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A second eigenvalue bound for the Dirichlet Schrödinger equation wtih a radially symmetric potential *

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Abstract

We study the time-independent Schrödinger equation with radially symmetric potential $k|x|^\alpha$, $k \geq 0$, $k \in \mathbb{R}$, $\alpha \geq 2$ on a bounded domain Ω in $\mathbb{R}^n$, $(n \geq 2)$ with Dirichlet boundary conditions. In particular, we compare the eigenvalue $\lambda_2(\Omega)$ of the operator $-\Delta + k|x|^\alpha$ on Ω with the eigenvalue $\lambda_2(S_1)$ of the same operator $-\Delta + kr^\alpha$ on a ball S_1, where S_1 has radius such that the first eigenvalues are the same $(\lambda_1(\Omega) = \lambda_1(S_1))$. The main result is to show $\lambda_2(\Omega) \leq \lambda_2(S_1)$. We also give an extension of the main result to the case of a more general elliptic eigenvalue problem on a bounded domain Ω with Dirichlet boundary conditions.

1 The Schrödinger eigenvalue equation with radially symmetric potential

In this paper we consider the Schrödinger eigenvalue equation with Dirichlet boundary conditions on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$:

$$-\Delta u + k|x|^\alpha u = \lambda u \quad \text{on } \Omega \subset \mathbb{R}^n \tag{1}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

with $\alpha \geq 2$, $k \geq 0$, $k \in \mathbb{R}$. Also of interest will be the generalization of the results to the case of a more general, second-order, elliptic, partial differential equation of the form

$$-\textstyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + k|x|^\alpha u = \lambda r(x) u \quad \text{on } \Omega \subset \mathbb{R}^n \tag{3}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{4}$$

with the assumption that the equation is uniformly elliptic on Ω. This means that there are positive numbers a and A such that the matrix $[a_{ij}]$ satisfies

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$0 < a \leq [a_{ij}] \leq A$ in the quadratic form sense throughout Ω. Also required is that the weight function $r(x)$ satisfy $0 < c \leq r(x) \leq C$ for some positive constants c and C and all $x \in \Omega$.

It is known that the set of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ corresponding to eigenfunctions u_k of the problems (1)-(2) and (3)-(4) are nonnegative and can be arranged in nondecreasing order as follows:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \leq \lambda_k \leq \ldots \to \infty.$$

The purpose of this paper is to compare the eigenvalues of (1)-(2) and (3)-(4) to those of problems on a ball. We will obtain comparison results for λ_2 utilizing the "fixed λ_1" technique of Ashbaugh and Benguria. This has been used to show for the Laplacian with no potential that $\lambda_2(\Omega) \leq \lambda_2(S_1)$ where S_1 is the n-ball with the same first eigenvalue as $-\Delta$ on Ω. Ashbaugh and Benguria then used the fact that the ratio of eigenvalues on balls for $-\Delta$ is constant to prove the Payne-Pólya-Weinberger conjecture (for $-\Delta$ the ratio of the first two eigenvalues is maximized for balls).

2 New results

In this section a "fixed λ_1" theorem for the Schrödinger problem (1)-(2), is stated along with a generalization of that result to the general elliptic eigenvalue problem (3)-(4). For the problem (1)-(2) the comparison eigenvalues considered will be those of the problem

$$-\Delta z + kr^{\alpha} z = \lambda z \quad \text{on } S_1 \in \mathbb{R}^n \tag{5}$$

$$z = 0 \quad \text{on } \partial S_1 \tag{6}$$

where S_1 is an n-dimensional ball of the appropriate radius so that we have $\lambda_1(\Omega) = \lambda_1(S_1)$.

Note: Here and throughout the paper we will use $k|x|^{\alpha}$ and kr^{α} somewhat interchangeably depending on whether we are talking about the potential function on a general domain or on a ball. It should be apparent that this only makes sense if the domain Ω allows us the make sense of the transformation of the potential function from Ω to $\Omega^{\star}$ (the n-ball having the same volume as Ω) or to S_1. That is, the domain Ω should have its center of mass at the origin, i.e., at $r = 0$, so that the origin of the potential on Ω and on $\Omega^{\star}$ or S_1 is unchanged. Throughout this paper we will assume this is the case.

For the general elliptic problem (3)-(4), the potential in (5) will be modified to $\frac{kr^{\alpha}}{a}$, where a is the "lower bound" of the matrix $[a_{ij}]$, and S_1 will be chosen so that $\lambda_1(S_1) = \frac{C}{a}\lambda_1(\Omega)$, where C is the upper bound of the weight function $r(x)$. The reason for these modifications will be discussed later in the paper.

The main result is now stated.

Theorem 2.1. *For equations (1) and (5), as specified above and with Dirichlet boundary conditions, the first two eigenvalues of the respective problems satisfy*

$$\lambda_2(\Omega) \le \lambda_2(S_1). \tag{7}$$

An extension of this theorem for equation (3) will also be derived:

Theorem 2.2. *For equations (3) and (5) specified previously (note, in particular, that S_1 is specified by $\lambda_1(S_1) = \frac{C}{a}\lambda_1(\Omega)$), and with Dirichlet boundary conditions, the first two eigenvalues of the respective problems satisfy*

$$\frac{\lambda_2}{\lambda_1}(\Omega) \le 1 + \frac{AC}{ac}\left(\frac{\lambda_2}{\lambda_1}(S_1) - 1\right). \tag{8}$$

3 Integral inequalities and rearrangements

We begin with the basic gap inequality

$$\lambda_2 - \lambda_1 \le \frac{\int_\Omega |\nabla P|^2 u_1^2 dx}{\int_\Omega P^2 u_1^2 dx} \text{ provided that } \int_\Omega P u_1^2 dx = 0 \text{ and } P \not\equiv 0, \tag{9}$$

derived using the Rayleigh-Ritz inequality and integration by parts.

