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ASYMPTOTIC STABILITY OF NONLINEAR CONTROL SYSTEMS DESCRIBED BY DIFFERENCE EQUATIONS WITH MULTIPLE DELAYS

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Abstract. In this paper we study nonlinear control systems with multiple delays on controls and states. To obtain asymptotic stability, we impose Hölder-type assumptions on the perturbing function, and show a Gronwall-type inequality for difference equations with delay. We prove that a nonlinear control system can be stabilized if its linear control system can be stabilized. Some examples are included in the last part of this paper.

1. Introduction

Consider a nonlinear control system described by discrete-time equations, with multiple delays on the controls and states, of the form

$$x(k+1) = L_{p,q}(x_k, u_k) + f_{p,q}(k, x_k, u_k), \quad k \in \mathbb{Z}^+, \tag{1}$$

where

$$L_{p,q}(x_k, u_k) = \sum_{j=1}^{p} A_j(k)x(k-p_j) + \sum_{i=1}^{q} B_i(k)u(k-q_i),$$

$$f_{p,q}(k, x_k, u_k) = f(k, x(k-p_1), x(k-p_2), \dots, x(k-p_p), u(k-q_1), .., u(k-q_q)),$$

$\mathbb{Z}^+ := \{0, 1, 2, \dots\}$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ with $n \geq m$, $A_j(k)$ and $B_i(k)$ are $n \times n$ and $n \times m$ matrices with $k \in \mathbb{Z}^+$, $f(k, .) : \mathbb{Z}^+ \times \mathbb{R}^{pn} \times \mathbb{R}^{qm} \to \mathbb{R}^n$ with $p, q \geq 1$, $q_q \leq p_p$, $0 = p_1 < p_2 < \dots < p_p$, $0 = q_1 < q_2 < \dots < q_q$.

We shall consider system (1) with the initial delay condition

$$x(k) = x_0, \quad k = -p_p, \dots, 0. \tag{2}$$

Unlike differential equations, discrete control system (1) with initial condition (2) has always solution for every control sequence $u(k)$, $k = -q_q, -q_q + 1, \dots, 0, 1, \dots$ Throughout this paper, we assume that $f(k, 0, \dots, 0) = 0$, $k \in \mathbb{Z}^+$. Associated with control system (1) we consider the delay system without controls

$$x(k+1) = \sum_{j=1}^{r} C_j(k)x(k-r_j) + g(k, x(k-r_1), x(k-r_2), \dots, x(k-r_r)), \tag{3}$$

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where $k \geq 0$, $C_j(k)$ is an $n \times n$ matrix, $r \geq 1$, $0 = r_1 < \cdots < r_r$, $g(k,.) : \mathbb{Z}^+ \times \mathbb{R}^{rn} \to \mathbb{R}^n$ is a given vector function satisfying $g(k,0,\dots,0) = 0$, $k \in \mathbb{Z}^+$. It has been shown in [3, 10] that for every $x_0 \in \mathbb{R}^n, k_0 \in \mathbb{Z}^+$, the system (3) has a solution $x(k)$ with the initial condition $x(k) = x_0$, $k = k_0 - r_r, \dots, k_0$ given by

$$x(k) = P_k x_0 + \sum_{s=k_0}^{k-1} G_{s+1}^k g(s, x(s-r_1), \dots, x(s-r_r)), \tag{4}$$

where the transition matrix $G_s^k, k, s \geq k_0$, satisfies

$$G_s^{k+1} = \sum_{i=1}^{r} C_i(k) G_s^{k-r_i},$$
$$G_k^k = I, \quad G_s^k = 0, \quad \text{for } k < s\,.$$

and

$$P_k := G_0^k + \sum_{i=2}^{r} \sum_{s=0}^{r_i-1} G_{s+1}^k C_i(s)\,. \tag{5}$$

Definition 1.1. The zero solution of system (3) is stable if for every $\epsilon > 0$ and for every $k_0 \in \mathbb{Z}^+$ there is $\delta > 0$ (depending on ϵ and k_0) such that $\|x(k)\| < \epsilon, k \geq k_0$, whenever $\|x_0\| < \delta$. The zero solution is asymptotically stable if it is stable and there is $\delta > 0$ such that $\lim_{k\to\infty} \|x(k)\| = 0$, whenever $\|x_0\| < \delta$. If δ is independent of k_0 then the zero solution is said to be uniformly stable and uniformly asymptotically stable.

Definition 1.2. The zero solution of system (3) is weakly asymptotically stable if there is a number $\delta > 0$ such that every solution of the system satisfies $\lim_{k\to\infty} \|x(k)\| = 0$, whenever $\|x_0\| < \delta$.

Definition 1.3. The control system (1) is stabilizable if there are matrices $D(k)$, $k \geq -q_q$, such that the system (1) with $u(k) = D(k)x(k)$ is asymptotically stable. The control $u(k) = D(k)x(k)$ is a feedback control of the system.

Definition 1.4. The system (1) is weakly stabilizable if there exist controls $u(k)$, $k \geq -q_q$ and a number $\delta > 0$, such that the solution $x(k)$ according to these controls of system (1) satisfies $\lim_{k\to\infty} \|x(k)\| = 0$, whenever $\|x_0\| < \delta$.

