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On the stability of convex symmetric polytopes of matrices *

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Abstract

In this article we investigate the stability properties of a convex symmetric time-varying second-order matrix's polytope depending on a real positive parameter. We apply the results obtained to the calculation of the stability radius of a second order matrix under affine general perturbations, and under linear structured multiple perturbations.

1 Introduction

In many control theory problems and in engineering applications, it is required to know the properties not only of the nominal model, but also of closely related models. This fact has contributed to develop the mathematical tools, such as robustness analysis, which allow the simultaneous analysis of the properties of a family of mathematical objects. An important problem in robustness analysis is that of determining the extent to which stability is preserved under various types of parameter perturbations.

In the early 80's a classical result by Kharitonov [1] motivated the investigation of the stability of intervals and other sets of polynomials and the study of stability of intervals of matrices [2]. An important result for second order matrices was obtained by Filippov [3]. There necessary and sufficient conditions are given for the stability of the time-varying matrices $A(t)$, $t \in \mathbb{R}^+$, which take values in the convex hull of a finite number of given matrices.

In this paper, following a Filippov type approach, we are concerned with the stability properties of families of time-varying second order matrices taking their values in a convex and symmetric polytope of matrices. However, in our case the polytope depends on a real positive parameter r whose growth indicates the expansion of the polytope. The Hurwitz stability of these families of matrices is investigated. The results obtained are applied to the determination

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of the stability radius as robustness measure of a matrix under affine general perturbations and under linear structured multiple perturbations.

We proceed as follows. In Section 2, we formulate the problem for the polytope and define the number r_t that measures the robustness of this family of matrices. In Section 3, we apply one of Filippov's theorems [3] for calculating the number above mentioned. In Section 4, we illustrate how to use the results obtained in the previous section for calculating the real radii of stability. Finally in Section 5, we show some applications of our results.

2 Formulation of the problem

Let $A \in \mathbb{R}^{2\times 2}$ be a stable matrix, and $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$, be a collection of matrices, not all zero. For each number $r > 0$ we consider the convex and symmetric polytope depending on the parameter r and formed by time-invariant matrices

$$\Pi(A, (B_i)_{i\in\underline{N}}, r) = \text{convex}\{A \pm rB_i \ \ , i \in \underline{N}\}$$

and the corresponding polytope of time-varying matrices

$$\Pi_t(A, (B_i)_{i\in\underline{N}}, r) = \{M(\cdot)/M(\cdot) : [0, +\infty) \to \Pi(A, (B_i)_{i\in\underline{N}}, r) \text{ measurable }\},$$

where $\underline{N}$ denotes the set $\{1, \dots, N\}$. For $\Pi_t(A, (B_i)_{i\in\underline{N}}, r)$ we formulate the following problem:

Find the values $r > 0$ such that the convex and symmetric polytope of time-varying matrices $\Pi_t(A, (B_i)_{i\in\underline{N}}, r)$ are stable; i.e., each matrix $M(\cdot) \in \Pi_t(A, (B_i)_{i\in\underline{N}}, r)$ is stable. Which means that the spectrum of $M(t)$, for each $t \in \mathbb{R}^+$, lies in $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$.

Let

$$\begin{aligned} r_t(A, (B_i)_{i\in\underline{N}}) \quad = \quad & \inf\{r > 0 : \Pi_t(A, (B_i)_{i\in\underline{N}}, r) \text{ contains at least} \\ & \qquad \text{one unstable matrix } M(\cdot)\}. \end{aligned}$$

It is clear that if we determine the number $r_t(A, (B_i)_{i\in\underline{N}})$, the stated problem is solved because the family of matrices $\Pi_t(A, (B_i)_{i\in\underline{N}}, r)$ is stable if and only if $r < r_t(A, (B_i)_{i\in\underline{N}})$.

3 Calculation of the number r_t

In this section we study the stability of the polytope $\Pi_t(A, (B_i)_{i\in\underline{N}}, r)$.

Let $C_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, m$, be given constants matrices, and let

$$C = \text{convex}\{C_i,\ i = 1, \dots, m\},$$
$$C_t = \{M(\cdot)/M(\cdot) : \mathbb{R}^+ \to C \text{ is measurable}\}.$$

Necessary and sufficient conditions for the stability of the family of matrices C_t are presented in [3].

Theorem 3.1 *For the stability of the polytope of time-varying matrices C_t it is necessary and sufficient that the matrices $C_i = (c^i_{pq})_{p,q=1,2}$, $i = 1, \dots, m$, that determine the polytope, satisfy the conditions a)-d).*

a) $\operatorname{tr} C_i < 0$, $\det C_i > 0$, $i = 1, \dots, m$

b) for each pair $i, j = 1, \dots, m$, $(i \neq j)$,

$$h_{ij} = c^i_{11}c^j_{22} - c^i_{12}c^j_{21} - c^i_{21}c^j_{12} + c^i_{22}c^j_{11} > -2\sqrt{\det C_i \det C_j}; \qquad (1)$$

c) if

$$c^j_{12} > 0 \text{ for some } j \in \{1, \dots, m\}, \qquad (2)$$

and if for each $k \in \mathbb{R}$ there exist $i \in \{1, \dots, m\}$ such that

$$c^i_{12}k^2 + (c^i_{11} - c^i_{22})k - c^i_{21} > 0, \qquad (3)$$

then

$$I \equiv \int_{-\infty}^{+\infty} \max_i \frac{c^i_{22}k^2 + (c^i_{12} + c^i_{21})k + c^i_{11}}{|c^i_{12}k^2 + (c^i_{11} - c^i_{22})k - c^i_{21}|} \frac{dk}{k^2 + 1} < 0\,, \qquad (4)$$

where the maximum is taken for those $i \in \{1, \dots, m\}$ that satisfy the inequality (3).

d) The same as condition c) when $>$ is replaced by $<$ in (2) and (3).

The integrals in c) and d) will be denoted by I^+ and I^- respectively. Our goal is to apply Theorem 3.1 to the study of the stability properties of the considered polytope. First with the matrices A and B_i, $i = 1, \dots, N$, we form matrices depending on the real parameter r.

$$C_{2i}(r) = A + rB_i, \quad C_{2i-1}(r) = A - rB_i, \; i = 1, \dots, N,$$

and with these matrices we form the families of matrices:

$$\begin{aligned} C(r) &= \text{convex}\{C_j(r), \; j = 1, \dots, 2N\}, \\ C_t(r) &= \{M(\cdot)/M(\cdot) : \mathbb{R}^+ \to C(r) \text{ is measurable}\}. \end{aligned}$$

Then, according to the defined polytopes in Section 2, we have

$$\Pi(A, (B_i)_{i \in \underline{N}}, r) = C(r), \quad \Pi_t(A, (B_i)_{i \in \underline{N}}, r) = C_t(r).$$

The last equalities show that Theorem 3.1 can be directly applied to the investigation of the stability of the family of polytopes. Note that when Theorem 3.1 is applied to the family of matrices $C_t(r)$, the integrals I^+, I^- depend on the parameter r. For this reason they will be denoted by $I^+(r)$ and $I^-(r)$ respectively.

To facilitate the application of the Theorem 3.1 to the family of matrices $C_t(r)$, $r > 0$, we state the following four lemmas, which allow us the verification of the conditions a)-d) of this theorem. The assertions of lemmas follow by straightforward computations and we will omit them.