To make use of inequality (9), we use not a single P but rather n different trial functions P_i, $i = 1, 2, \ldots, n$, where again n is the dimension of the space. We define the P_i by

$$P_i = g(r)\frac{x_i}{r}, \quad i = 1, 2, \ldots, n,$$

where g is chosen to be a nonnegative, nontrivial, continuous, differentiable, and bounded function of the radial variable $r = |x|$. The exact choice of g will be made later. Following [3] it can be shown that the side conditions in (9) of $\int_\Omega P_i {u_1}^2 dx = 0$ for $i = 1, 2, \ldots, n$ can all be satisfied. Accepting this, rewrite (9) as

$$(\lambda_2 - \lambda_1)\int_\Omega {P_i}^2 u_1^2 dx \le \int_\Omega |\nabla P_i|^2 u_1^2 dx$$

and sum on i, arriving at

$$\lambda_2 - \lambda_1 \le \frac{\int_\Omega (\sum_{i=1}^n |\nabla P_i|^2) u_1^2 dx}{\int_\Omega (\sum_{i=1}^n {P_i}^2) u_1^2 dx}.$$

Again following Ashbaugh and Benguria [3] the angular dependence of the P_i's drops out, leading to the new gap inequality

$$\lambda_2 - \lambda_1 \le \frac{\int_\Omega [g'(r)^2 + (n-1)g(r)^2/r^2] u_1^2 dx}{\int_\Omega g(r)^2 u_1^2 dx}. \tag{10}$$

The next step is to choose g such that there is equality in (10) if Ω is a ball. This implies the choice of

$$g(r)=\begin{cases}\frac{y_2(r)}{y_1(r)} & \text{for } 0\le r<r^*\\ \lim\limits_{t\nearrow r^*} g(t) & \text{for } r\ge r^*,\end{cases} \tag{11}$$

where r^* is an appropriate radius (less than or equal to that of Ω^*, with equality if $\Omega=\Omega^\star$) and y_1 and y_2 are solutions to certain one-dimensional radial eigenvalue problems obtained from (5)-(6) via separation of variables. By standard separation of variables techniques in spherical coordinates and a result due to Baumgartner, Gross, and Martin for $\mathbb{R}^3$ [7], extended by Ashbaugh-Benguria [1] to all n and Ω a ball, one finds that for radial potentials $W(r)$ with $[rW]''\ge 0$ (certainly satisfied in our case) the first two eigenvalues of $-\Delta+W$ on a ball B of radius r^* are the respective first eigenvalues of certain one-dimensional problems. In our case these are

$$y_1{}''+\frac{n-1}{r}y_1{}'+(\lambda_1^\star-kr^\alpha)y_1=0 \tag{12}$$

with $y_1(0)$ finite, $y_1(r^*)=0$, and

$$y_2{}''+\frac{n-1}{r}y_2{}'+(\lambda_2^\star-\frac{n-1}{r^2}-kr^\alpha)y_2=0 \tag{13}$$

with $y_2(0)=y_2(r^*)=0$. Here $\lambda_1^\star$ and $\lambda_2^\star$ are defined as the first eigenvalues of the respective problems. Note that by construction this makes g continuously differentiable on $[0,\infty)$.

Finally, by defining the function $B(r)$ by

$$B(r)\equiv {g'(r)}^2+(n-1)\frac{{g(r)}^2}{r^2}. \tag{14}$$

the gap inequality (10) can be put into the form

$$\lambda_2-\lambda_1\le\frac{\int_\Omega B(r)u_1^2dx}{\int_\Omega {g(r)}^2u_1^2dx}. \tag{15}$$

Next our attention is turned to proving two key properties of B and g. Both functions are nonnegative, and now two important facts about their respective derivatives need to be shown. Let r^* be the radius of the ball of definition for g and B. Then

(1) $g(r)$ is increasing for $r\in[0,r^*]$, and
(2) $B(r)$ is decreasing for $r\in[0,r^*]$.

The idea of the proof is based on the properties of the function $q(r)$ defined by

$$q(r)=\frac{rg'(r)}{g(r)}. \tag{16}$$

With q as above and B defined by (14) an easy substitution shows $B=[q^2+(n-1)]{(g/r)}^2$ and hence $B'=2[qq'+(q-1)(q^2+(n-1))/r]{(g/r)}^2$. It is clear that properties (1) and (2) will be met by showing that $0 \leq q \leq 1$ and $q' \leq 0$ for $0 \leq r \leq r^*$.

To prove these results one uses a Riccati differential equation satisfied by q on $0 < r < r^*$ to analyze the behavior of q both in the interval and at the boundary points. To derive this differential equation, start from (16) and compute

$$q = r[\frac{y_2'}{y_2} - \frac{y_1'}{y_1}].$$

Next take the derivative

$$q' = q/r - q(\frac{y_1'}{y_1} + \frac{y_2'}{y_2}) + r(\frac{y_2''}{y_2} - \frac{y_1''}{y_1})$$

and use the fact that $\frac{q}{r} + \frac{y_1'}{y_1} = \frac{y_2'}{y_2}$ to eliminate $\frac{y_2'}{y_2}$ from the equation. Then use the differential equations for y_1 and y_2 to eliminate second derivatives of y_1 and y_2 from the q' equation. This gives a Riccati differential equation

$$q' = (\lambda_1^\star - \lambda_2^\star)r + \frac{(1-q)(q+(n-1))}{r} - 2q\frac{y_1'}{y_1}. \tag{17}$$

Now consider the behavior of q at the endpoints $r = 0$ and $r = r^*$. Straightforward calculations using L'Hopital's rule show that

$$q(0) = 1, \quad q'(0) = 0, \quad q''(0) = 2\lambda_1^\star/n - 2\lambda_2^\star/(n+2), \tag{18}$$

and at the other endpoint,

$$q(r^*) = 0, \quad q'(r^*) = \frac{1}{3}[(\lambda_1^\star - \lambda_2^\star)r^* + (n-1)/r^*]. \tag{19}$$

This leads to the following lemma.

