity relations to reduce the inverse problem in $\mathbb{R}^3$ to the Marchenko equation, which is subsequently shown to be uniquely solvable under a variety of conditions on the potential. Newton's method has subsequently been generalized from R^3 to R^2 by Cheney [4] and to $\mathbb{R}^n$ by Weder [22]. Other authors have been successful in generalizing the class of admissible potentials to include certain types of singularities (see Päivärinta, Serov, and Somersalo [15], Serov [20], [21]). Our interest lies specifically in Schrödinger operators associated with measure potentials. Delta potentials and other highly singular measure potentials have captured the interest of many authors, (e.g. [1], [3], [6] and the references therein). The direct scattering problem has been solved in

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$\mathbb{R}^n$ for a broad class of measure potentials [8], [9], [10]. Inverse scattering for measure potentials is not so well developed and few results are currently available. One approach that has produced results is that of Saitō, but modified to accommodate reconstruction of the measure potential in the sense of distributions [11]. The results only apply in 3 dimensions and to potentials with bounded support. Most of the inversion methods we have mentioned above depend on the decay properties of the boundary value of the resolvent operator in operator norm when viewed as an operator from weighted to unweighted L^p spaces (usually $p = 2$). In stark contrast, the modified Saitō method relies on certain Riemann-Lebesgue like properties of convolutions of the kernel of the unperturbed resolvent operator. Essentially, the vibrations of the convolutions annihilate the Born remainder shedding additional light on the importance of the Born approximation. It has long been known that the Born approximation to the scattering amplitude carries all of the essential information for reconstruction for a wide variety of potentials. The kernel convolutions and their properties provide evidence that this variety may be substantially wider than results to date indicate. This paper shall address extending the results of [11] regarding the kernel convolutions from 3 to $n \geq 3$ dimensions. Applications to the inverse scattering problem shall be illustrated by obtaining new reconstruction results for measure potentials with unbounded support. An explicit example is provided showing that even highly singular delta potentials can be recovered through the modified Saitō's method.

2 Results

Let H be the standard self-adjoint realization of the Laplacian on $\mathbb{R}^n$ and let $R(z)$ denote the associated resolvent operator, $(z-H)^{-1}$. It is well known that $R(z)$ is an integral operator with kernel given by $G(x,y;\kappa) = G(|x-y|,\kappa)$ where $G(r,\kappa) = \frac{i}{4}(\frac{\kappa}{2\pi r})^{\frac{n-1}{2}} H^1_{\frac{n-1}{2}}(r\kappa)$, $\kappa^2 = z$, $\operatorname{im}\kappa > 0$ and H^1_ν is the Hankel function of the first kind. Now let

$$J(r) = \int_{S^{n-1}} e^{ir\omega\cdot\omega'}\, d\omega' . \tag{2.1}$$

By Alsholm and Schmidt [2] we have that $\frac{4\pi}{i}\left(\frac{2\pi}{k}\right)^{n-2}(G(x,k) - G(x,-k)) = J(k|x|)$ where $k > 0$. The primary result of this work is the following.

Theorem 2.1 *Let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$. Then for all $x \neq y$ we have*

$$\lim_{k\to\infty} k^{n-1} \int_{\mathbb{R}^n} J(k|x-\xi|)J(k|\xi-y|)\psi(\xi)\, d\xi = 0 . \tag{2.2}$$

The impact of this theorem on the inverse scattering problem lies in the following observations. In the case of ordinary Schrödinger Operator scattering with short range potentials and valid generalized eigenfunctions, $\phi_{\pm}(x, k\omega)$, the scattering matrix, $S(k)$, is given by

$$[S(k)f](\omega) = f(\omega) - \int\int e^{ik\omega\cdot x}\phi_{+}(x, k\omega')f(\omega')V(x)\,dx\,d\omega'\,. \tag{2.3}$$

When one views the the potential, $|V(x)|$ as a weight defining a Hilbert space, $\mathbf{K} = L^2(\mathbb{R}^n, |V(x)|)$, this operator takes the form,

$$S(k) = 1 - \gamma(k)(1 - Q^{+}(k^2))^{-1}\gamma(k)^*\,,$$

where $\gamma(k)$ is a mapping from $\mathbf{K}$ to $L^2(S^{n-1})$ where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. This mapping is sometimes referred to as a *trace* operator and is given by

$$\gamma(k)f(\omega) = \int_{\mathbb{R}^n} e^{-ik\omega\cdot x}f(x)V(x)\,dx \tag{2.4}$$

and $Q^{+}(k^2)$ is the boundary value of the modified unperturbed resolvent operator mapping $\mathbf{K}$ to $\mathbf{K}$ given by

$$Q^{+}(k^2)f(x) = \int_{\mathbb{R}^n} G(x-y;k)f(y)V(y)\,dy \tag{2.5}$$

Under various limiting absorbtion principles the perturbed resolvent admits a boundary value of its associated Green's function, $G_1^{+}(x, y : k)$. The associated modified resolvent operator is then given by

$$Q_1^{+}(k^2)f(x) = \int_{\mathbb{R}^n} G_1^{+}(x, y; k)f(y)V(y)\,dy. \tag{2.6}$$

The Marchenko-Newton, Saitō, and other inverse scattering methods essentially require that this modified perturbed resolvent vanish in some sense as $k \to \infty$. The Riemann-Lebesgue properties established in Theorem 2.1 can weakly annihilate the Born remainder terms in the inner product between the scattered and unscattered plane waves. This remainder is given in weak formulation by

$$-\pi\left(\frac{k}{2\pi}\right)^{n-1}\int_{\mathbb{R}^n}\langle[\gamma(k)Q_1^{+}(k^2)\gamma(k)^*e^{-ik(\cdot)\cdot x}](\omega), e^{-ik\omega\cdot x}\rangle_{\omega}\psi(x)\,dx\,, \tag{2.7}$$

with $\psi(x)$ in $C_0^{\infty}(\mathbb{R}^n)$, and where the integration $\langle\cdot,\cdot\rangle_{\omega}$ is taken over the surface of the unit sphere. As we will show in our application to measure potentials, this remainder will reduce to

$$\text{const.} \times k^{n-1}\int\int G_1(x, y; k)J(k|\xi - y|)J(k|x-y|)\psi(x)V(d\xi)V(dy)\,dx \tag{2.8}$$

which will vanish as $k \to \infty$ under appropriate conditions on G_1 due to the Riemann-Lebesgue properties of J (which is independent of the potential) rather than the decay of modified *perturbed* resolvent. The potential can therefore be reconstructed weakly through the knowledge of the high energy scattering data with the Saitō method. We will first carry out the details of this process in the next section. The proof of Theorem 2.1 will then be carried out in section 4.