Lemma 3.2 *Let $A = (a_{pq})_{p,q=1,2} \in \mathbb{R}^{2\times 2}$ be a stable matrix, $B_i = (b^i_{pq})_{p,q=1,2} \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$, be matrices, not all zero. Then the corresponding matrices $C_i(r)$, $i = 1, \dots, 2N$, $r > 0$, satisfy the condition a) of Theorem 3.1 if and only if*

$$r < \widetilde{r} := \min\{r_1(A, (B_i)_{i\in\underline{N}}), \quad r_2(A, (B_i)_{i\in\underline{N}})\},$$

where

$$\begin{aligned} r_1(A, (B_i)_{i\in\underline{N}}) &:= \inf\{r > 0 : \operatorname{tr} A + r|\operatorname{tr} B_i| = 0 \text{ for some } i \in \underline{N}\}, \\ r_2(A, (B_i)_{i\in\underline{N}}) &:= \inf\{r > 0 : \det A - r|\sigma_i| + r^2 \det B_i = 0 \text{ for some } i \in \underline{N}\}, \\ \sigma_i &= a_{11}b^i_{22} - a_{12}b^i_{21} - a_{21}b^i_{12} + a_{22}b^i_{11}. \end{aligned}$$

Lemma 3.3 *Let $A \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$ be matrices that satisfy the conditions of the Lemma 3.2. If the corresponding matrices $C_i(r) = (c^i_{pq}(r))_{p,q=1,2}$, $i = 1, \dots, 2N$ satisfy condition a) of the Theorem 3.1, then these matrices satisfy condition b) of the same theorem if and only if $r < \widetilde{\widetilde{r}}(A, (B_i)_{i\in\underline{N}})$, where*

$$\begin{aligned} \widetilde{\widetilde{r}}(A, (B_i)_{i\in\underline{N}}) \quad := \quad & \inf\big\{r > 0 :\ c^i_{11}(r)c^j_{22}(r) - c^i_{12}(r)c^j_{21}(r) - c^i_{21}(r)c^j_{12}(r) \\ & +c^i_{22}(r)c^j_{11}(r) \le -2\sqrt{\det[C_i(r)]\det[C_j(r)]},\ i,j \in \underline{N}\big\}. \end{aligned}$$

Lemma 3.4 *Let $A \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$. Then the corresponding matrices $C_i(r) = (c^i_{pq}(r))_{p,q=1,2}$, $i = 1, \dots, 2N$, satisfy the conditions*

i) $c^i_{12}(r) > 0$ *for some* $i \in \{1, \dots, 2N\}$

ii) *for each* $k \in \mathbb{R}$, $c^i_{12}(r)k^2 + (c^i_{11}(r) - c^i_{22}(r))k - c^i_{21}(r) > 0$ *for some* $i \in \{1, \dots, 2N\}$

if and only if

$$r > \varrho(A, (B_i)_{i\in\underline{N}}) := \max\{\varrho_1(A, (B_i)_{i\in\underline{N}}), \varrho_2(A, (B_i)_{i\in\underline{N}})\},$$

where

$$\begin{aligned} \varrho_1(A, (B_i)_{i\in\underline{N}}) &:= \inf\{r \ge 0 :\ a_{12} + r|b^i_{12}| > 0 \text{ for some } i \in \underline{N}\}, \\ \varrho_2(A, (B_i)_{i\in\underline{N}}) &:= \inf\{r \ge 0 : \forall k \in \mathbb{R},\ \gamma(k) + r|\delta^i(k)| > 0 \text{ for some } i \in \underline{N}\}, \\ \gamma(k) &= a_{12}k^2 + (a_{11} - a_{22})k - a_{21}, \\ \delta^i(k) &= b^i_{12}k^2 + (b^i_{11} - b^i_{22})k - b^i_{21},\ i = 1, \dots, N. \end{aligned}$$

Lemma 3.5 *Let $A \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$. Then the corresponding matrices $C_i(r) = (c^i_{pq}(r))_{p,q=1,2}$, $i = 1, \dots, 2N$, satisfy the conditions*

i) $c^i_{12}(r) < 0$ *for some* $i \in \{1, \dots, 2N\}$

ii) *for each* $k \in \mathbb{R}$, $c^i_{12}(r)k^2 + (c^i_{11}(r) - c^i_{22}(r))k - c^i_{21}(r) < 0$ *for some* $i \in \{1, \dots, 2N\}$.

if and only if

$$r > \zeta(A,(B_i)_{i\in\underline{N}}) := \max\{\zeta_1(A,(B_i)_{i\in\underline{N}}),\zeta_2(A,(B_i)_{i\in\underline{N}})\},$$

where

$$\begin{aligned}\zeta_1(A,(B_i)_{i\in\underline{N}}) &:= \inf\{r\geq 0:\ a_{12}-r|b^i_{12}|<0 \text{ for some } i\in\underline{N}\},\\ \zeta_2(A,(B_i)_{i\in\underline{N}}) &:= \inf\{r\geq 0: \forall k\in\mathbb{R},\ \gamma(k)-r|\delta^i(k)|<0 \text{ for some } i\in\underline{N}\}.\end{aligned}$$

¿From condition c) of Theorem 3.1 and Lemma 3.4 it follows that when $\varrho(A,(B_i)_{i\in\underline{N}}) \geq \widehat{r}(A,(B_i)_{i\in\underline{N}})$ condition c) is satisfied for the matrices $C_i(r)$, $i\in\{1,\dots,2N\}$, $r\in(0,\widehat{r}(A,(B_i)_{i\in\underline{N}}))$. While if $\varrho(A,(B_i)_{i\in\underline{N}}) < \widehat{r}(A,(B_i)_{i\in\underline{N}})$, then this assertion is true for $r\in(0,\varrho(A,(B_i)_{i\in\underline{N}}))$ and it will be true for $r\in(\varrho(A,(B_i)_{i\in\underline{N}}),\widehat{r}(A,(B_i)_{i\in\underline{N}}))$ if and only if $I^+(r)<0$.

A similar result can be stated for the condition d) of the Theorem 3.1 making use of Lemma 3.5.

If now we put

$$\widehat{r}(A,(B_i)_{i\in\underline{N}}) := \min\{\widetilde{r}(A,(B_i)_{i\in\underline{N}}),\ \widetilde{\widetilde{r}}(A,(B_i)_{i\in\underline{N}})\},$$

where $\widetilde{r}(A,(B_i)_{i\in\underline{N}})$ and $\widetilde{\widetilde{r}}(A,(B_i)_{i\in\underline{N}})$ are defined in the Lemmas 3.2 and 3.3, then from the Theorem 3.1, Lemmas 3.2-3.5 and the last comments the following result is obtained directly.

Theorem 3.6 *The polytope of matrices $\Pi_t(A,(B_i)_{i\in\underline{N}},r)$ with $r>0$ is stable if and only if the following conditions hold*

i) $r<\widehat{r}(A,(B_i)_{i\in\underline{N}})$

ii) $r\leq\varrho(A,(B_i)_{i\in\underline{N}})$ *or* $I^+(r)<0$

iii) $r\leq\zeta(A,(B_i)_{i\in\underline{N}})$ *or* $I^-(r)<0$.