Lemma 3.1. *For q defined as above, $q \geq 0$ for $0 \leq r \leq r^*$, and hence $q'(r^*) \leq 0$.*

Proof. Assume not. Then there exist two points r_1 and r_2 with $0 < r_1 < r_2 \leq r^*$ and such that $q(r_1) = q(r_2) = 0$ but $q'(r_1) \leq 0$ and $q'(r_2) \geq 0$. Suppose first that $r_2 < r^*$. Then by (17) we have

$$0 \geq q'(r_1) = (\lambda_1^\star - \lambda_2^\star)r_1 + (n-1)/r_1 > (\lambda_1^\star - \lambda_2^\star)r_2 + (n-1)/r_2 = q'(r_2) \geq 0, \tag{20}$$

a contradiction. If $r_2 = r^*$, then using (17) and (19) a similar argument to the above yields

$$0 \geq q'(r_1) = (\lambda_1^\star - \lambda_2^\star)r_1 + (n-1)/r_1 > (\lambda_1^\star - \lambda_2^\star)r^* + (n-1)/r^* = 3q'(r^*) \geq 0 \tag{21}$$

which is again a contradiction. $\square$

Moving on, we make some further definitions to simplify our notation. Denote $E = (\lambda_2^\star - \lambda_1^\star)/2$ and $b = (n-2)/2$; also let $p = y_1'/y_1$. Then from (16) and (17) one has $q' = 2T$ with T defined to be

$$T(r,q) = -pq - \frac{bq}{r} - \frac{q^2}{2r} + \frac{2b+1}{2r} - Er. \tag{22}$$

To analyze the behavior of T for fixed values of q, we will study the behavior of $\frac{\partial T}{\partial r}$ at points where $T = 0$ (fixed q) and the behavior of $T(r, \text{fixed } q)$ near $r = 0$ and $r = r^*$.

For r approaching zero, p is $O(r)$, and thus from (22)

$$T(r,q) = \frac{1}{2r}[(2b+1+q)(1-q)] + O(r).$$

This implies that for q fixed, $0 < q < 1$, $T(r) \to +\infty$ as $r \to 0$. For $q = 1$, $T(r) \to 0$ as $r \to 0$. For any fixed q and $r \to r^*$ the boundary conditions on y_1 show $p \to -\infty$, and all other terms in (22) being finite, $T(r,q)|_{r \to r^*, q\,\text{fixed}} \to +\infty$. Now, taking the partial derivative of T with respect to r yields

$$\frac{\partial T}{\partial r} = -qp' + \frac{bq}{r^2} + \frac{N}{r^2} - E \tag{23}$$

where $N \equiv \frac{q^2-(2b+1)}{2}$. Since from here on we are interested in $T(r,q)$ only when q is a fixed constant, it is permissible to consider T as a function of the single variable r and abuse notation somewhat by referring to $\partial T/\partial r = T'$, $\partial^2 T/\partial r^2 = T''$, etc. When $T = 0$, we have, solving for p in (22) that

$$p|_{T=0} = -\frac{b}{r} - \frac{N}{qr} - \frac{Er}{q}. \tag{24}$$

Also, from the equation for y_1 we get the Riccati equation

$$p' + p^2 + \frac{2b+1}{r}p + \lambda_1^\star - kr^\alpha = 0. \tag{25}$$

Now, substituting (24) into (25) and the combination into (23) we obtain

$$T'|_{T=0} = \frac{M}{r^2} + \frac{E^2r^2}{q} + Q - qkr^\alpha \tag{26}$$

where

$$\begin{aligned} M &= \frac{1}{q}(N^2 - q^2b^2), \\ Q &= q\lambda_1^\star + \frac{2EN}{q} - 2E. \end{aligned}$$

We then define $T'|_{T=0} = Z(r)$ by

$$Z(r) = \frac{M}{r^2} + \frac{E^2r^2}{q} + Q - qkr^\alpha. \tag{27}$$

(As with T, Z is really a function of r and q, but is treated as a function of r only since q is considered fixed.) The behavior of Z at zero is determined by M. To see what values this takes on for different $q \in (0,1]$, consider

$$qM = N^2 - q^2b^2 \equiv d(q) = \frac{1}{4}(q^2-1)[(q-1)-(n-2)][(q+1)+(n-2)] \quad (28)$$

This implies that

$$d(q) \text{ is} \begin{cases} >0 & \text{if } 0<q<1 \\ =0 & \text{if } q=1 \end{cases}$$

Furthermore, we can show that $\frac{M}{r^2} \to 0$ as $r \to 0$. This shows that

$$Z(0) \text{ is} \begin{cases} +\infty & \text{if } 0<q<1 \\ Q & \text{if } q=1. \end{cases} \quad (29)$$

Using this information leads to the following lemmas.

Lemma 3.2. *Given $q(0)=1$, $q'(0)=0$, and $q(r^*)=0$, it follows that q must first become less than one as r increases from zero.*

Proof. Suppose not. Then certainly $q''(0) \geq 0$, else there is an immediate contradiction. Then since $q(r^*)=0$ there must exist at least one point $0 < r_1 < r^*$ such that $q(r_1)=1$ and $q'(r_1) \leq 0$, with q crossing below one to the right of r_1.

Fix $q \equiv 1$ in T, then $T(0)=0$ and recall that $T(r^*,q)=+\infty$. Thus T must start at zero, become positive to the right of zero, and at r_1 be either 0 or negative, and then tend to positive infinity. If $T(r_1)$ is zero, then T must still cross below zero just to the right of r_1 since q must cross below one just to the right of r_1. Hence T must have the following sign changes: positive-negative-positive. Likewise, $Z(r,q)=T'(r,q\,\text{fixed},T=0)$, which has the representation

$$Z(r,q) = \frac{M}{r^2} + \frac{E^2r^2}{q} + Q - qkr^\alpha$$

must start out positive as T increases from zero, then change to negative as T decreases near r_1, and become positive again as T heads off to infinity. Summing up, Z should have the sign changes positive-negative-positive. (Note we have used the fact that higher derivatives of Z match those of T when $T=0$.) When q is fixed at one M disappears so Z becomes

$$Z(r,q\equiv 1) = E^2r^2 + Q - kr^\alpha$$

with $Z(0,q\equiv 1) \geq 0$. Taking the derivative of Z with respect to r we get

$$Z'(r,q\equiv 1) = 2E^2r - \alpha kr^{\alpha-1}.$$

Looking at this when $Z=0$ and making substitutions we get

$$Z'(r,q\equiv 1, Z=0) = (2-\alpha)kr^{\alpha-1} - \frac{2Q}{r} \quad (30)$$

and hence $Z'(r, q \equiv 1, Z = 0) \leq 0$ since $Q = Z(0) \geq 0$ and $\alpha \geq 2$.