3 Application to Inverse Scattering

Theorem 2.1 shall now be applied to the inverse scattering problem for a broad class of measure potentials. This section will conclude with an explicit example, the delta function on a sphere. Let us consider the Schrödinger equation with real potential in $\mathbb{R}^n$ $(n > 2)$. Throughout this section our measure potentials, $V(dx)$, will satisfy the following:

Assumption 1.

$$\sup_{y>M} \int_{|x-y|>1} |x-y|^{\frac{1-n}{2}} |V|(dx) \to 0 \quad \text{as } M \to \infty\,, \tag{3.1}$$

$$\sup_{y>M} \int_{|x-y|\le 1} |x-y|^{2-n} |V|(dx) \to 0 \quad \text{as } M \to \infty\,, \tag{3.2}$$

$$\sup_{y} \int_{|x-y|<\delta} |x-y|^{2-n} |V|(dx) \to 0 \quad \text{as } \delta \to 0\,. \tag{3.3}$$

Our conditions in Assumption 1 will allow for the possibility of highly singular measure potentials, $V(dx)$, such as the delta function on a manifold. We denote by $\mathbf{K}$ the space of square integrable functions under the measure, $|V|(dx)$ with norm and inner product given respectively by

$$\|f\|_K = \left\{ \int |f(x)|^2 |V|(dx) \right\}^{1/2}, \quad \langle f, g\rangle_K = \int f(x)\overline{g(x)} |V|(dx)\,. \tag{3.4}$$

Let $\mathcal{S}$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and $\mathcal{S}'$ the corresponding set of tempered distributions. Let $\mathcal{K}$ be the space of $C_0^\infty(\mathbf{R}^n)$-functions with the usual "test function" topology, i.e. $f_k \to 0$ in $\mathcal{K}$ if and only if some bounded set contains all of their supports and the ∞-norm of f_k and all of its derivatives vanish. Let $\mathcal{K}'$ be the set of continuous linear functionals on $\mathcal{K}$. We shall define A as the identification operator, $A\,f(x) = f(x)$, from $\mathcal{K}$ to $\mathbf{K}$. It is shown in [8] that under Assumption 1, $AR(z)$ extends to a bounded operator from $L^2(\mathbb{R}^n)$ to $\mathbf{K}$. An important operator on the space, $\mathbf{K}$ is the modified resolvent given by $Q(z) = A[AR(\overline{z})]^*$ so that for $z = \kappa^2$, $\operatorname{im}\kappa > 0$,

$$Q(z)f(x) = \int G(x, y; \kappa) f(y) V(dy)\,. \tag{3.5}$$

It is also shown in [8] that there is a self-adjoint operator H_1 satisfying

$$(H_1 u, v) = (u, Hv) + \int u(x)\overline{v(x)} V(dx) \quad u \in D(H_1),\ v \in D(H)\,. \tag{3.6}$$

We denote by $R_1(z)$ the associated resolvent operator, $[z - H_1]^{-1}$ and let $Q_1(z) = A[AR_1(\overline{z})]^*$. Under Assumption 1 the wave operators

$$W_\pm(H_1, H) = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH_1} e^{-itH}$$

exist and are strongly complete and the scattering operator, $S = W_+^* W_-$ exists and is unitary. It is shown in [10] that there exists a family of unitary operators, $\{S(k)\}_{k>0}$ on $L^2(S^{n-1})$ such that

$$[S(k)f(k,\cdot)](\omega) = \hat{S} f(k\omega) \quad f \in \mathcal{K} \tag{3.7}$$

where $\hat{S} = \mathcal{F}S\mathcal{F}^*$, $\mathcal{F}$ is the unitary Fourier Transform and $k\omega$ are polar coordinates in $\mathbb{R}^n$. Modified *trace* operators, $\gamma(k)$ and $\gamma_\pm(k)$ are defined by:

$$\gamma(k)f(\omega) = \int e^{-ik\omega\cdot x} f(x)V(dx) \tag{3.8}$$

$$\gamma_\pm(k)f(\omega) = \int \phi_\pm(x, k\omega)f(x)V(dx)\,, \tag{3.9}$$

where $\phi_\pm$ are the generalized eigenfunctions associated with H_1. It is known [10] that these are bounded operators from $\mathbf{K}$ to $L^2(S^{n-1})$ under the conditions in Assumption 1. Furthermore, we have the following representation of the scattering matrix.

$$1 - S(k) = \frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2} \gamma_+(k)\gamma(k)^*. \tag{3.10}$$

We will denote by V the linear functional associated with $V(dx)$,

$$(V, u) = \int u(x)V(dx), \quad u \in \mathcal{S}. \tag{3.11}$$

We denote by ν the linear functional defined by ,

$$(\nu, f) = \int\int |x-y|^{1-n} f(x)V(dy)\,dx, \quad u \in \mathcal{S}. \tag{3.12}$$

Finally, we define the function, $h(k, x)$ on $(\mathbb{R}^+ \times \mathbb{R}^n)$ by

$$h(k, x) = 2\pi ik\langle(1 - S(k))e^{-ik\omega\cdot x}, e^{-ik\omega\cdot x}\rangle_\omega \tag{3.13}$$

where $\langle\cdot,\cdot\rangle_\omega$ is the inner product in $L^2(S^{n-1})$. Noting that $h(k, x)$ is bounded for each fixed k, we consider $h(k)$ as the associated distribution in $\mathcal{S}'$ given by

$$(h(k), u) = \int h(k, x)\overline{u(x)}\,dx \tag{3.14}$$

Assumption 2A. There exists some positive k_0 such that $Q_1(k^2 + i\epsilon)$ admits a boundary value, $Q_1^+(k^2) = \lim_{\epsilon\searrow 0} Q_1(k^2 + i\epsilon)$ with integral kernel, $G_1^+(x, y; k)$ for all $k > k_0$. Furthermore, there exists a function $F(x, y)$ on $\mathbf{R}^{2n}$ with $F(x, y) > |G_1^+(x, y; k)|$ for all $k > k_0$ and $\int\int F(x, y)|V|(dx)|V|(dy) < \infty$. $G_1^+(x, y; k)$ is the Green's function and is known to exist under a variety of conditions and is described as the response at x to a point source at y (see [5] and the references provided). As an alternative to Assumption 2A, we can offer

Assumption 2B. There exists a positive number, $\alpha < 1$, some k_0, and a function, $F(x, y)$ on $\mathbb{R}^{2n}$ such that for all $k > k_0$, $F(x, y) \geq |G(|x - y|; k)|$ and

$$\sup_y \int F(x, y)|V|(dx) \leq \alpha \tag{3.15}$$

We define $\Omega_V = \int |V|(dx)$ and assume it is finite. We now provide;

Theorem 3.1 *If V satisfies Assumption 1 above, then the following holds:*
1. *V and ν as given above are both continuous linear functionals on $\mathcal{S}$, hence in $\mathcal{S}'$.*
2. *The linear functional, $\Lambda\nu$, defined on $\mathcal{K}$ by*

$$(\Lambda\nu, \psi) = (\nu, \mathcal{F}^*(|\xi|\mathcal{F}\psi)) \tag{3.16}$$

extends to a well-defined element of $\mathcal{S}'$.