To investigate the integrands that appear in the expressions of $I^+(r)$ and $I^-(r)$, we introduce the following notation.

$$N^i(k,r) = c^i_{22}(r)k^2+(c^i_{12}(r)+c^i_{21}(r))k+c^i_{11}(r),\ \ i=1,\dots,2N, \tag{5}$$

$$D^i(k,r) = c^i_{12}(r)k^2+(c^i_{11}(r)-c^i_{22}(r))k-c^i_{21}(r),\ \ i=1,\dots,2N, \tag{6}$$

$$g_i(k,r)=\tfrac{N^i(k,r)}{D^i(k,r)},\ i=1,\dots,2N, \tag{7}$$

$$\Gamma^+(k,r)=\{i\in\{1,\dots,2N\}:\ D^i(k,r)>0\}, \tag{8}$$

$$\Gamma^-(k,r)=\{i\in\{1,\dots,2N\}:\ D^i(k,r)<0\}, \tag{9}$$

$$P^{j,i}(r)=c^j_{22}(r)c^i_{12}(r)-c^j_{12}(r)c^i_{22}(r),\ \ i,j=1,\dots,2N\ (i\neq j), \tag{10}$$

$$\begin{aligned}Q^{j,i}(r) &= c^j_{22}(r)c^i_{11}(r)-c^j_{11}(r)c^i_{22}(r)-c^j_{12}(r)c^i_{21}(r)+c^j_{21}(r)c^i_{12}(r),\\ &\qquad i,j=1,\dots,2N\ (i\neq j),\end{aligned} \tag{11}$$

$$S^{j,i}(r)=c^j_{21}(r)c^i_{11}(r)-c^j_{11}(r)c^i_{21}(r),\ \ i,j=1,\dots,2N\ (i\neq j), \tag{12}$$

$$H^{j,i}(k,r)=P^{j,i}(r)k^2+Q^{j,i}(r)k+S^{j,i}(r),\ \ i,j=1,\dots,2N\ (i\neq j). \tag{13}$$

Now we can write

$$
\begin{aligned}
I^{+}(r) &= \int_{-\infty}^{+\infty} \max_{i\in\Gamma^{+}(k,r)} g_i(k,r)\frac{dk}{k^2+1}, && (14)\\
I^{-}(r) &= \int_{-\infty}^{+\infty} \max_{i\in\Gamma^{-}(k,r)} \{-g_i(k,r)\}\frac{dk}{k^2+1}, && (15)\\
g_j(k,r)-g_i(k,r) &= \frac{H^{j,i}(k,r)(1+k^2)}{D^j(k,r)D^i(k,r)},\quad i,j=1,\dots,2N, && (16)
\end{aligned}
$$

and moreover formulate the following auxiliary lemmas that will be useful for the investigation of the integrals $I^{+}(r)$ and $I^{-}(r)$.

Lemma 3.7 *Let $\rho\in(0,\widehat{r}(A,(B_i)_{i\in\underline{N}}))$ and $\widetilde{k}\in\mathbb{R}$ be such that $D^i(\widetilde{k},\rho)=0$, $i\in\{1,\dots,2N\}$. Then $N^i(\widetilde{k},\rho)<0$, where $N^i(k,r)$ and $D^i(k,r)$ are defined in (5) and (6) respectively.*

Proof. After suitable ordering in expressions (5),(6) we obtain

$$
\begin{aligned}
D^i(\widetilde{k},\rho) &= [c^i_{12}(\rho)\widetilde{k}+c^i_{11}(\rho)]\widetilde{k}-[c^i_{22}(\rho)\widetilde{k}+c^i_{21}(\rho)], && (17)\\
N^i(\widetilde{k},\rho) &= [c^i_{22}(\rho)\widetilde{k}+c^i_{21}(\rho)]\widetilde{k}+[c^i_{12}(\rho)\widetilde{k}+c^i_{11}(\rho)]. && (18)
\end{aligned}
$$

The condition $D^i(\widetilde{k},\rho)=0$ and (17) imply

$$
[c^i_{22}(\rho)\widetilde{k}+c^i_{21}(\rho)]=[c^i_{12}(\rho)\widetilde{k}+c^i_{11}(\rho)]\widetilde{k}\,. \tag{19}
$$

Making use of this equality in (18) we obtain

$$
N^i(\widetilde{k},\rho)=[c^i_{12}(\rho)\widetilde{k}+c^i_{11}(\rho)](k^2+1). \tag{20}
$$

On the other hand, from (17) and the condition $D^i(\widetilde{k},\rho)=0$, we have that the vector $C_i(\rho)\begin{pmatrix}1\\ \widetilde{k}\end{pmatrix}$ is parallel to the vector $\begin{pmatrix}1\\ \widetilde{k}\end{pmatrix}$, and so there exists a real number λ such that

$$
C_i(\rho)\begin{pmatrix}1\\ \widetilde{k}\end{pmatrix}=\lambda\begin{pmatrix}1\\ \widetilde{k}\end{pmatrix}, \tag{21}
$$

i.e., λ is an eigenvalue of $C_i(\rho)$ and thus $\lambda<0$, because, due to Lemma 3.2, the matrix $C_i(\rho)$ is stable for $\rho<\widehat{r}(A,(B_i)_{i\in\underline{N}})$. But from the equality of the first components in (21) it follows that $\lambda=c^i_{12}(\rho)\widetilde{k}+c^i_{11}(\rho)$ and thus for (20) we conclude that $signN^i(\widetilde{k},\rho)=\operatorname{sign}\lambda=-1$. ♢

Lemma 3.8 *Let $A\in\mathbb{R}^{2\times2}$, $B_i\in\mathbb{R}^{2\times2}$, $i=1,\dots,N$, and r be a fixed positive number. Let $\mathcal{P}$ be the partition of the real axis determined by the real roots of the polynomials $H^{j,i}(k,r)$, $i,j\in\{1,\dots,2N\}$ in the variable k. Then for each interval (α,β) of the partition $\mathcal{P}$ there exists $m\in\{1,\dots,2N\}$ such that $\max_{i\in\Gamma^{+}(k,r)} g_i(k,r)=g_m(k,r)$ for each $k\in(\alpha,\beta)$.*

Proof. Suppose that for $k_1 \in \mathbb{R}$ there exist a real number $\epsilon > 0$ and indexes $s, t \in \{1, \dots, 2N\}$, $s \neq t$, such that

i) $\max\limits_{i \in \Gamma^+(k,r)} g_i(k,r) = g_s(k,r)$, $k \in (k_1 - \epsilon, k_1)$

ii) $\max\limits_{i \in \Gamma^+(k,r)} g_i(k,r) = g_t(k,r)$, $k \in (k_1, k_1 + \epsilon)$.

Thus $D^s(k_1, r) \neq 0$, otherwise, by virtue of the Lemma 3.7, $N^s(k_1, r) < 0$ and $g_s(k_1 - \epsilon, r) \to -\infty$ when $k_1 - \epsilon \nearrow k_1$. This fact contradicts i). Analogously $D^t(k_1, r) \neq 0$.

Now from i), ii) follows the equality $g_s(k_1, r) = g_t(k_1, r)$. On the other hand, from expression (16) we have that

$$0 = g_s(k_1, r) - g_t(k_1, r) = \frac{(1 + k_1^2) H^{s,t}(k_1, r)}{D^s(k_1, r) D^t(k_1, r)},$$

and so $H^{s,t}(k_1, r) = 0$. $\diamondsuit$

Applying Lemma 3.8, for given matrices $A, (B_i)_{i \in \underline{N}}$ we can construct a function $k \to \max\limits_{i \in \Gamma^+(k,r)} g_i(k,r)$ over all the real axis. On each interval (α, β) of the partition $\mathcal{P}$, this function is given by a function $k \to g_m(k,r)$, where the index m is determined fixing an arbitrary $\widetilde{k} \in (\alpha, \beta)$ and checking the condition $\max\limits_{i \in \Gamma^+(\widetilde{k},r)} g_i(\widetilde{k}, r) = g_m(\widetilde{k}, r)$. In a similar way can be constructed the function $k \to \max\limits_{i \in \Gamma^-(k,r)} \{-g_i(k,r)\}$.