If $\alpha > 2$ this is seen to be an immediate contradiction of the necessary sign change of Z from negative to positive. If $\alpha = 2$ we consider $Z'' = 2E^2 - 2k$. If $E^2 - k$ is less than zero we are done since that would make Z'' negative. If $E^2 - k$ is zero then $Z \equiv Q$, which would mean Z has no sign changes, an obvious contradiction. If $E^2 - k$ is greater than zero then $Z' > 0$, another contradiction.

Thus T cannot have the required sign changes if we assume q increases away from one as r increases away from zero, so q must decrease from one as r increases from zero. □

With the end behavior of q established, the next step is to establish the condition that q is decreasing and hence $q \leq 1$.

Lemma 3.3. *Given $q(0) = 1$, q first decreases from one as r increases from zero, $q(r^*) = 0$, and $q'(r^*) \leq 0$, it follows that $q'(r) \leq 0$ for $0 \leq r \leq r^*$.*

Corollary 3.4. *Given that $q'(r) \leq 0$ on $0 \leq r \leq r^*$ and $q(0) = 1$, it follows that $q(r) \leq 1$ on $0 \leq r \leq r^*$.*

Proof. (Of Lemma.) Suppose not. Then there exist three points $0 < r_1 < r_2 < r_3 < r^*$ with $0 < q(r_1) = q(r_2) = q(r_3) < 1$ and $q'(r_1) < 0$, $q'(r_2) > 0$, $q'(r_3) < 0$. Using the fixed value of $q < 1$ at these points in the function T, recall that T should be positive for small and large values of r. Also, since $T = q'/2$, we have $T(r_1) < 0, T(r_2) > 0$, and $T(r_3) < 0$. Hence T must have at least four sign changes: positive-negative-positive-negative-positive. The value of Z at each of the roots of T must alternate in a similar way (positive to negative in T implies negative Z, etc.) Starting from $Z(0, q < 1) = +\infty$, this shows that Z must also alternate positive-negative-positive-negative-positive. Recall from (27) that

$$Z(r, q < 1) = \frac{M}{r^2} + \frac{E^2 r^2}{q} + Q - qkr^\alpha$$

and thus

$$\begin{aligned} Z'(r, q < 1) &= \frac{-2M}{r^3} + \frac{2E^2 r}{q} - q\alpha k r^{\alpha - 1} \\ Z''(r, q < 1) &= \frac{6M}{r^4} + \frac{2E^2}{q} - q\alpha(\alpha - 1)kr^{\alpha - 2}. \end{aligned}$$

Since q less than one implies $M > 0$ and it follows that $Z''(r, q < 1)$ is strictly decreasing, $Z(r, q < 1)$ could not have the changes in concavity necessary for its changes in sign, a contradiction to the behavior of $T(r, q < 1)$, and thus $q' \leq 0$ on $[0, r^*]$. This proves the lemma and thus the corollary. □

Combining our work, we have $0 \leq q \leq 1$ and $q' \leq 0$ on $0 \leq r \leq r^*$, which in turn shows that $g(r)$ is increasing and $B(r)$ is decreasing. These results are fundamental in using the rearrangement techniques of Hardy, Littlewood, and Pólya [11].

Rearrangements have several useful properties. In particular, if f and h are nonnegative functions, then we have the integral inequalities

$$\int_{\Omega^\star} f_\star h^\star dx \leq \int_\Omega fh dx \leq \int_{\Omega^\star} f^\star h^\star dx.$$

Also, if f is a nonnegative, radially symmetric decreasing (resp. increasing) function on Ω then $f^\star(r) \leq f(r)$ and respectively $f_\star(r) \geq f(r)$ for r between 0 and the radius of $\Omega^\star$. The previous lemmas showed that $g(r)$ is increasing and $B(r)$ is decreasing (they are both nonnegative and radially symmetric), hence

$$\int_\Omega B(r) u_1^2 dx \leq \int_{\Omega^\star} B(r)^\star {u_1^\star}^2 dx \leq \int_{\Omega^\star} B(r) {u_1^\star}^2 dx, \tag{31}$$

and

$$\int_\Omega g(r)^2 u_1^2 dx \geq \int_{\Omega^\star} g(r)_\star^2 {u_1^\star}^2 dx \geq \int_{\Omega^\star} g(r)^2 {u_1^\star}^2 dx. \tag{32}$$

Note also, concerning our potential function, or any potential function $V(x) = V(r)$ that is radially symmetric and increasing, that

$$\int_\Omega V(x) u_1^p dx \geq \int_{\Omega^\star} V_\star(r) {u_1^\star}^p dx \geq \int_{\Omega^\star} V(r) {u_1^\star}^p dx$$

where $p = 1, 2$.

With the inequalities (31), (32) established, it is time to show the last set of inequalities necessary for completing the theorem. These are

$$\int_{\Omega^\star} B(r) {u_1^\star}^2 dx \leq \int_{S_1} B(r) z_1^2 dx \tag{33}$$

and

$$\int_{\Omega^\star} g(r)^2 {u_1^\star}^2 dx \geq \int_{S_1} g(r)^2 z_1^2 dx \tag{34}$$

where z_1 is the first eigenfunction of the comparison problem (5)-(6). In order to derive these two inequalities we establish a crossing result for $u_1^\star$ and z_1 similar to the type found in [8, 9, 19, 3]. The result will resemble the work in these papers, with the exception that the differential equation satisfied by z_1 includes a potential term, whereas those of the previous works had no potential.