If, in addition V satisfies either of assumptions 2A or 2B then we also have
3. *$lim_{k\to\infty} h(k) = -2\pi\nu$ in $\mathcal{S}'$ and*
4. *V can be recovered in the sense of distributions through:*

$$\langle V, \psi\rangle = (2\pi)^{-\frac{n}{2}}\alpha^{-1}(\Lambda\nu, \psi) \quad \psi \in \mathcal{S}\,, \tag{3.17}$$

where $\alpha = 2^{\frac{2-n}{2}}\sqrt{\pi}\Gamma(\frac{n-1}{2})^{-1}$.

Proof: To show that V is in $\mathcal{S}'$ we let $\psi_k(x)$ be a sequence of functions vanishing in $\mathcal{S}$. We therefore have for any $\alpha > 0$ a vanishing sequence C_k^α such that

$$|x^\alpha \psi_k(x)| \le C_k^\alpha \quad \forall\, x \in \mathbb{R}^n \tag{3.18}$$

We now have that

$$\begin{aligned} |(V, \psi_k)| &\le \int_{|x|\ge 1} |\psi_k(x)||V|(dx) + \int_{|x|<1} |\psi_k(x)|V|(dx) \\ &\le C_k^\alpha \int_{|x|\ge 1} |x|^{-\alpha}|V|(dx) + C_k^0 \int_{|x|<1} |V|(dx). \end{aligned}$$

Taking $\alpha = (n-1)/2$ and applying (3.1) and (3.2) we see this vanishes as $k \to \infty$ showing that V is in $\mathcal{S}'$. Turning our attention to ν we write

$$\begin{aligned} (\nu, \psi_k) &= \int\int_{|x-y|\le 1} |x-y|^{1-n}\psi_k(x)\, dx V(dy) \\ &\quad + \int\int_{|x-y|>1} |x-y|^{1-n}\psi_k(x)\, dx V(dy) \\ &= I_1(k) + I_2(k) \end{aligned}$$

Since ψ_k vanishes in $\mathcal{S}$ we also have for each $\alpha \ge 0$ vanishing sequences of constants, c_k^α such that

$$|\psi_k(x)| \le c_k^\alpha (2 + |x|)^\alpha \quad \forall x \in \mathbb{R}^n. \tag{3.19}$$

Thus we have that

$$\begin{aligned} |I_1| &\le \int \left\{ \sup_{x:|x-y|\le 1} c_k^\alpha (2+|x|)^{-\alpha} \right\} \int_{|x-y|\le 1} |x-y|^{1-n}\, dx |V|(dy) \\ &\le c_k^\alpha \Omega^n \int (1+|y|)^{-\alpha}|V|(dy)\,, \end{aligned}$$

where Ω^n is the surface area of S^{n-1}. Again by (3.1) and (3.2) we see by taking $\alpha \geq \frac{n-1}{2}$ that $I_1(k)$ vanishes as $k \to \infty$. The verification that $I_2(k)$ vanishes is similar and follows from the observation that for $|x-y| > 1$, we have $|x-y|^{1-n} \leq |x-y|^{(1-n)/2}$. This completes the proof of item 1 of Theorem 3.1. Item 2 requires that we show that $\Lambda\nu \in \mathcal{S}'$. Here we will borrow some ideas from Saitō [19]. We define for $s \geq 0$ a norm $\|\cdot\|_s$ on $\mathcal{S}$ by

$$\|\psi\|_s = \sum_{|\beta|\leq n} \sum_{\alpha\leq\beta} \int |\xi|^{s-|\beta|+|\alpha|} \, |D^\alpha \psi(\xi)| \; d\xi \tag{3.20}$$

where α and β are multi-indices. As noted by Saitō the topology induced by this norm is weaker than the proper topology on $\mathcal{S}$. Furthermore, it follows from the proof of lemma A.1 in [19] that for any $\psi \in \mathcal{S}$

$$|\mathcal{F}^*(|\xi|\psi)(x)| \leq C_s(2+|x|)^{-n}\|\psi\|_s \tag{3.21}$$

where C_s is a constant depending on s. With ψ_k as before, we have

$$|(\Lambda\nu, \psi_k)| \leq C_s \|\mathcal{F}\psi_k\|_s \left| \int\int \; |x-y|^{1-n}(2+|x|)^{-n} V(dy)\, dx \right| \tag{3.22}$$

By the method of proof applied to V and ν one can verify the integral remaining is finite, hence the right hand side will vanish as $k \to \infty$ proving item 2 of the theorem.

Moving on to item 3, we let $\psi(x) \in \mathcal{K}$. By the definition of $h(k,x)$ above and (3.10) we have that

$$(h(k), \psi) = -\pi \left(\frac{k}{2\pi}\right)^{n-1} \int \langle \gamma_+(k)\gamma(k)^* e^{-ik\omega\cdot x}, e^{-ik\omega\cdot x}\rangle_\omega \overline{\psi(x)}\, dx\,. \tag{3.23}$$

By Assumption 1 and the results of [8] we know that $1 - Q(\kappa^2)$ has a bounded inverse given by $1 + Q_1(\kappa^2)$ for im κ^2 sufficiently large and where $Q_1(\kappa^2)$ is defined by,

$$Q_1(\kappa^2)f(x) = \int G_1(x,y;\kappa) f(y) V(dy)\,, \tag{3.24}$$

where $G_1(x,y;\kappa)$ is the kernel of the perturbed resolvent operator. By Assumption 2A, analytic continuation, and continuity we have that

$$\left[1 - Q^+(k^2)\right]^{-1} = 1 + Q_1^+(k^2) \text{ for } k > k_0\ . \tag{3.25}$$

By [10] (Eqn. 4.23) we have that $\gamma_+(k) = \gamma(k)(1 + Q_1^+(k^2))$. Inserting this into (3.23) we obtain,

$$(h(k), \psi(x)) = (h_1(k), \psi(x)) + (h_2(k), \psi(x)) \tag{3.26}$$

where

$$(h_1(k), \psi(x)) = -\pi \left(\frac{k}{2\pi}\right)^{n-1} \int \langle \gamma(k)\gamma(k)^* e^{-ik\omega\cdot x}, e^{-ik\omega\cdot x}\rangle_\omega \overline{\psi(x)}\, dx \tag{3.27}$$

and

$$(h_2(k),\psi(x)) = -\pi\left(\frac{k}{2\pi}\right)^{n-1}\int\langle\gamma(k)Q_1^+(k^2)\gamma(k)^* e^{-ik\omega\cdot x}, e^{-ik\omega\cdot x}\rangle_\omega\overline{\psi(x)}\,dx \tag{3.28}$$