Lemma 3.9 *Let $A \in \mathbb{R}^{2\times2}$, $B_i \in \mathbb{R}^{2\times2}$, $i = 1, \dots, N$, $r \in (\varrho(A, (B_i)_{i \in \underline{N}}), \widehat{r}(A, (B_i)_{i \in \underline{N}}))$. For given $k \in \mathbb{R}$ and $j \in \Gamma^+(k,r)$ such that $\max\limits_{i \in \Gamma^+(k,r)} g_i(k,r) = g_j(k,r)$ we define*

$$v^j(r) = C_j(r) \begin{pmatrix} 1 \\ k \end{pmatrix}, \qquad v^{\widetilde{j}}(r) = C_{\widetilde{j}}(r) \begin{pmatrix} 1 \\ k \end{pmatrix},$$
$$\Theta_j(r) = \angle(v^j(r), (1,k)^T), \qquad \Theta_{\widetilde{j}}(r) = \angle(v^{\widetilde{j}}(r), (1,k)^T),$$

where

$$\widetilde{j} = \begin{cases} j+1 & \text{if } j \text{ is odd} \\ j-1 & \text{if } j \text{ is even,} \end{cases}$$

and $\angle(u, (1,k)^T)$ denotes the angle (in the positive direction) from the direction $(1,k)^T$ to the direction u. Then $\Theta_j(r) < \pi$ and $\Theta_{\widetilde{j}}(r) - \Theta_j(r) < \pi$.

Proof. First we prove that $\Theta_j(r) < \pi$. By virtue of (17) and the fact that $j \in \Gamma^+(k,r)$ we have

$$D^j(k,r) = [c^j_{12}(r)k + c^j_{11}(r)]k - [c^j_{22}(r)k + c^j_{21}(r)] = \langle (k,-1)^T, v^j(r) \rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. So that $\angle((k,-1)^T, v^j(r)) < \frac{\pi}{2}$ from what, taking into account that $\angle((k,-1)^T, (1,k)^T) = \frac{\pi}{2}$, it follows that $\Theta_j(r) < \pi$. Now suppose that the second statement of the lemma is false, i.e. $\Theta_{\widetilde{j}}(r) - \Theta_j(r) \geq \pi$. Then, noting that $\Theta_j(r) < \pi$, there exists a convex combination of the vectors $v^j(r)$ and $v^{\widetilde{j}}(r)$, which has the same direction as the vector $\begin{pmatrix} 1 \\ k \end{pmatrix}$, i. e., there are real constants α, β and $\gamma > 0$ such that

$$\alpha C_j(r) \begin{pmatrix} 1 \\ k \end{pmatrix} + \beta C_{\widetilde{j}}(r) \begin{pmatrix} 1 \\ k \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ k \end{pmatrix},$$

i.e., γ is an eigenvalue of the matrix $\alpha C_j(r) + \beta C_{\widetilde{j}}(r)$. This contradicts the fact that matrix $\alpha C_j(r) + \beta C_{\widetilde{j}}(r)$ is stable. $\diamondsuit$

Next we prove that the functions $I^+(r)$, $r \in (\varrho(A, (B_i)_{i \in \underline{N}}), \widehat{r}(A, (B_i)_{i \in \underline{N}}))$ and $I^-(r)$, $r \in (\zeta(A, (B_i)_{i \in \underline{N}}), \widehat{r}(A, (B_i)_{i \in \underline{N}}))$ increase monotonically and that they are negative for values of r in these intervals sufficiently close to $\varrho(A, (B_i)_{i \in \underline{N}})$ and $\zeta(A, (B_i)_{i \in \underline{N}})$ respectively.

Before passing to demonstrate the following lemmas let us make the following agreement. If a vector $v^j(r)$ has coordinates $(v_1^j(r), v_2^j(r))$, we will denote by $(v^j(r))^\perp$ the vector with coordinates $(-v_2^j(r), v_1^j(r))$.

Lemma 3.10 *Let $A \in \mathbb{R}^{2\times 2}$ be a stable matrix, let $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$. Then for each $k \in \mathbb{R}$, the function*

$$r \to \max_{i \in \Gamma^+(k,r)} g_i(k,r), \quad r \in (\varrho(A, (B_i)_{i \in \underline{N}}), \widehat{r}(A, (B_i)_{i \in \underline{N}})),$$

increases monotonically.

Proof. Let $k \in \mathbb{R}$ and $r,\ r + \Delta r \in (\varrho(A, (B_i)_{i \in \underline{N}}), \widehat{r}(A, (B_i)_{i \in \underline{N}}))$, $\Delta r > 0$. According to Lemma 3.8, there exists $p \in \{1, \dots, 2N\}$ such that $\max_{i \in \Gamma^+(k,r)} g_i(k,r) = g_p(k,r)$. So it suffices to prove that

$$g_p(k, r + \Delta r) \geq g_p(k,r)\,.$$

The function $s \to g_p(k,s),\ s \in \mathbb{R}^+$, is defined and differentiable every where except at those points s where $D^p(k,s) = 0$. Furthermore by straightforward calculation and considering the expressions (5)-(9) we obtain

$$\frac{\partial}{\partial s} g_p(k,s) = \frac{(-1)^p (1+k^2) H^p(k)}{(D^p(k,s))^2}, \tag{22}$$

where

$$H^p(k) = (a_{12}b_{22}^p - a_{22}b_{12}^p)k^2 + (a_{11}b_{22}^p - a_{22}b_{11}^p + a_{12}b_{21}^p - a_{21}b_{12}^p)k + (a_{11}b_{21}^p - a_{21}b_{11}^p),$$

for $p \in \{1, \dots, 2N\}$. Taking into account the definitions of $v^p(r)$, $(v^{\widetilde{p}}(r))^\perp$, $p \in \Gamma^+(k,r)$, by straightforward calculation we obtain

$$2r(-1)^p H^p(k) = \langle v^p(r), (v^{\widetilde{p}}(r))^\perp \rangle,$$

where

$$\widetilde{p} = \begin{cases} p+1 & \text{if p is odd} \\ p-1 & \text{if p is even.} \end{cases}$$