4 Chiti and Talenti type arguments

Since the results of this section apply to any potential function that is continuous, radially symmetric, increasing, and has its origin at the center of mass for Ω, we will make our proof for a general potential function $V(x) = V(r)$, and keep in mind that our potential $V(x) = k|x|^\alpha = kr^\alpha = V(r)$ is a special case.

As a first step we follow Talenti [19], Chiti [8], Ashbaugh and Benguria [3] in integrating both sides of $-\Delta u + Vu = \lambda u$ over the level set $\{x \in \Omega : u(x) > t\}$ to get

$$-\int_{u(x)>t} \Delta u dx = \int_{u(x)>t} (\lambda - V(x))u dx.$$

Using Gauss's Divergence Theorem we have

$$-\int_{u(x)>t} \Delta u dx = \int_{u(x)=t} (|\nabla u|) H_{n-1}(dx),$$

where H_{n-1} denotes $(n-1)$-dimensional measure. Now consider the distribution function $\mu(t) = \text{meas}\{x \in \Omega : |u(x)| > t\}$. Taking the derivative of $\mu(t)$ we can derive the following special case of the coarea formula:

$$-\mu'(t) = \int_{u(x)=t} \frac{1}{|\nabla u|} H_{n-1}(dx). \tag{35}$$

Using (35) and the Cauchy-Schwarz inequality we find

$$\begin{aligned}
H_{n-1}\{x \in \Omega : u(x) = t\} &= \int_{u=t} H_{n-1}(dx) \\
&= \int_{u=t} \frac{|\nabla u|^{1/2}}{|\nabla u|^{1/2}} H_{n-1}(dx) \\
&\leq \left(\int_{u=t} |\nabla u| H_{n-1}(dx)\right)^{1/2} \left(\int_{u=t} \frac{1}{|\nabla u|} H_{n-1}(dx)\right)^{1/2} \\
&\leq \left[-\mu'(t) \int_{u=t} |\nabla u| H_{n-1}(dx)\right]^{1/2}.
\end{aligned}$$

Also, the n-dimensional isoperimetric inequality gives us

$$H_{n-1}\{x \in \Omega : u(x) = t\} \geq n{C_n}^{1/n} \mu(t)^{1-1/n}$$

where C_n is the volume of the n-dimensional unit ball. Combining these two inequalities yields

$$\int_{u=t} |\nabla u| H_{n-1}(dx) \geq n^2 {C_n}^{2/n} \mu(t)^{2-2/n} \left(\frac{-1}{\mu'(t)}\right).$$

Combining all these inequalities together for $u = u_1$ and $\lambda = \lambda_1$, one finds

$$\int_{u_1(x)>t} [\lambda_1 - V(x)] u_1 dx \geq n^2 {C_n}^{2/n} \mu_1(t)^{2-2/n} (-1/\mu_1'(t)).$$

We then use rearrangement properties to pass from u_1 to u_1^*. Finally, considering the fact that u_1^* and μ_1 are essentially inverse functions of each other, we get the key inequality

$$-\frac{du_1^*}{ds} \leq n^{-2} {C_n}^{-2/n} s^{-2+2/n} \int_0^s (\lambda_1 u_1^*(w) - V(w) u_1^*(w)) dw, \tag{36}$$

which is similar to equations of Chiti [8], Ashbaugh and Benguria [2], and Talenti [19].

Next consider the following lemma, an extension of the Faber-Krahn inequality [10, 12, 13].

Lemma 4.1 (Extension of Faber-Krahn). *Suppose V is continuous, radially symmetric and increasing. Then the first eigenvalue for the operator $-\Delta+V$ on $\Omega^\star$ is less than or equal to $\lambda_1(\Omega)$. Equivalently $|S_1| \leq |\Omega|$, with equality if and only if $\Omega = \Omega^\star = S_1$.*

Proof. By the Rayleigh-Ritz inequality and rearrangement properties we have

$$\begin{aligned}
\lambda_1 &= \frac{\int_\Omega (|\nabla u_1|^2 + V u_1^2)dx}{\int_\Omega u_1^2 dx} \\
&\geq \frac{\int_{\Omega^\star} (|\nabla u_1^\star|^2 + V {u_1^\star}^2)dx}{\int_{\Omega^\star} {u_1^\star}^2 dx} \\
&\geq [\text{first eigenvalue of } -\Delta + V \text{ on } \Omega^\star].
\end{aligned}$$

Thus

$$\lambda_1(S_1) = \lambda_1(\Omega) \geq [\text{first eigenvalue of } -\Delta + V \text{ on } \Omega^\star] = \lambda_1(\Omega^\star).$$

Since λ_1 is exactly the first eigenvalue of $-\Delta + V$ on S_1 it follows by the monotonicity property of Dirichlet eigenvalues with respect to domains that $|S_1| \leq |\Omega^\star| = |\Omega|$, and thus $S_1 \subseteq \Omega^\star$, with equality if and only if $\Omega = \Omega^* = S_1$. □

The groundwork has now been laid to prove a modified form of the Chiti comparison result.

Theorem 4.2 (Modified Chiti Comparison Theorem). *Let V be continuous, radially symmetric and increasing. Let u_1 and z_1 be the first eigenfunctions of $-\Delta+V$ on Ω and S_1 respectively, with Dirichlet Boundary conditions, assumed positive and normalized such that*

$$\int_\Omega u_1^2 dx = \int_{\Omega^\star} {u_1^\star}^2 dx = \int_{S_1} z_1^2 dx. \tag{37}$$

Then, making a change of variables so that z_1 and $u_1^\star$ can be viewed as functions of $s = C_n r^n$, (so that we are actually looking at u_1^*), there will exist a point $s_1 \in (0, |S_1|)$ such that $u_1^*(s_1) = z_1(s_1)$ and

$$\begin{cases} u_1^*(s) \leq z_1(s) & \text{for } 0 \leq s \leq s_1 \\ u_1^*(s) \geq z_1(s) & \text{for } s_1 \leq s \leq |S_1|. \end{cases} \tag{38}$$