The first term, h_1, corresponds to the Born approximation while the second term, h_2 is the Born remainder. We will look at h_2 first to illustrate how Theorem 2.1 applies. By writing all of the integrals involved including the integral form of $Q_1(k)$, and employing Assumption 2A and Fubini's theorem we arrive at

$$(h_2(k),\psi(x)) = Ck^{n-1}\int G_1(x,y;k)J_k^\psi(x,y)V(dx)V(dy)\,, \tag{3.29}$$

where C is a constant and

$$J_k^\psi(x,y) = \int J(k|x-\xi|)J(k|\xi-y|)\overline{\psi(\xi)}\,d\xi. \tag{3.30}$$

By Assumption 2A we can apply the Lebesque dominated convergence theorem and Theorem 2.1 to conclude this term vanishes as $k\to\infty$. If Assumption 2B holds instead of Assumption 2A, then the Neumann series for $[1-Q^+(k^2)]^{-1}$ will converge uniformly for $k>k_0$. In this case, we have the expansion

$$\begin{aligned}&(h_2(k),\psi(x)) \\ &= -\pi\left(\frac{k}{2\pi}\right)^{n-1}\sum_{N=1}^{\infty}\int\langle\gamma(k)Q^+(k)^N\gamma(k)^* e^{-ik(\cdot)\cdot x}(\omega), e^{-ik\omega\cdot x}\rangle_\omega\overline{\psi(x)}\,dx\,.\end{aligned} \tag{3.31}$$

For each fixed N, we can construct the iterated kernel, $Q^N(x,y,k)$ for $Q^+(k)^N$ given by

$$Q^0(x,y,k) = G(|x-y|;k), \quad Q^N(x,y;k) = \int Q^{N-1}(x,\xi,k)G(|\xi-y|;k)V(d\xi)\,. \tag{3.32}$$

We next define $F^N(x,y)$ similarly by replacing G with F in (3.32). By induction we can see that $F^N(x,y)\ge|Q^N(x,y;k)|$ for all k and furthermore,

$$\int F^N(x,y)|V|(dx)|V|(dy)\le\Omega_V\alpha^{N+1}. \tag{3.33}$$

We therefore have

$$\begin{aligned}&\left|-\pi\left(\frac{k}{2\pi}\right)^{n-1}\int\langle\gamma(k)Q^+(k)^N\gamma(k)^*(k), e^{-ik(\cdot)\cdot x}(\omega), e^{-ik\omega\cdot x}\rangle_\omega\overline{\psi(x)}\,dx\right| \\ &= \left|\pi\left(\frac{k}{2\pi}\right)^{n-1}\int Q^N(x,y,k)J_k^\psi(x,y)V(dx)V(dy)\right| \\ &\le \pi\left(\frac{k}{2\pi}\right)^{n-1}\int\left|F^N(x,y)J_k^\psi(x,y)\right||V|(dx)|V|(dy)\,.\end{aligned}$$

The last integral will vanish as $k \to \infty$ by dominated convergence and the fact that $J_k^\psi(x,y)$ vanishes for all $x \neq y$ by Theorem 2.1. Furthermore, by the uniform convergence of the Neumann series for $k > k_0$ we see that the infinite sum yielding $h_2(k)$ will vanish as well. We now turn our attention to the Born approximation, $h_1(k)$. By again writing out the integrals involved and employing Fubini's theorem we obtain,

$$(h_1(k), \psi(x)) = -\pi \left(\frac{k}{2\pi}\right)^{n-1} \int\int |J(k|x-y|)|^2\, \overline{\psi(x)} V(dy)\, dx. \tag{3.34}$$

Setting $\Delta(k) = J(k) - 2(\frac{k}{2\pi})^{\frac{1-n}{2}} \cos(k - \frac{(n-1)\pi}{4})$ we have by [19] equation (2.17)

$$|\Delta(k)| \le C \min\{k^{-\frac{(n-1)}{2}}, k^{-\frac{(n+1)}{2}}\}\,, \tag{3.35}$$

where C is a constant. We can now write,

$$\begin{aligned}
&-(h_1(k), \psi(x)) \\
&\quad= \pi \left(\frac{k}{2\pi}\right)^{n-1} \int\int \left|2\left(\frac{k}{2\pi}\right)^{\frac{1-n}{2}} \cos\left(k - \frac{(n-1)\pi}{4}\right) + \Delta(k|x-y|)\right|^2 \\
&\qquad\qquad \times \overline{\psi(x)} V(dy)\, dx \\
&\quad= \pi \left(\frac{k}{2\pi}\right)^{n-1} \int\int |\Delta(k|x-y|)|^2 \overline{\psi(x)} V(dy)\, dx \\
&\qquad +4\pi \left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} \mathrm{Re} \int\int \Delta(k|x-y|) \cos\left(k - \frac{(n-1)\pi}{4}\right) \overline{\psi(x)} V(dy)\, dx \\
&\qquad +4\pi \int\int |x-y|^{1-n} \cos\left(k - \frac{(n-1)\pi}{4}\right) \overline{\psi(x)} V(dy)\, dx\,.
\end{aligned}$$

We set these three terms equal to $L_1(k)$, $L_2(k)$, and $L_3(k)$ respectively. Our proof of item 3 of Theorem 3.1 will be completed by showing that $L_1(k)$ and $L_2(k)$ vanish as $k \to \infty$ and that $\lim_{k\to\infty} L_3(k) = 2\pi(\nu, \psi)$. Regarding $L_2(k)$, by using (3.3), (3.35) and the fact that for $|x-y| < 1$, $|x-y|^{\frac{1-n}{2}}| \le |x-y|^{2-n}$ we see that

$$\begin{aligned}
&4\pi \left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} \int\int_{|x-y|\le\delta} \left|\Delta(k|x-y|) \cos\left(k - \frac{(n-1)\pi}{4}\right) \overline{\psi(x)}\right| |V|(dy)\, dx \\
&\quad\le \text{constant} \times \int\int_{|x-y|<\delta} |x-y|^{2-n} |\psi(x)||V|(dy)\, dx \to 0 \quad \text{as } \delta \to 0
\end{aligned}$$