But, by Lemma 3.9, it is easy to see that $\angle((v^{\widetilde{p}}(r))^{\perp},(1,k)^T) = \Theta_{\widetilde{p}} - \frac{\pi}{2}$ and $\angle(v^p(r),(v^{\widetilde{p}}(r))^{\perp}) \equiv \Theta_{\widetilde{p}} - \frac{\pi}{2} - \Theta_p < \frac{\pi}{2}$. This implies that $\langle v^p(r),(v^{\widetilde{p}}(r))^{\perp}\rangle > 0$ and so $2r(-1)^pH^p(k) > 0$. Now taking into account that $\text{sign}[2r(-1)^pH^p(k)] = sign[(-1)^pH^p(k)]$ for all $s \in \mathbb{R}^+$, from this equality and expression (22) it is obtained that the function $s \to g_p(k,s),\ s \in \mathbb{R}^+$, increases in an interval $[r,\rho)$, where $\rho = +\infty$ or $D^p(k,\rho) = 0$ and $D^p(k,s) \neq 0$ for $s \in [r,\rho)$. Let us see that $\rho \geq \widehat{r}(A,(B_i)_{i\in\underline{N}})$. If $\rho = +\infty$ there is nothing to demonstrate. If ρ is finite, taking into account that $p \in \Gamma^+(k,r),\ s \in [r,\rho)$, and Lemma 3.7, we obtain that $\lim_{s\nearrow\rho} g_p(k,s) = -\infty$. But on the other hand, due to monotony, we have that $\lim_{s\nearrow\rho} g_p(k,s) \geq g_p(k,r)$, and like $g_p(k,r)$ is a finite number, we arrive so to a contradiction. For everything previous we conclude that the function $s \to g_p(k,s)$ increases for $s \in [r,\widehat{r}(A,(B_i)_{i\in\underline{N}}))$ and so $g_p(k,r) \leq g_p(k,r+\Delta r)$. $\diamondsuit$

Lemma 3.11 *Let $A \in \mathbb{R}^{2\times2}$ be a stable matrix, let $B_i \in \mathbb{R}^{2\times2},\ i = 1,\dots,N$. Then for each $k \in \mathbb{R}$, the function*

$$r \to \max_{i\in\Gamma^-(k,r)}\{-g_i(k,r)\},\ \ r \in (\zeta(A,(B_i)_{i\in\underline{N}}),\widehat{r}(A,(B_i)_{i\in\underline{N}})),$$

increases monotonically.

The proof of above lemma is analogous to the proof of the previous lemma.

Lemma 3.12 *Let $A \in \mathbb{R}^{2\times2}$ be a stable matrix, $B_i \in \mathbb{R}^{2\times2},\ i = 1,\dots,N$, $\varrho(A,(B_i)_{i\in\underline{N}}) \in [0,\widehat{r}(A,(B_i)_{i\in\underline{N}}))$. Then $I^+(r) < 0$ for r greater than the number $\varrho(A,(B_i)_{i\in\underline{N}})$ and sufficiently close to it.*

Proof. To simplify notation we will write ϱ in place of $\varrho(A,(B_i)_{i\in\underline{N}})$. Let us take a decreasing sequence r_n of real numbers such that $r_n \to \varrho$ when $n \to +\infty$. Then the sequence of continuous functions

$$h_n(k) = \max_{i\in\Gamma^+(k,r_n)} g_i(k,r_n),\ k \in \mathbb{R},$$

is decreasing and converges to a measurable function $h(k),\ k \in \mathbb{R}$. Therefore, due to the theorem about monotonous convergent,

$$\int_{-\infty}^{+\infty} h_n(k)\frac{dk}{k^2+1} \longrightarrow \int_{-\infty}^{+\infty} h(k)\frac{dk}{k^2+1} \quad \text{if} \quad n \to +\infty.$$

So according with (14) in order to prove the Lemma 3.12 it is enough to show that

$$\int_{-\infty}^{+\infty} h(k)\frac{dk}{k^2+1} = -\infty. \tag{23}$$

Before passing to prove the equality (23) will be proved some auxiliary propositions about properties of the function $h(k)$. We know that $\Gamma^+(k,r) \neq \emptyset$ for all real k and $r > \varrho$; while for $r = \varrho$ there exists a finite set of real numbers $\widetilde{k}$ such that $\Gamma^+(\widetilde{k}, r) = \emptyset$. The set of such $\widetilde{k}$ will be denoted by $\mathcal{K}$.

Proposition 3.13 *If $k \notin \mathcal{K}$, then*

$$h(k) = \max_{i \in \Gamma^+(k,\varrho)} g_i(k,\varrho). \tag{24}$$

Proof. First we prove that for each $k \notin \mathcal{K}$ there exist $\epsilon > 0$ and $j \in \{1, \dots, 2N\}$ such that

$$\max_{i \in \Gamma^+(k,r)} g_i(k,r) = g_j(k,r) \text{ for each } r \in (\varrho, \varrho + \epsilon). \tag{25}$$

On the contrary, assume that there exist $p, q \in \{1, \dots, 2N\}$ and decreasing sequences r_n, s_n converging to ϱ such that

$$\max\nolimits_{i \in \Gamma^+(k,r_n)} g_i(k,r_n) = g_p(k,r_n), \tag{26}$$

$$\max\nolimits_{i \in \Gamma^+(k,s_n)} g_i(k,s_n) = g_q(k,s_n), \tag{27}$$

$$\text{with } g_p(k,r) \not\equiv g_q(k,r) \text{ as functions of } r. \tag{28}$$

From (26) and (27) we have that $g_p(k,r_n) \geq g_q(k,r_n)$ and $g_q(k,s_n) \geq g_p(k,s_n)$ for each natural number n. Due to the continuity of the functions $r \to g_p(k,r)$ and $r \to g_q(k,r)$ on a right neighborhood of ϱ, there exist infinitely many values of r for which these two functions are equal. However this equality implies that the functions are identically equal, because, in this case, $g_p(k,r)$ and $g_q(k,r)$ in the numerator as in the denominator have polynomials of second degree. This fact contradicts (28). Now we show that

$$\max_{i \in \Gamma^+(k,\varrho)} g_i(k,\varrho) = g_j(k,\varrho), \tag{29}$$

where j has been chosen so that it satisfies (25). Let $i_0 \in \Gamma^+(k,\varrho)$ such that

$$\max_{i \in \Gamma^+(k,\varrho)} g_i(k,\varrho) = g_{i_0}(k,\varrho). \tag{30}$$

Then as $i_0 \in \Gamma^+(k,\varrho)$, we have also that $i_0 \in \Gamma^+(k,r)$ for $r \in (\varrho, \varrho + \epsilon)$ if $\epsilon > 0$ is taken sufficiently small, and thus, due to (25) it is obtained that $g_j(k,r) \geq g_{i_0}(k,r)$. If here we pass to the limit when $r \searrow \varrho$ we obtain that $g_j(k,\varrho) \geq g_{i_0}(k,\varrho)$. This inequality and (30) imply

$$g_j(k,\varrho) = g_{i_0}(k,\varrho). \tag{31}$$

This prove the equality (29). Finally, when we pass to the limit in (25) as $r \searrow \varrho$, taking into account (31) and definition of $h(k)$, we obtain equality (24). $\diamondsuit$

Proposition 3.14 *For each $\widetilde{k} \in \mathcal{K}$ there exist a left neighborhood $O_1(\widetilde{k})$ of $\widetilde{k}$, a right neighborhood $O_2(\widetilde{k})$ of $\widetilde{k}$ and $j_1, j_2 \in \{1, \dots, 2N\}$, such that*

$$\begin{aligned} h(k) &= g_{j_1}(k,\varrho), \text{ for } k \in O_1(\widetilde{k}),\ k \neq \widetilde{k}, \\ h(k) &= g_{j_2}(k,\varrho), \text{ for } k \in O_2(\widetilde{k}),\ k \neq \widetilde{k}. \end{aligned}$$

Proof. For reasons of analogy, we will prove only the statement relative to right neighborhood of $\widetilde{k}$. Suppose the opposite, i.e., there exist $p, q \in \{1, \dots, 2N\}$ and decreasing sequences k_n, l_n converging to $\widetilde{k}$ such that

i) $h(k_n) = g_p(k_n, \varrho)$

ii) $h(l_n) = g_q(l_n, \varrho)$

iii) $g_p(k, \varrho)$ is not identical to $g_q(k, \varrho)$ as function of k.