Proof. The arguments depend upon the continuity of u_1^* and z_1. (The absolute continuity of u_1^* is established in the arguments of Talenti [19] or Chiti [8]. See, in particular, Lemma 1 in [8].) It should also be noted that z_1 is positive and decreasing on $[0, |S_1|]$. To show this, consider the Rayleigh quotient for λ_1 and the related rearrangement inequality.

$$\lambda_1 = \frac{\int_{S_1}(|\nabla z_1|^2 + V z_1^2)dx}{\int_{S_1} z_1^2 dx} \geq \frac{\int_{S_1}(|\nabla z_1^\star|^2 + V {z_1^\star}^2)dx}{\int_{S_1} {z_1^\star}^2 dx} \tag{39}$$

Since V is increasing, if z_1 is not decreasing, then we have strict inequality in (39). However, $z_1^\star$ is a valid trial function for z_1, so we have using Rayleigh-Ritz

$$\lambda_1 \leq \frac{\int_{S_1}(|\nabla z_1^\star|^2 + V {z_1^\star}^2)dx}{\int_{S_1} {z_1^\star}^2 dx} \tag{40}$$

which is a contradiction.

The proof for (38) then consists of basically two parts, each following the same line of reasoning. The first step is to show that if z_1 is normalized so that $z_1(0) = \text{ess sup}\, u_1$, then $z_1(s) \leq u_1^*(s)$ for all $s \in [0, |S_1|]$. To prove this, suppose not. Assuming $|S_1| \neq |\Omega|$ and $z_1 \not\equiv u_1^*$ (otherwise it is trivial), there are two possibilities. The first is that z_1 first starts out above u_1^*, and, since $|S_1| < |\Omega|$, z_1 eventually drops below u_1^*. If this is the situation, multiply u_1^* by some constant $c > 1$ such that there is a point s_0 sufficiently close to zero so that $cu_1^*(s_0) = z_1(s_0)$, and $\lambda_1 - V(s_0) \geq 0$. (This can be done since $\lambda_1 - V(0)$ is positive and V is continuous at zero.) Now, define a new function $h(s)$ such that

$$h(s) = \begin{cases} cu_1^*(s) & \text{for } 0 \leq s \leq s_0 \\ z_1(s) & \text{for } s_0 < s \leq |S_1| \end{cases}$$

and notice that h satisfies the inequality given in (36). This is true for $s \leq s_0$ by (36) and for $s \geq s_0$ by the fact that $z_1(s)$ satisfies (36) with equality, and replacing z_1 by cu_1^* on the interval $(0, s_0)$ will only increase the right-hand side since $cu_1^* \geq z_1$ and $\lambda_1 - V(s) \geq 0$. (Note that with equality (36) is just an integrated form of the radial differential equation for z in the variable $s = C_n r^n$, hence the equality for $h \equiv z_1$, see for example Talenti [19].) Since $u_1^*(0) = z_1(0)$, we must have $cu_1^* > z_1$ on some subinterval of $(0, s_0)$, and hence there is strict inequality in (36) beyond s_0.

The other possibility is that z_1 starts out below u_1^*, crosses above u_1^* at some point $\tilde{s}_0$, and then crosses back below u_1^* at some point $\tilde{s}_1$. If $\lambda_1 - V(\tilde{s}_0) \geq 0$, form $h(s)$ exactly as above with $\tilde{s}_0$ replacing s_0 and the arguments that h satisfies the inequality in (36) hold as before. If $\lambda_1 - V(\tilde{s}_0) < 0$, then on $[0, \tilde{s}_0]$ pick h to be whichever function (u_1^* or z_1) yields the greater value in the functional

$$I_0^{\tilde{s}_0}[f] = \int_0^{\tilde{s}_0} (\lambda_1 - V(w)) f(w) dw. \tag{41}$$

Choose h on the whole interval $[0, |S_1|]$ as follows:

$$h(s) = \begin{cases} \left\{ \begin{array}{ll} u_1^*(s) & \text{if } I_0^{\tilde{s}_0}[u_1^*] \geq I_0^{\tilde{s}_0}[z_1] \\ z_1(s) & \text{if } I_0^{\tilde{s}_0}[u_1^*] < I_0^{\tilde{s}_0}[z_1] \end{array} \right\} & \text{for } s \in [0, \tilde{s}_0] \\ \min[u_1^*, z_1] & \text{for } s \in (\tilde{s}_0, \tilde{s}_1) \\ z_1(s) & \text{for } s \in [\tilde{s}_1, |S_1|]. \end{cases}$$

Again we must check that (36) is satisfied. Certainly it is on $(0,\tilde{s_0})$ for whichever function is h. For $s\in(\tilde{s_0},\tilde{s_1})$, if $h(s)=u_1^*(s)$ we have

$$\begin{aligned}
-\frac{dh}{ds} &= -\frac{du_1^*}{ds}\\
&\le n^{-2}{C_n}^{-2/n}s^{-2+2/n}\left(\int_0^{\tilde{s_0}}(\lambda_1-V)u_1^*dw+\int_{\tilde{s_0}}^{s}(\lambda_1-V)u_1^*dw\right)\\
&\le n^{-2}{C_n}^{-2/n}s^{-2+2/n}\left(\int_0^{\tilde{s_0}}(\lambda_1-V)hdw+\int_{\tilde{s_0}}^{s}(\lambda_1-V)hdw\right)
\end{aligned}$$

the first replacement on $(0,\tilde{s_0})$ true by the construction that h would give the greater integral for (41) and the second replacement true since $h\le u_1^*$ and $\lambda_1-V(s)<0$ on $(\tilde{s_0},\tilde{s_1})$. (Note that $\lambda_1-V(s)$ is a decreasing function.) Similarly, if $h(s)=z_1(s)$ for some $s\in(\tilde{s_0},\tilde{s_1})$, then at this value of s we have

$$-\frac{dh}{ds}=-\frac{dz_1}{ds}\le n^{-2}{C_n}^{-2/n}s^{-2+2/n}\left(\int_0^{\tilde{s_0}}(\lambda_1-V)hdw+\int_{\tilde{s_0}}^{s}(\lambda_1-V)hdw\right).$$

Finally, note that $h=u_1^*<z_1$ on some subinterval of $(\tilde{s_0},\tilde{s_1})$, else $z_1\le u_1^*$ on all $[0,|S_1|]$.