Furthermore, for fixed $\delta > 0$ we also have that

$$\begin{aligned}
&4\pi \left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} \int\int_{|x-y|>\delta} \left|\Delta(k|x-y|) \cos\left(k - \frac{(n-1)\pi}{4}\right) \overline{\psi(x)}\right| |V|(dy)\, dx \\
&\le \mathit{constant} \times k^{-1} \int\int_{|x-y|>\delta} |x-y|^{\frac{1-n}{2}} |\psi(x)||V|(dy)\, dx \\
&\le \mathit{constant} \times k^{-1} ||\psi||_{L^2} \left\{ \sup_x \int_{|x-y|>\delta} |x-y|^{\frac{1-n}{2}} |\psi(x)||V|(dy) \right\}
\end{aligned}$$

which vanishes as $k \to \infty$. This shows that $\lim_{k\to\infty} L_2(k) = 0$. A similar approach can be applied to show that $\lim_{k\to\infty} L_1(k) = 0$. Considering $L_3(k)$ we first note that applying an elementary trig identity we have

$$\begin{aligned} L_3(k) &= 2\pi \int\int |x-y|^{1-n}\left(1+\cos(2k|x-y| - \frac{(n-1)\pi}{2})\right)\overline{\psi(x)}V(dy)\,dx \\ &= 2\pi(\nu,\psi) + L_4(k) + L_5(k)\,, \end{aligned}$$

where

$$\begin{aligned} L_4(k) &= 2\pi \int\int_{|x-y|\le 1} |x-y|^{1-n}\cos\left(2k|x-y| - \frac{(n-1)\pi}{2}\right)\overline{\psi(x)}V(dy)\,dx \\ L_5(k) &= 2\pi \int\int_{|x-y|> 1} |x-y|^{1-n}\cos\left(2k|x-y| - \frac{(n-1)\pi}{2}\right)\overline{\psi(x)}V(dy)\,dx \end{aligned}$$

Concerning $L_4(k)$ we set

$$F_1(k,y) = \int_{|x-y|\le 1} |x-y|^{1-n}\cos(2k|x-y| - \frac{(n-1)\pi}{2})\psi(x)\,dx \tag{3.36}$$

It is easy to see that there is a bounded set containing the supports of $F_1(k,y)$ for all k, for if the support of ψ is contained in $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ then for any k, $F_1(k,y) = 0$ for $|y| > R+1$ It is also easy to see that $F_1(k,y)$ is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^n$. Furthermore, by the Riemann-Lebesgue lemma, for each fixed y, $F_1(k,y)$ vanishes as $k \to \infty$. Applying the bounded convergence theorem we have $\lim_{k\to\infty} L_4(k) = 0$. Turning our attention to $L_5(k)$ we have

$$\begin{aligned} |L_5(k)| &\le \int\int_{|x-y|>1} |x-y|^{1-n}|\psi(x)||V|(dy)\,dx \\ &\le \left\{\sup_x \int_{|x-y|>1} |x-y|^{\frac{1-n}{2}}|V|(dy)\right\} ||\psi(x)||_{L^1}\,, \end{aligned}$$

which is finite by (3.1). We can conclude by Fubini-Tonelli that if $F(y) = \int_{|x-y|>1} |x-y|^{1-n}|\psi(x)|\,dx$ then $F(y)$ is in $L^1(\mathbb{R}^n; V(dy))$. Since ψ is in $\mathcal{K}$ we can conclude that for all y, $|x-y|^{1-n}\psi(x)$ is in $L^1(\mathbb{R}^n; dx)$. Setting

$$F_2(k,y) = \int_{|x-y|>1} |x-y|^{1-n}\cos(2k|x-y| - \frac{(n-1)\pi}{2})\overline{\psi(x)}\,dx \tag{3.37}$$

we can conclude by the Riemann-Lebesque lemma that $F_2(k,y)$ vanishes as $k \to \infty$ for almost all y. Since $F(y) \ge F(k,y)$ we can apply dominated convergence to conclude

$$|L_5(k)| = 2\pi\left|\int F(k,y)V(dy)\right| \to 0 \quad \text{as } k \to \infty\,. \tag{3.38}$$

Since $\mathcal{K}$ is dense in $\mathcal{S}$, this completes the verification that $h_1(k) \to -2\pi\nu$ proving item 3 of Theorem 3.1.

To verify item 4 of the theorem we let $\psi \in \mathcal{S}$ and set $\varphi(\xi) = \alpha^{-1}(2\pi)^{-\frac{n}{2}}|\xi|\mathcal{F}\psi(\xi)$ where $\alpha = 2^{\frac{2-n}{2}}\sqrt{\pi}\Gamma(\frac{n-1}{2})^{-1}$. We claim that

$$\left[|x|^{1-n} * \mathcal{F}^*\varphi\right](x) = \psi(x) \quad \forall x \in \mathbb{R}^n. \tag{3.39}$$

By using the fact that $\mathcal{F}(|x|^{1-n})(\xi) = \alpha|\xi|^{-1}$ (e.g. see [12]) we have for any $\phi \in \mathcal{K}$,

$$\begin{aligned}
\langle \phi, |x|^{1-n} * \mathcal{F}^*\varphi\rangle &= \alpha^{-1}(2\pi)^{-\frac{n}{2}}\langle |x|^{1-n} * \phi, \mathcal{F}^*|\xi|\mathcal{F}\psi(\xi)\rangle \\
&= \alpha^{-1}(2\pi)^{-\frac{n}{2}}\langle \mathcal{F}[|x|^{1-n} * \phi], |\xi|\mathcal{F}\psi(\xi)\rangle \\
&= \langle |\xi|^{-1}\mathcal{F}\phi, |\xi|\mathcal{F}\psi(\xi)\rangle \\
&= \langle \phi, \psi\rangle
\end{aligned}$$

This together with the fact that both sides of (3.39) are continuous proves our claim. We now have;

$$\begin{aligned}
\langle \nu, \mathcal{F}^*\varphi\rangle &= \langle |x|^{1-n} * V, \mathcal{F}^*\varphi\rangle \\
&= \langle V, |x|^{1-n} * \mathcal{F}^*\varphi\rangle \\
&= \langle V, \psi\rangle
\end{aligned}$$

Thus in the sense of distributions, $\alpha^{-1}(2\pi)^{-\frac{n}{2}}\mathcal{F}^*|\xi|\mathcal{F}\nu = V$ completing the proof of item 4. Note that once V is recovered, it is a simple matter to reconstruct the associated measure potential, $V(dx)$ (see [11] for details).