¿From i), ii) and (24) we obtain that $g_p(k_n, \varrho) \geq g_q(k_n, \varrho)$ and $g_q(l_n, \varrho) \geq g_p(l_n, \varrho)$ for each natural number n. Thus, considering the continuity of the functions $k \to g_p(k, r)$, $k \to g_q(k, r)$ in a right neighborhood of $\widetilde{k}$, excluding this point, we will have that there exist in this neighborhood infinite values of k for which these functions are equal implicating that they should be identically equal by the same reason that in the Proposition 3.13. This contradicts iii). $\diamondsuit$

Proposition 3.15 *If for each $\widetilde{k} \in \mathcal{K}$ is taken a neighborhood $O(\widetilde{k})$, then outside of the union $\mathcal{V}$ of these neighborhoods the function $h(k)$ is bounded from above.*

Proof. See first that the function $h(k)$ is bounded in each compact set F disjoint with $\mathcal{V}$. For this, according to Proposition 3.13, it is sufficient to prove that the set $\{g_i(k, \varrho)/k \in F,\ i \in \Gamma^+(k, \varrho)\}$ is bounded above. Suppose the opposite. Then there exist a sequence of points $k_n \in F$ and $i \in \Gamma^+(k_n, \varrho)$ for all natural n, such that $g_i(k_n, \varrho) \to +\infty$. But for the compactness of F, the sequence k_n can be taken convergent to $k_0 \in F$, and passing to the limit when $n \to +\infty$ it is obtained that $\dfrac{N^i(k_n, \varrho)}{D^i(k_n, \varrho)} \to +\infty$, and so $D^i(k_0, \varrho) = 0$ and $N^i(k_0, \varrho) \geq 0$ but this is a contradiction to Lemma 3.7.

Due to the definition of ϱ, the functions $k \to g_i(k, \varrho)$, $i \in \Gamma^+(k, \varrho)$, are bounded above in any compact set disjoint with the neighborhoods $O(\widetilde{k})$, $\widetilde{k} \in \mathcal{K}$.

Suppose now that the statement of the lemma is not true. Then there exists a sequence $\{k_n\}$, $k_n \to +\infty$ or $k_n \to -\infty$ such that $h(k_n) \to +\infty$. Without losing of generality we can assume that $k_n \to +\infty$ (the prove in the other case is technically the same). Therefore there is a sequence j_n, $j_n \in \{1, \dots, 2N\}$, for which $g_{j_n}(k_n, \varrho) \to +\infty$, but like $j_n \in \{1, \dots, 2N\}$ we have that there exists $j \in \Gamma^+(k_{\hat{n}}, \varrho)$ such that $g_j(k_{\hat{n}}, \varrho) \to +\infty$, from where it follows that

$$\frac{c_{22}^j(\varrho)k_n^2 + (c_{12}^j(\varrho) + c_{21}^j(\varrho))k_n + c_{11}^j(\varrho)}{c_{12}^j(\varrho)k_n^2 + (c_{11}^j(\varrho) - c_{22}^j(\varrho))k_n - c_{21}^j(\varrho)} \to +\infty.$$

But this is not possible since

$$\lim_{k_n \to +\infty} \frac{c_{22}^j(\varrho)k_n^2 + (c_{12}^j(\varrho) + c_{21}^j(\varrho))k_n + c_{11}^j(\varrho)}{c_{12}^j(\varrho)k_n^2 + (c_{11}^j(\varrho) - c_{22}^j(\varrho))k_n - c_{21}^j(\varrho)} = \begin{cases} \dfrac{c_{22}^j(\varrho)}{c_{12}^j(\varrho)}, & \text{if } c_{12}^j(\varrho) \neq 0 \\ -\infty, & \text{if } c_{12}^j(\varrho) = 0, \end{cases}$$

because $c^j_{22}(\varrho) < 0$ when $c^j_{12}(\varrho) = 0$ due to the stability of the matrix $C_j(\varrho)$ and that in this case $c^j_{11}(\varrho) - c^j_{22}(\varrho) \geq 0$, which follows from the inclusion $j \in \Gamma^+(k_{\hat{n}}, \varrho)$. $\diamondsuit$

Making use of the Propositions 3.13-3.15 we shall prove (23) and so the proof of Lemma 3.12 will be complete. By Proposition 3.13 and Lemma 3.7 for each $\widetilde{k} \in \mathcal{K}$ there exist a left neighborhood $O_1(\widetilde{k})$ of $\widetilde{k}$, a right neighborhood $O_2(\widetilde{k})$ of $\widetilde{k}$ and $j_1, j_2 \in \{1, \dots, 2N\}$, such that $h(k) = g_{j_1}(k, \varrho)$, for $k \in O_1(\widetilde{k})$, $k \neq \widetilde{k}$, $h(k) = g_{j_2}(k, \varrho)$, for $k \in O_2(\widetilde{k})$, $k \neq \widetilde{k}$. Note, that according to Lemma 3.7, the numerator of $h(k)$ is negative if these neighborhoods are sufficiently small. Because the denominator of $h(k)$ is a polynomial with a zero in $\widetilde{k}$ and positive in the considered neighborhood, we have

$$\int_{O_1(\widetilde{k})} h(k) \frac{dk}{k^2+1} = -\infty\,, \quad \int_{O_2(\widetilde{k})} h(k) \frac{dk}{k^2+1} = -\infty\,.$$

If we denote by $O(\widetilde{k})$ the union of $O_1(\widetilde{k})$, $O_2(\widetilde{k})$, and denote by $\mathcal{U}$ the complement of the union of the neighborhoods $O(\widetilde{k})$, $\widetilde{k} \in \mathcal{K}$, then we will have by virtue of Proposition 3.15, that the integral of $\frac{h(k)}{k^2+1}$ on $\mathcal{U}$ is convergent, therefore the integral of $\frac{h(k)}{k^2+1}$ over all $\mathbb{R}$ is $-\infty$. $\diamondsuit$

Lemma 3.16 *Let $A \in \mathbb{R}^{2\times 2}$ be a stable matrix, $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$, $\zeta(A, (B_i)_{i\in \underline{N}}) \in [0, \widehat{r}(A, (B_i)_{i\in \underline{N}}))$. Then $I^-(r) < 0$ for r greater than the number $\zeta(A, (B_i)_{i\in \underline{N}})$ and sufficiently close to it.*

The proof is similar to the proof of Lemma 3.12. $\diamondsuit$

An immediate consequence of Theorem 3.6 and Lemmas 3.10-3.12, 3.16 is the following fundamental result of this work.