For whichever version of h being used, $\Phi(x)=h(C_n|x|^n)$ is a valid trial function for $-\Delta+V$ on S_1, and it follows by Rayleigh-Ritz that

$$\lambda_1<\frac{\int_{S_1}(|\nabla\Phi|^2+V\Phi^2)dx}{\int_{S_1}\Phi^2dx},\tag{42}$$

the strict inequality since $h\not\equiv z_1$, or the theorem is proved. Also notice that by definition of $\Phi(x)=h(C_n|x|^n)$ we have

$$\int_{S_1}\Phi^2dx=\int_0^{|S_1|}h^2(s)ds.$$

Taking the gradient of Φ and using the change of variable $s=C_nr^n$ leads to the integral identity

$$\int_{S_1}|\nabla\Phi|^2dx=n^2{C_n}^{2/n}\int_0^{|S_1|}s^{2-2/n}h'(s)^2ds.$$

Now use the inequality (36) to substitute for one of the $h'(s)$'s above to get

$$\int_{S_1}|\nabla\Phi|^2dx\le-\int_0^{|S_1|}h'(s)\int_0^s(\lambda_1-V(w))h(w)dwds.$$

Next we integrate by parts, and observing that the boundary terms vanish, we have

$$\begin{aligned}
\int_{S_1}|\nabla\Phi|^2dx &\le \lambda_1\int_0^{|S_1|}h^2(s)ds-\int_0^{|S_1|}V(s)h^2(s)ds\\
&= \lambda_1\int_{S_1}\Phi^2(x)dx-\int_{S_1}V(r)\Phi^2(x)dx.
\end{aligned}$$

It therefore follows that

$$\lambda_1 \geq \frac{\int_{S_1}(|\nabla\Phi|^2 + V\Phi^2)dx}{\int_{S_1}\Phi^2(x)dx}, \tag{43}$$

a contradiction to (42). Hence $z_1(s) \leq u_1^*(s)$ for all $s \in [0, |S_1|]$.

Now normalize the functions as in (37). From above, it follows that $z_1(0) \geq u_1^*(0)$. If $z_1(0) = u_1^*(0)$, then by the integral condition and part one just proved, the identities $|\Omega| = |S_1|$ and $z_1(s) = {u_1}^*(s)$ for all $s \in [0, |\Omega|]$ would be forced, and thus the theorem is proved trivially with any $s_1 \in (0, |S_1|)$ sufficing.

Next, if $z_1(0) > u_1^*(0)$, then by the extension of the Faber-Krahn inequality $|S_1| < |\Omega|$, and, by the Dirichlet boundary conditions $u_1^*(|\Omega|) = z_1(|S_1|) = 0$, it is clear that there is at least one point $s \in (0, |S_1|)$ at which $u_1^*(s) = z_1(s)$.

Take s_1 to be the largest $s \in (0, |S_1|)$ such that the condition $u_1^*(w) \leq z_1(w)$ for all $0 \leq w \leq s$ holds. Then it remains to be proven that ${u_1}^*(s) > z_1(s)$ for all $s \in (s_1, |S_1|]$. If not, there is a point s_2, with $s_1 < s_2 < |S_1|$, which we define to be the largest s such that $u_1^*(w) \geq z_1(w)$ for all $s_1 \leq w \leq s$. If s_2 exists, there also exists a point s_3, $s_2 < s_3 < |S_1|$, defined to be the largest s such that $u_1^*(w) \leq z_1(w)$ for all $s_2 \leq w \leq s$. Now, as before, we piece together a function $v(s)$ from u_1^* and z_1 which will satisfy the key integral inequality (36). This construction is done again based upon the interval where the decreasing function $\lambda_1 - V(s)$ changes sign from positive to negative.

The possibilities are as follows:

Case 1. $\lambda_1 - V(s)$ changes sign in $[s_2, |S_1|]$ or not at all. Then define the trial function v as

$$v(s) = \begin{cases} z_1(s) & \text{for } s \in [0, s_1] \cup [s_2, |S_1|] \\ \max[u_1^*, z_1] & \text{for } s \in (s_1, s_2) \end{cases}$$

Case 2. $\lambda_1 - V(s)$ changes sign in (s_1, s_2). Then

$$v(s) = \begin{cases} z_1(s) & \text{for } s \in [0, s_1] \\ \left\{ \begin{array}{ll} u_1^*(s) & \text{if } I_{s_1}^{s_2}[u_1^*] \geq I_{s_1}^{s_2}[z_1] \\ z_1(s) & \text{if } I_{s_1}^{s_2}[u_1^*] < I_{s_1}^{s_2}[z_1] \end{array} \right\} & \text{for } s \in (s_1, s_2) \\ \min[u_1^*, z_1] & \text{for } s \in (s_2, s_3) \\ z_1(s) & \text{for } s \in [s_3, |S_1|]. \end{cases}$$

Case 3. $\lambda_1 - V(s)$ changes sign in $[0, s_1]$. Then

$$v(s) = \begin{cases} \left\{ \begin{array}{ll} u_1^*(s) & \text{if } I_0^{s_1}[u_1^*] \geq I_0^{s_1}[z_1] \\ z_1(s) & \text{if } I_0^{s_1}[u_1^*] < I_0^{s_1}[z_1] \end{array} \right\} & \text{for } s \in [0, s_1] \\ \min[u_1^*, z_1] & \text{for } s \in (s_1, s_3) \\ z_1(s) & \text{for } s \in [s_3, |S_1|]. \end{cases}$$