A Singular Example

We now provide a concrete example illustrating the use of these results. We will let $n = 3$ and define $V(dx)$ as the measure, $\beta\delta(|x| - R)$, the delta function of the sphere of radius R and strength parameter β. For any $\psi \in \mathcal{K}$ we have

$$\langle V, \psi\rangle = \int_{S_R} \beta\psi(x)\, d\omega(x)\,, \tag{3.40}$$

where S_R is the sphere of radius R and $d\omega(x)$ is the inherited surface measure. The verification that Assumption 1 of Theorem 3.1 holds is straightforward. In this explicit example $G(x,k) = -\frac{1}{4\pi}\frac{e^{i\kappa|x|}}{|x|}$ and so $Q(z)$ is given by;

$$Q(\kappa^2)f(x) = -\frac{\beta}{4\pi}\int_{S_R} \frac{e^{i\kappa|x-y|}}{|x-y|} f(y)\, d\omega(x)\,. \tag{3.41}$$

Taking $F(x,y) = \frac{1}{4\pi|x-y|} = |G(x-y;k)|$ we note that the $\sup_y \int F(x,y)|V|(dx)$ occurs when $|y| = R$, and therefore

$$\sup_y \int F(x,y)|V|(dx) = \frac{\beta}{4\pi}\int_{S_R} \frac{1}{|x-y|}\, d\omega(x) = \beta\, R\,. \tag{3.42}$$

In addition we have $\int |V|(dx) = \int_{S^2} d\omega = \Omega^3$. Thus we see that Assumption 2B holds whenever $\beta < R$. Theorem 3.1 then provides us the following:

$$\lim_{k\to\infty} \big(-2\pi ik\langle(1-S(k))e^{-ik\omega\cdot x}, e^{-ik\omega\cdot x}\rangle, \psi(x)\big) = \int_{S_R} \frac{\beta\overline{\psi(y)}}{|x-y|}\, d\omega(y) \tag{3.43}$$

Furthermore, (3.17) holds as well giving

$$\begin{aligned}&\lim_{k\to\infty} \big(-(2\pi)^{-\frac{n}{2}} ik\alpha^{-1}\langle(1-S(k))e^{-ik\omega\cdot x}, e^{-ik\omega\cdot x}\rangle_\omega, \mathcal{F}^*(|\xi|\mathcal{F}\psi(x)\big)\\ &\quad = \int_{S_R} \beta\overline{\psi(x)}\, d\omega(x)\,. \end{aligned}\tag{3.44}$$

We summarize this example by stating the following:

Theorem 3.2 *Fix $R > 0$ and let $0 < \beta < R$. Let H be the self-adjoint realization of the Laplacian in $\mathbb{R}^3$. Then the following hold:*

1. *There exists a self-adjoint operator, H_1, satisfying*

$$(H_1 u, v) = (u, Hv) + \beta \int_{S_R} u(x)\overline{v(x)}\, d\omega(x)$$

for all u in $D(H_1)$ and v in $D(H)$.

2. *The associated wave operators, $W_\pm(H_1, H)$ exist and are strongly complete.*

3. *There exist generalized eigenfunctions, $\phi_\pm(x,\xi)$ satisfying*

$$\phi_\pm(x,\xi) = e^{-ix\cdot\xi} - 4\pi\beta \int_{S_R} \frac{\phi_\pm(y,\xi)e^{\pm i|\xi||x-y|}}{|x-y|} d\omega(y).$$

4. *The scattering matrix, $S(k)$, exists as a unitary operator on $L^2(S^2)$ satisfying*

$$[S(k)f(k,\cdot)](\omega) = \left[\mathcal{F}W_+^* W_- \mathcal{F}^* f\right](k\omega) \quad \forall\ f \in C_0^\infty(\mathbb{R}^3)\,.$$

5. *The scattering matrix admits the representation,*

$$S(k) = 1 - \frac{ik}{8\pi^2}\gamma_+(k)\gamma(k)\,,$$

where γ and γ_+ are given by (3.8) and (3.9).

6. *The inverse scattering results, equations (3.43) and (3.44) hold.*

The remainder of this work is devoted to the proof of main theorem.

4 Proof of Theorem 2.1

We will prove the theorem through a series of lemmas.

Lemma 4.1 *Fix $z \neq 0$ in $\mathbb{R}^n$. Let $(\rho, \theta_1, \theta_2, \ldots, \theta_{n-1})$ be spherical coordinates with θ_1 the polar angle measured from the positive z direction. Let Ω be an open n-rectangle in $\mathbb{R}^n$, i.e.*

$$\Omega = \{(\rho, \theta_1, \theta_2, \ldots, \theta_{n-1}) | \rho \in (a_0, b_0), \theta_1 \in (a_1, b_1), \ldots, \theta_{n-1} \in (a_{n-1}, b_{n-1})\},$$

where $0 \leq a_0 < b_0 \leq \infty$, $0 \leq a_{n-1} < b_{n-1} \leq 2\pi$, and $0 \leq a_i < b_i \leq \pi$, $i = 1, 2, \ldots, n-2$. Let $\mathbf{X}_\Omega(x)$ be the characteristic function for $\Omega \subset \mathbb{R}^n$. Set

$$c(k, r) = \cos(kr - \frac{(n-1)\pi}{4}) \quad k, r \in \mathbb{R}, \tag{4.1}$$

and set

$$I(k, z) = \int_{\mathbb{R}^n} \frac{c(k, |x|)}{|x|^{n-2}} \frac{c(k, |z-x|)}{|z-x|} (\sin\theta_1)^{3-n} \mathbf{X}_\Omega(x)\, dx\,. \tag{4.2}$$

Then for all $z \neq 0$

$$\lim_{k \to \infty} I(k, z) = 0\,. \tag{4.3}$$

Proof: Noting that the integrand in (4.2) is dependent only on ρ and θ_1, and calculating the integral in spherical coordinates, we have that

$$I(k, z) = C \int_{a_0}^{b_0} \int_{a_1}^{b_1} \frac{c(k, \rho)}{\rho^{n-2}} \frac{c(k, u)}{u} \rho^{n-1} \sin\theta_1\, d\theta_1\, d\rho\,, \tag{4.4}$$

where C is the constant obtained by integrating over the angular variables, θ_i, $i \geq 2$ and where $u = |z - x|$. We make a change of coordinates in the θ_1-variable by setting

$$u = |z - x| = (|z|^2 + |x|^2 - 2|z||x| \cos\theta_1)^{1/2}$$

to obtain

$$\begin{aligned} I(k, z) &= C \int_{a_0}^{b_0} \int_{u_a}^{u_b} \frac{c(k, \rho) c(k, u)}{|z|}\, d\theta_1\, d\rho \\ &= \frac{C}{k|z|} \int_{a_0}^{b_0} c(k, \rho) \left(s(k, u_b) - s(k, u_a)\right)\, d\rho\,, \end{aligned} \tag{4.5}$$

where u_a and u_b are the appropriate new limits of integration in the u-variable. And where

$$s(k, \rho) = \sin(k\rho - \frac{(n-1)\pi}{4})$$

We now can see that the final integral in (4.5) is uniformly bounded in k, hence $|I(k, z)| \leq$ constant $/k$ which vanishes as $k \to \infty$. □

The following corollary is immediate.