Theorem 3.17 *Let $A \in \mathbb{R}^{2\times 2}$, $B_i \in \mathbb{R}^{2\times 2}$, $i = 1, \dots, N$. Then it holds that*

$$r_t(A, (B_i)_{i\in \underline{N}}) = \min\{r^+(A, (B_i)_{i\in \underline{N}}), r^-(A, (B_i)_{i\in \underline{N}})\},$$

where

$$r^+(A, (B_i)_{i\in \underline{N}}) = \begin{cases} \widehat{r}(A, (B_i)_{i\in \underline{N}}) & \text{if } \varrho(A, (B_i)_{i\in \underline{N}}) \notin (0, \widehat{r}(A, (B_i)_{i\in \underline{N}})) \text{ or} \\ & \exists\, r_n \nearrow \widehat{r}(A, (B_i)_{i\in \underline{N}}) :\ I^+(r_n) < 0\,, \\ \text{root of } I^+(r) = 0 & \text{otherwise;} \end{cases}$$

$$r^-(A, (B_i)_{i\in \underline{N}}) = \begin{cases} \widehat{r}(A, (B_i)_{i\in \underline{N}}) & \text{if } \zeta(A, (B_i)_{i\in \underline{N}}) \notin (0, \widehat{r}(A, (B_i)_{i\in \underline{N}})) \text{ or} \\ & \exists\, r_n \nearrow \widehat{r}(A, (B_i)_{i\in \underline{N}}) :\ I^-(r_n) < 0, \\ \text{root of } I^-(r) = 0 & \text{otherwise.} \end{cases}$$

Theorem 3.17 allows us to calculate with arbitrary given accuracy, the number $r_t(A, (B_i)_{i\in \underline{N}})$ for all set of data $A, (B_i)_{i\in \underline{N}}$.

4 Application to radius of stability

In [4] and [5] the concept of the radius of stability of a matrix was introduced for different perturbation classes. Radius of stability represents a measure of how large could be the perturbations that conserve the property of stability of the matrix.

Methods for the calculation of the complex radius of stability for structured perturbations non depending on time are obtained in [6], and for the real case in [7]. For time-varying perturbations some important results are obtained in the works [8],[9],[10],[11]. Next we will apply the results given in Section 3 to the calculation of the stability radius of a matrix for time-varying structured general affine perturbations and linear structured multiple perturbations.

Let the nominal system be

$$\Sigma : \ \dot{x} = Ax,$$

where the matrix $A \in \mathbb{R}^{2\times 2}$ is Hurwitz-stable. Together with the system Σ we consider the perturbed system

$$\Sigma_P : \ \dot{x} = (A + P(t))x,$$

where the perturbation $P(\cdot)$ belongs to a perturbation class D defined below. Let $(E, \|\cdot\|)$ a normed space, and $\Phi : E \to \mathbb{R}^{2\times 2}$ a given linear map. If we put

$$D = \{\Phi(\Delta(t)) : \ \Delta(t) \in L^{\infty}(\mathbb{R}^+, E)\}, \tag{32}$$

then on D there is also induced a structure of normed space taking $\|P(\cdot)\|_D$ as

$$\|\Delta(\cdot)\|_{\infty} = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \|\Delta(t)\|.$$

Definition 4.1 *The real radius of stability of the matrix $A \in \mathbb{R}^{2\times 2}$ for perturbations of the class D given in (32) is defined by the number*

$$r_{\mathbb{R}}(A) = \inf\{\|P(\cdot)\|_D : \ P(\cdot) \in D, \ \Sigma_p \textit{is not asymptotically stable } \}.$$

Lemma 4.2 *Let E be a normed space with a polytopic norm, that is to say, the unit ball is the convex hull of a finite set of points $\{M_i, \ -M_i, \ i = 1, \dots, m\}$. Let $\Phi : E \to \mathbb{R}^{2\times 2}$ be a linear map. Then for the stable matrix $A \in \mathbb{R}^{2\times 2}$ it is true that*

$$r_{\mathbb{R}}(A) = r_t(A, (\Phi(M_i))_{i=1,\dots,m}),$$

where the last number was defined in the Section 2 and can be calculated by the method exposed in the Section 3.

Proof. This lemma is a direct consequence of the definitions of the numbers $r_{\mathbb{R}}(A)$, $r_t(A, (\Phi(M_i))_{i=1,\dots,m})$ and the fact that linear maps transform convex hull of a finite number of points in the convex hull of the image points. $\diamondsuit$

Calculation of the real radius of stability of a second order matrix under affine general time-varying perturbations

Let $A, B_i \in \mathbb{R}^{2\times 2}$, $i \in \{1, \dots, N\}$, where A is a Hurwitz stable matrix and consider the nominal system

$$\Sigma:\ \dot{x} = Ax,$$

and the perturbed system

$$\Sigma_\Delta:\ \dot{x} = (A + \sum_{i=1}^{N} \delta_i(t) B_i)x,$$

with the perturbation $\Delta(t) = (\delta_1(t), \dots, \delta_N(t))$ in the space $L^\infty(\mathbb{R}^+, \mathbb{R}^N)$ with the norm $\|\Delta(\cdot)\|_\infty = ess\ sup_{t\in\mathbb{R}^+} \|\Delta(t)\|$, where $\|\cdot\|$ is some norm on $\mathbb{R}^N$ such that the unit ball of the space $(E, \|\cdot\|)$ is the convex hull of a finite set of points $\{M_i,\ -M_i,\ i = 1, \dots, m\}$.

Definition 4.3 *We define the real radius of stability of the stable matrix A with respect to affine general time-varying perturbations determined by the matrices $B_i \in \mathbb{R}^{2\times 2}$, $i \in \{1, \dots, N\}$, as the number*

$$\begin{aligned} r^-_{\mathbb{R},\,t}(A, (B_i)_{i\in\underline{N}}) &= \inf\{\|\Delta(t)\|_\infty:\ \Delta(t) \in L^\infty(\mathbb{R}^+, \mathbb{R}^N), \\ &\qquad \Sigma_\Delta \textit{ is not asymptotically stable}\}. \end{aligned}$$

Our goal is to give a method for the calculation of this stability radius. First we take $E = \mathbb{R}^N$ and define the linear map $\Phi:\ \mathbb{R}^N \to \mathbb{R}^{2\times 2}$ by the expression

$$\Phi(\Delta) = \Phi(\delta_1, \dots, \delta_N) = \sum_{i=1}^{N} \delta_i B_i. \tag{33}$$

Then according to Definition 4.1 and expression (32) it is easy to see that

$$r^-_{\mathbb{R},\,t}(A, (B_i)_{i\in\underline{N}}) = r_\mathbb{R}(A),$$

and so if we apply now the Lemma 4.2 we obtain the equality

$$r^-_{\mathbb{R},\,t}(A, (B_i)_{i\in\underline{N}}) = r_t(A, (\Phi(M_i))_{i\in\underline{N}}), \tag{34}$$

which allows the application of the fundamental result of Section 3 for the calculation of $r^-_{\mathbb{R},\,t}(A, (B_i)_{i\in\underline{N}})$. Below we consider two particular norms on $\mathbb{R}^N$.