As before, let $F(x) = v(C_n|x|^n)$ for whichever v is applicable. Then as long as s_2 exists F cannot be the ground-state eigenfunction on $|S_1|$, which gives the left-hand side of the inequality below. However, using the same methods that allowed us to arrive at (43) for Φ, one can derive the right-hand side of the inequality

$$\lambda_1 < \frac{\int_{S_1}(|\nabla F|^2 + VF^2)dx}{\int_{S_1} F^2 dx} \le \lambda_1, \tag{44}$$

a contradiction. $\square$

The fact that $g(r)$ is increasing and the modified Chiti Comparison Theorem can be used to prove the inequality (34). Keeping in mind that g and $u_1^\star$ are functions of the radial variable only, let r^* be the radius of S_1. Also, let r_1 correspond to s_1 from the Chiti theorem via the change of variable $s_1 = C_n r_1^n$, and let R be the radius of $\Omega^\star$. Then we have

$$\begin{aligned}
&\int_{S_1} g(r)^2 z_1^2 dx - \int_{\Omega^\star} g(r)^2 u_1^{\star 2} dx \\
&= nC_n \Big[\int_0^{r_1} g(r)^2 (z_1^2 - u_1^{*2}) r^{n-1} dr + \int_{r_1}^{r^*} g(r)^2 (z_1^2 \\
&\quad -u_1^{*2}) r^{n-1} dr - \int_{r^*}^{R} g(r)^2 u_1^{*2} r^{n-1} dr \Big] \\
&\le nC_n \Big[\int_0^{r_1} g(r_1)^2 (z_1^2 - u_1^{*2}) r^{n-1} dr \\
&\quad + \int_{r_1}^{r^*} g(r_1)^2 (z_1^2 - u_1^{*2}) r^{n-1} dr - \int_{r^*}^{R} g(r_1)^2 u_1^{*2} r^{n-1} dr \Big] \\
&= g(r_1)^2 nC_n \left[\int_0^{r^*} z_1^2 r^{n-1} dr - \int_0^{R} u_1^{*2} r^{n-1} dr \right] \\
&= g(r_1)^2 \left[\int_{S_1} z_1^2 dx - \int_{\Omega^\star} u_1^{\star 2} dx \right] \\
&= 0
\end{aligned}$$

the last line by virtue of the normalization hypothesis (37).

By a similar calculation, using the fact that $B(r)$ is radially decreasing and the Chiti Theorem, we can show the inequality (33):

$$\int_{\Omega^\star} B(r) u_1^{\star 2} dx \le \int_{S_1} B(r) z_1^2 dx.$$

5 Conclusion of the proof of the main theorem

Combining the results (31), (32), (33), (34) yields the string of inequalities

$$\int_\Omega B(r) u_1^2 dx \le \int_{\Omega^\star} B(r)^\star u_1^{\star 2} dx \le \int_{\Omega^\star} B(r) u_1^{\star 2} dx \le \int_{S_1} B(r) z_1^2 dx$$

and

$$\int_{\Omega} g(r)^2 u_1^2 dx \geq \int_{\Omega^\star} g(r)_\star^2 u_1^{\star 2} dx \geq \int_{\Omega^\star} g(r)^2 u_1^{\star 2} dx \geq \int_{S_1} g(r)^2 z_1^2 dx.$$

Together with (15) this gives

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{S_1} B(r) z_1^2 dx}{\int_{S_1} g(r)^2 z_1^2 dx}.$$

Since B and g were defined in such a way as to give equality in the case of Ω equal to a ball, the right hand side yields

$$\lambda_2(S_1) - \lambda_1(S_1) = \frac{\int_{S_1} B(r) z_1^2 dx}{\int_{S_1} g(r)^2 z_1^2 dx}$$

and so this reduces to the main inequality

$$\lambda_2(\Omega) \leq \lambda_2(S_1),$$

since S_1 was chosen so that $\lambda_1(S_1) = \lambda_1(\Omega)$.

6 Extension to general elliptic equation

This section gives an outline of how to extend the bound given in Theorem 2.1 for the Schrödinger problem to the bound (8) referring to the general elliptic problem of Theorem 2.2.

Proceeding as in the case of the Schrödinger operator we find B and g are defined as before with the exception that kr^α becomes $\frac{kr^\alpha}{a}$. Equality to $\lambda_2 - \lambda_1$ in the integral ratio

$$\frac{\int_\Omega B(r) u_1^2 dx}{\int_\Omega g(r)^2 u_1^2 dx}$$

again comes for the Schrödinger operator on a ball. Since B and g differ only by a constant in kr^α, their properties of g increasing and B decreasing remain unchanged. The Chiti and Talenti type arguments proceed with little change, the key differences being that S_1 is chosen so that

$$\begin{cases} -\Delta z + \frac{kr^\alpha}{a} z = \lambda z & \text{on } S_1 \\ z = 0 & \text{on } \partial S_1, \end{cases}$$

has $\frac{C}{a}\lambda_1(\Omega)$ as its first eigenvalue as opposed to $\lambda_1(\Omega)$. Another extension of the Faber-Krahn result shows that $|S_1| \leq |\Omega^*| = |\Omega|$.

Corresponding to the change from λ_1 to $\frac{C}{a}\lambda_1$, the key integral inequality (36) changes to

$$-\frac{du_1^*}{ds} \leq \frac{n^{-2} {C_n}^{-2/n} s^{-2+2/n}}{a} \int_0^s (C\lambda_1 u_1^*(w) - kw^\alpha u_1^*(w)) dw.$$

(In this inequality we see the necessity for the change in potential and first eigenvalue.) The Chiti comparison result follows as before, leading to

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{A}{c} \frac{\int_{S_1} B(r) z_1^2 dx}{\int_{S_1} g(r)^2 z_1^2 dx} = \frac{A}{c}(\lambda_2(S_1) - \lambda_1(S_1))$$

which simplifies to the ratio inequality

$$\frac{\lambda_2}{\lambda_1}(\Omega) \leq 1 + \frac{AC}{ac}\left(\frac{\lambda_2}{\lambda_1}(S_1) - 1\right)$$

which is the result stated in Theorem 2.2.

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