Corollary 4.2 *Let $\phi(x)$ be any step function on $\mathbb{R}^n$ of the form $\sum_{j=1}^m a_j \mathbf{X}_{\Omega_j}(x)$ where each $\Omega_j(x)$ is an open n-rectangle as in lemma (4.1). Then for all $z \neq 0$*

$$\int_{\mathbb{R}^n} \frac{c(k,|x|)}{|x|^{n-2}} \frac{c(k,|z-x|)}{|z-x|} (\sin\theta_1)^{3-n} \phi(x)\, dx \to 0 \quad \text{as } k \to \infty\,. \tag{4.6}$$

Lemma 4.3 *Let $z \neq 0$ be fixed in $\mathbb{R}^n$ and set $\Psi(x,k) = \frac{c(k,|x|)}{|x|^{n-2}} \frac{c(k,|z-x|)}{|z-x|} (\sin\theta_1)^{3-n}$ where θ_1 and $c(k,r)$ are as in Lemma 4.1. Then for $0 < p < \frac{n}{n-2}$, and any bounded, measurable set, $\Omega \subset \mathbb{R}^n$, $\Psi(x,k) \in L^p(\Omega)$ for each k. Furthermore, there exists a constant, $C(\Omega, z, p)$ depending on Ω, z, and p but independent of k such that $||\Psi(x,k)||_p < C(\Omega, z, p)$.*

Proof: If we fix, $\Omega \subset \mathbb{R}^n$, and let $M = \sup\{|x| \,:\, x \in \Omega\}$. We have

$$\int_\Omega |\Psi(x,k)|^p\, dx = I_1 + I_2 + I_3\,, \tag{4.7}$$

where

$$\begin{aligned} I_1 &= \int_{\Omega_1} |\Psi(x,k)|^p\, dx\,, \quad \Omega_1 = \Omega \cap \left\{ x \,:\, |x| < \frac{|z|}{2}\,, \right\} \\ I_2 &= \int_{\Omega_2} |\Psi(x)|^p\, dx\,, \quad \Omega_2 = \Omega \cap \left\{ x \,:\, |x-z| < \frac{|z|}{2} \right\}\,, \end{aligned}$$

and I_3 is the integral over the remaining region. Considering I_1 we switch to spherical coordinates taking the z direction as the polar axis and letting θ_1 be as before. We obtain

$$\begin{aligned} |I_1| &\leq \int_{|x|<\frac{|z|}{2}} |x|^{(2-n)p} \left(\frac{2}{|z|}\right)^p (\sin\theta_1)^{(3-n)p}\, dx \qquad (4.8) \\ &\leq C \left(\frac{2}{|z|}\right)^p \int_{\rho=0}^{\frac{|z|}{2}} \int_{\theta_1=0}^{\pi} \rho^{(2-n)p+n-1} (\sin\theta_1)^{(3-n)p+n-2}\, d\theta_1\, d\rho\,, \end{aligned}$$

where C is as in Lemma 4.1 with $a_i = 0$ for each $i = 1, 2, \ldots, n-1$ and $b_i = \pi$ for $i = 1, 2, \ldots, n-2$ and $b_{n-1} = 2\pi$. The singularities at $\rho = 0$ and $\theta_1 = 0$ are integrable provided that

$$(2-n)p + n - 1 > -1 \text{ and } (3-n)p + n - 2 > -1 \tag{4.9}$$

both of which are satisfied when $p < \frac{n}{n-2}$. Note that for such p we have that $|I_1|$ is bounded by a constant depending only on z. Looking at I_2 we make a change of variable and let $w = z - x$. We will again switch to spherical coordinates taking the z direction as the polar axis, but setting $\tilde{\theta_1}$ to be the angle between z and w. By the law of sines we have that $\sin\theta_1 = \frac{|w|}{|x|} \sin\tilde{\theta_1}$ and so we obtain

$$|I_2| \leq \int_{|w|<\frac{|z|}{2}} |x|^{p(2-n)} |w|^{-p} (\frac{|w|}{|x|} \sin\tilde{\theta_1})^{(3-n)p}\, dx$$

$$\begin{aligned} &= \int_{|w|<\frac{|z|}{2}} |x|^{-1}|w|^{(2-n)p}(\sin\theta_1)^{(3-n)p}\,dx \qquad (4.10)\\ &\le C\int_{\rho=0}^{\frac{|z|}{2}}\int_{\tilde{\theta_1}=0}^{\pi}\left(\frac{2}{|z|}\right)\rho^{(2-n)p+n-1}(\sin\tilde{\theta_1})^{(3-n)p+n-2}\,d\tilde{\theta_1}\,d\rho\,. \end{aligned}$$

Note that the integrability conditions on p are identical with those found in I_1. Thus for $p<\frac{n}{n-2}$, $|I_2|$ is also bounded by a constant depending only on z.

Finally, considering I_3 we note that over the remaining region we have

$$|\Psi(x,k)|^p \le \left(\frac{2}{|z|}\right)^{(n-1)p} \qquad (4.11)$$

and so $|I_3|$ is bounded by this constant times the volume of the sphere of radius M in $\mathbb{R}^n$ Combining these results for I_1, I_2, and I_3 we have that $\int_\Omega|\Psi(x,k)|^p\,dx$ is bounded by a constant depending only on z and Ω. □

Lemma 4.4 *Let z be fixed in $\mathbb{R}^n$, $n\ge 3$ and let $\psi(x)\in C_0^\infty(\mathbb{R}^n)$. Let θ_1 be the angle between z and x as in Lemma 4.3. Then $F(x,\psi)=(\sin\theta_1)^{n-3}|z-x|^{\frac{3-n}{2}}|x|^{\frac{n-3}{2}}\psi(x)$ is a bounded function with compact support and hence is in $L^p(\mathbf{R}^n)$ for all $p\ge 1$.*

Proof: To prove the lemma it is enough to verify that $|z-x|^{\frac{3-n}{2}}(\sin\theta_1)^{n-3}$ is bounded. To see this we first set for $r>0$,

$$\Theta(r)=\sup\{\theta_1 : |z-x|=r\}\,.$$

By elementary geometric considerations we have for any x such that $|z-x|<|z|$ that $|z-x|=|z|\sin\Theta(|z-x|)$. This fact in turn implies that

$$|z-x|\ge|z|\sin\theta_1 \quad \text{whenever } |z-x|<|z|\,. \qquad (4.12)$$

From this we see that for x such that $|z-x|<|z|$ we have,

$$|z-x|^{\frac{3-n}{2}}|\sin\theta_1|^{n-3}\le|z|^{\frac{3-n}{2}}|\sin\theta_1|^{\frac{n-3}{2}}\le|z|^{\frac{3-n}{2}} \qquad (4.13)$$

and trivially for x such that $|z-x|\ge|z|$, we have

$$|z-x|^{\frac{3-n}{2}}|\sin\theta_1|^{n-3}\le|z|^{\frac{3-n}{2}}. \qquad (4.14)$$

□

Lemma 4.5 *Let $\psi(x)\in C_0^\infty(\mathbb{R}^n)$ and $c(k,r)$ be defined as in Lemma 4.1 and set*

$$J(x,y,k,\psi)=\int_{\mathbb{R}^n}\frac{c(k,|x-\xi|)}{|x-\xi|^{\frac{n-1}{2}}}\frac{c(k,|\xi-y|)}{|\xi-y|^{\frac{n-1}{2}}}\psi(\xi)\,d\xi\,. \qquad (4.15)$$

Then for all $x\ne y$, $\lim_{k\to\infty}J(x,y,k,\psi)=0$.