Case 4.4 $\|\Delta\| = \max\{|\delta_i|,\ i = 1, \dots, N\}$. *In the space $\mathbb{R}^N$ with this norm the unit ball is the convex hull of all the points $T = (t_1, \dots, t_N)$, such that each t_j, $j \in \{1, \dots, N\}$, is equal to 1 or -1. The set of these points has 2^N elements, which will be denoted by T_i, $i = 1, 2, \dots, 2^N$. Let $\{M_i,\ i = 1, 2, \dots, 2^{N-1}\}$ be such that $\{M_i, -M_i,\ i = 1, 2, \dots, 2^{N-1}\} = \{T_i,\ i = 1, 2, \dots, 2^N\}$. Now if we put the points M_i, $i = 1, 2, \dots, 2^{N-1}$, in the expression (34) we obtain the method for the calculation of the stability radius.*

Case 4.5 $\|\Delta\| = \sum_{i=1}^{N} |\delta_i|$. *In the space* $\mathbb{R}^N$ *with this norm the unit ball is the convex hull of points* $T = (t_1, \dots, t_N)$*, where each* t_j *is equal to zero, with the exception of one that is equal to* 1 *or* -1*. The set of these points has* $2N$ *elements, which will be denoted by* T_i*,* $i = 1, 2, \dots, 2N$*. Thus from expression (33) it follows that* $\{\Phi(T_i),\ i = 1, \dots, 2N\} = \{B_i, -B_i,\ i = 1, \dots, N\}$ *and finally making use of (34) we conclude that*

$$r^-_{\mathbb{R},\,t}(A, (B_i)_{i\in\underline{N}}) = r_t(A, (B_i)_{i=1,\dots,2N}).$$

Calculation of the real radius of stability of a second order matrix under linear structured time-varying multiple perturbations

Let $B_i \in \mathbb{R}^{2\times p_i}$, $C_i \in \mathbb{R}^{q_i\times 2}$, $i \in \{1, \dots, N\}$ and let $A \in \mathbb{R}^{2\times 2}$ be a Hurwitz stable matrix. Consider as nominal system

$$\Sigma :\ \dot{x} = Ax,$$

and as perturbed system

$$\Sigma_\Delta :\ \dot{x} = (A + \sum_{i=1}^{N} B_i \Delta_i(t) C_i)x. \tag{35}$$

On the space of matrices $\Delta = (\Delta_1, \dots, \Delta_N) \in \prod_{i=1}^{N} \mathbb{R}^{p_i\times q_i}$ we consider the norm

$$\|\Delta\| = \max_i\{\|\Delta_i\|_i\},\ \ \|\Delta_i\|_i = \max_{p,q}\{|\delta^i_{p,q}|\},\ \text{where}\ \ \Delta_i = (\delta^i_{pq}),$$

and the perturbations $\Delta(\cdot)$ in (35) are taking in the space $L^\infty(\mathbb{R}^+, \prod_{i=1}^{N} \mathbb{R}^{p_i\times q_i})$ with the norm

$$\|\Delta(\cdot)\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \|\Delta(t)\|.$$

Definition 4.6 *The real radius of stability of the stable matrix* A *for linear structured time-varying multiple perturbations determined by the matrices* $B_i \in \mathbb{R}^{2\times p_i}$*,* $C_i \in \mathbb{R}^{q_i\times 2}$*,* $i \in \{1, \dots, N\}$*, is defined as the number*

$$\begin{aligned} r^-_{\mathbb{R},\,t}(A, (C_i, B_i)_{i\in\underline{N}}) &= \inf\{\|\Delta(t)\|_\infty :\ \Delta(t) \in L^\infty(\mathbb{R}^+, \prod_{i=1}^{N} \mathbb{R}^{p_i\times q_i}), \\ &\qquad \Sigma_\Delta \textit{ is not asymptotically stable}\}. \end{aligned}$$

Let $E = \prod_{i=1}^{N} \mathbb{R}^{p_i\times q_i}$ and $\Phi : \prod_{i=1}^{N} \mathbb{R}^{p_i\times q_i} \to \mathbb{R}^{2\times 2}$ be the linear map given by

$$\Phi(\Delta) = \Phi(\Delta_1, \dots, \Delta_N) = \sum_{i=1}^{N} B_i \Delta_i C_i.$$

In the space $\prod_{i=1}^{N}\mathbb{R}^{p_i\times q_i}$ with the considered norm the unit ball is the convex hull of the set of all points $T=(\Delta_1,\dots,\Delta_N)$ such that the matrices $\Delta_j,\ j\in\{1,\dots,N\}$, have all their elements equal to 1 or -1. The set of these points has $s=\prod_{i=1}^{N}2^{p_iq_i}$ elements, which will be denoted by $T_i,\ i=1,2,\dots,s$. Let $\{M_i,\ i=1,2,\dots,s/2\}$ be such that $\{M_i,-M_i,\ i=1,2,\dots,s/2\}=\{T_i,\ i=1,2,\dots,s\}$. ¿From Definitions 4.1, 4.6 and Lemma 4.2 we conclude that

$$r^-_{\mathbb{R},\,t}(A,(C_i,B_i)_{i\in\underline{N}})=r_t(A,(\Phi(M_j))_{j=1,\dots,s}).$$

So, also in this case, we can calculate the considered stability radius by application of the fundamental result of Section 3.

5 Examples

In this Section we apply Theorem 3.17 and results of the Section 4 to concrete values of the data $A,(B_i)_{i\in\underline{N}}$.

Example 5.1 *Let*

$$A=\begin{pmatrix}-1 & -1\\ 3 & -2\end{pmatrix},\quad B_1=\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix},\quad B_2=\begin{pmatrix}2 & -3\\ 3 & 1\end{pmatrix}$$

and consider perturbations of the form $A\to A+\sum_{i=1}^{2}\delta_i(t)B_i$, *where*

$$\Delta(t)=(\delta_1(t),\delta_2(t))\in L^\infty(\mathbb{R}^+,\mathbb{R}^2).$$

On the space $\mathbb{R}^2$ *we fix the norm* $\|\Delta\|=\sum_{i=1}^{2}|\delta_i|$. *In this case we can apply the results of Section 4.1 and Section 3 for the calculation of the stability radius* $r^-_{\mathbb{R},\,t}(A,(B_1,B_2))$. *Simple calculations showed that* $\widehat{r}(A,(B_1,B_2))=1$, $\varrho(A,(B_1,B_2))=1$, $\zeta(A,(B_1,B_2))=0$. *Making use of the computational system "Mathematica" it was obtained that the root of the Equation* $I^-(r)=0$ *is between the numbers* 0.752926 *and* 0.752927 *and so was finally obtained that* $r^-_{\mathbb{R},\,t}(A,(B_1,B_2))\approx 0.752926$.

Example 5.2 *Let*

$$A=\begin{pmatrix}-1 & -1\\ 3 & -2\end{pmatrix},\quad B_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad B_2=\begin{pmatrix}-1\\ 0\end{pmatrix},$$
$$C_1=\begin{pmatrix}0 & 1\end{pmatrix},\quad C_2=\begin{pmatrix}1 & 0\end{pmatrix},$$

and consider perturbations of the form $A\to A+\sum_{i=1}^{2}B_i\Delta_i(t)C_i$, *where*

$$\Delta(t)=(\Delta_1(t),\Delta_2(t))\in L^\infty(\mathbb{R}^+,\mathbb{R}^{2\times 1}\times\mathbb{R}^{1\times 1}).$$

On the space $\mathbb{R}^{2\times1}\times\mathbb{R}^{1\times1}$ *we fix the norm* $\|\Delta\|=\max_i\{\|\Delta_i\|_i\}$, $\|\Delta_i\|_i=\max_{p,q}\{|\delta^i_{pq}|\}$. *In this case we can apply the results of Section 4.2 and Section 3 for the calculation of the stability radius* $r^-_{\mathbb{R},\,t}(A,(B_1,B_2,C_1,C_2))$. *Simple calculations showed that* $\widehat{r}(A,(B_1,B_2,C_1,C_2))=1$, $\varrho(A,(B_1,B_2,C_1,C_2))=+\infty$, $\zeta(A,(B_1,B_2,C_1,C_2))=0$. *Making use of the computational system "Mathematica" it was obtained that the root of the Equation* $I^-(r)=0$ *is approximately* 0.920898 *and so was finally obtained that* $r^-_{\mathbb{R},\,t}(A,(B_1,B_2,C_1,C_2))\approx 0.920898$.

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