Proof: Setting $w = x - \xi$ and $z = x - y$ we obtain

$$\begin{aligned} J(x,y,k,\psi) &= \int_{\mathbb{R}^n} \frac{c(k,|w|)}{|w|^{\frac{n-1}{2}}} \frac{c(k,|z-w|)}{|z-w|^{\frac{n-1}{2}}} \psi_x(w) dw \\ &= \int_{\mathbb{R}^n} \Psi(k,w) F(w,\psi_x)\, dw\,, \end{aligned} \tag{4.16}$$

where Ψ is as defined in Lemma 4.3 and $F(w,\psi_x)$ is as defined in Lemma 4.4 with $\psi_x(w) = \psi(x-w)$. We now fix $p \in (1, \frac{n}{n-2})$ and let p' satisfy $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\epsilon > 0$ be arbitrary but fixed. Let $C(\Omega_x, z, p)$ be the uniform bound from Lemma 4.3 on the p-norm of $\Psi(k,w)$ and a step function, $\phi(w)$ such that

$$||F(w,\psi_x) - \phi(w)||_{L^p(\Omega_x)} < \epsilon\,.$$

Then by Lemmas 4.3 and 4.4, we have

$$\begin{aligned} &\lim_{k\to\infty} |J(x,y,k,\psi)| \\ &\le \lim_{k\to\infty} \left| \int_{\mathbb{R}^n} \Psi(k,w) F(w,\psi_x)\, dw \right| \\ &\le \lim_{k\to\infty} \left| \int_{\mathbb{R}^n} \Psi(k,w)\phi(w) dw \right| + \left| \int_{\mathbb{R}^n} \Psi(k,w)(F(w,\psi_x) - \phi(w)) dw \right| \\ &\le C(\Omega_x, z, p) ||F(w,\psi_x) - \phi(w)||_{L^{p'}(\Omega_x)} \qquad (4.17) \\ &\le C(\Omega_x, z, p)\epsilon\,. \end{aligned}$$

The present proof follows from the fact that ϵ was arbitrary. □

Proof of Theorem 2.1 With the aid of this last lemma we now now provide the proof of the main theorem. We begin by setting

$$\begin{aligned} C(r) &= 2 \left(\tfrac{2\pi}{r}\right)^{\frac{n-1}{2}} \cos\left(r - \tfrac{(n-1)\pi}{4}\right)\,, \\ \Delta(r) &= J(r) - C(r)\,, \end{aligned}$$

where $J(r)$ is defined by (2.1). The asymptotic behavior of $J(r)$ is given (see Saitō [19]) by

$$|\Delta(r)| = |J(r) - C(r)| \le \mathbf{A} r^{-\frac{(n+1)}{2}}\,,$$

where $\mathbf{A}$ is a constant. With this notation, we have that

$$\begin{aligned} &k^{n-1} \int_{\mathbb{R}^n} J(k|x-\xi|) J(k|\xi - y|) \psi(\xi) d\xi \\ &= k^{n-1} \int_{\mathbb{R}^n} \left(C(k|x-\xi|) + \Delta(k|x-\xi|)\right) \left(C(k|\xi - y|) + \Delta(k|\xi - y|)\right) \psi(\xi)\, d\xi \\ &= k^{n-1} \int_{\mathbb{R}^n} C(k|x-\xi) C(k|\xi - y|) \psi(\xi)\, d\xi \end{aligned}$$

$$
\begin{aligned}
&+k^{n-1}\int_{\mathbb{R}^n}\Delta(k|x-\xi|)C(k|\xi-y|)\psi(\xi)\,d\xi\\
&+k^{n-1}\int_{\mathbb{R}^n}C(k|x-\xi|)\Delta(k|\xi-y|)\psi(\xi)\,d\xi\\
&+k^{n-1}\int_{\mathbb{R}^n}\Delta(k|x-\xi|)\Delta(k|\xi-y|)\psi(\xi)\,d\xi\\
=\;& I_1+I_2+I_3+I_4\,.
\end{aligned}
$$

We have that I_1 vanishes by Lemma 4.5. Looking at I_2 we have

$$
\begin{aligned}
|I_2| &\le k^{n-1}\int_{\mathbb{R}^n}\mathbf{A}(k|x-\xi|)^{-\frac{(n+1)}{2}}|C(k|\xi-y|)||\psi(\xi)|\,d\xi\\
&\le k^{n-1}\int_{\mathbb{R}^n}\mathbf{A}(k|x-\xi|)^{-\frac{(n+1)}{2}}2\left(\frac{2\pi}{k|\xi-y|}\right)^{\frac{n-1}{2}}|\psi(\xi)|\,d\xi\\
&\le \frac{\mathbf{A}}{k}\int_{\mathbb{R}^n}\frac{|\psi(\xi)|}{|x-\xi|^{\frac{n+1}{2}}|\xi-y|^{\frac{n-1}{2}}}\,d\xi
\end{aligned}
$$

Clearly the last integral is finite and independent of k, hence I_2 vanishes as $k\to\infty$. I_3 behaves just like I_2, hence it vanishes as $k\to\infty$. For the final term, we have

$$
\begin{aligned}
|I_4| &\le k^{n-1}\left|\int_{\mathbb{R}^n}\Delta(k|x-\xi|)\Delta(k|\xi-y|)\psi(\xi)\,d\xi\right|\\
&\le \frac{\mathbf{A}^2}{k^2}\int_{\mathbb{R}^n}|x-\xi|^{-\frac{(n+1)}{2}}|\xi-y|^{-\frac{(n+1)}{2}}|\psi(\xi)|\,d\xi
\end{aligned}
$$

This integral is also finite and independent of k, hence I_4 vanishes as $k\to\infty$. This completes the proof of our theorem.

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Richard Ford
Department of Mathematics
California State University
Chico, CA 95929, USA
e-mail: rford@csuchico.edu