Qualitative theory of dynamical systems described by the difference equations has attracted a good deal of interest in the last decade due to the various applications of their qualitative properties [3, 6, 7, 10, 15]. Consequently, the stability, which is one of the essential qualitative properties, has been widely studied for discrete-time equations; see, for example, [4, 8, 9, 11, 16]. Most publications considered the stability of nonlinear discrete systems, whereas the stability and applications of nonlinear discrete systems with delays have received little attention. In [4, 5, 13, 17] sufficient conditions for the controllability and stability of discrete systems with time-delays are developed. In this paper, we extend the system of discrete delay inequalities of Gronwall type, then study the asymptotic stability of nonlinear system (3) with multiple delays, and then give new stabilizability conditions for nonlinear control system (1).

This paper is organized as follows. Section 2 gives the Gronwall-type inequality for difference equations with delays. In Section 3, we establish asymptotic stability conditions. In Section 4, we give new conditions for stabilizability for nonlinear control systems with multiple delays on controls and states. The paper concludes some illustrative examples.

2. Discrete Gronwall-type inequality

It is well known that the Gronwall integral inequality plays an important role in the study of qualitative properties of differential systems of various kinds. The classical integral Gronwall's inequality claims that if $z(t), a(t) : \mathbb{R}^+ \to \mathbb{R}^+$ are non-negative continuous functions satisfying $z(0) \le C$, and $z(t) \le C + \int_0^t a(s)z(s)\,ds$, $t \ge 0$, then

$$z(t) \le C \exp[\int_0^t a(s)\,ds], \quad t \ge 0.$$

The first discrete analog of the integral Gronwall inequality (see, e.g. [1] and references therein) is that if $z(k), a(k) : \mathbb{Z}^+ \to \mathbb{R}^+$, $C \ge 0$ satisfy the condition $z(0) \le C$ and

$$z(k) \le C + \sum_{i=0}^{k-1} a(i)z(i),$$

then

$$z(k) \le C \exp \sum_{i=0}^{k-1} a(i), \quad \text{or} \quad z(k) \le C \prod_{i=0}^{k-1} [1 + a(i)].$$

Later, in [8, 12], the Gronwall-type inequality was extended to the system of discrete equations of the form

$$z(k) \le C + \sum_{i=0}^{k-1} a(i)z^m(i).$$

where m is an arbitrary positive number. Some other discrete versions of the Gronwall inequality can be found in [7, 13]. In this section, we present some discrete versions of the Gronwall-type inequality that will be used in studying the stability properties of nonlinear delay systems. We first need the following technical lemma.

Lemma 2.1. *Let $a \ge 0$, $x \ge 0$. Then $(1 + x)^a(1 - ax) \le 1$.*

Proof. Consider the continuous function $f(x) = (1+x)^a(1-ax)$. Note that $f(0) = 1$ and

$$\begin{aligned}\frac{d}{dx} f(x) &= a(1 + x)^{a-1}(1 - ax) - a(1 + x)^a \\ &= -ax(1 + a)(1 + x)^{a-1} \le 0,\end{aligned}$$

which implies that $f(x)$ is decreasing and hence $f(x) \le 1$, for all $x \ge 0$.

Theorem 2.1 (Generalized discrete Gronwall's inequality). *Let $z(k) : \mathbb{Z}^+ \to \mathbb{R}^+$. Assume that*

$$z(k) \le C + \sum_{s=0}^{k-1} \sum_{j=1}^{p} a_j(s)z(s - p_j)^{m_1} + \sum_{s=0}^{k-1} \sum_{i=1}^{q} b_i(s)z(s - q_i)^{m_2}, \tag{6}$$

where $m_1, m_2 > 0$; $p, q \geq 1$; $p_p \geq q_q$, $a_j(k), b_i(k) : \mathbb{Z}^+ \to \mathbb{R}^+$; $z(k) \leq C \leq 1$, $k = -p_p, \dots, 0$ *and* $0 = p_1 < p_2 < \cdots < p_p$; $0 = q_1 < q_2 < \cdots < q_q$. *Let* $m = \min\{m_1, m_2\}$, $d(s) = \sum_{j=1}^{p} a_j(s) + \sum_{i=1}^{q} b_i(s)$.

(a) If $m_1, m_2 \leq 1$, *then*

$$z(k) \leq C^{m^k} \prod_{s=0}^{k-1} [1 + d(s)]. \tag{7}$$

(b) If $m_1 \leq 1 < m_2$, *then*

$$z(k) \leq C^{m_1^k} \prod_{s=0}^{k-1} [1 + d(s)]^{m_2^{k-s-1}}. \tag{8}$$

(c) If $m_1, m_2 > 1$, *then*

$$z(k) \leq \frac{C}{\{1 - (m-1)C^{m-1} \sum_{s=0}^{k-1} d(s)\}^{1/(m-1)}}, \tag{9}$$

whenever

$$1 - (m-1)C^{m-1} \sum_{s=0}^{k-1} d(s) > 0. \tag{10}$$

Proof. (a) Case $m_1, m_2 \leq 1$: We shall prove the theorem by induction on $k \in \mathbb{Z}^+$. Letting $k = 1$, the inequality (6) gives

$$z(1) \leq C + \sum_{j=1}^{p} a_j(0) C^{m_1} + \sum_{i=1}^{q} b_i(0) C^{m_2}$$

Since $C \leq 1$, $m_i \geq m$, $C^{m_i} \leq C^m$, $i = 1, 2$, we have

$$z(1) \leq C^m + d(0)C^m \leq C^m[1 + d(0)],$$

which implies (7) for $k = 1$. Let us assume that (7) holds for $1, 2, \dots, k-1$. Using (6) for the step k we have

$$\begin{aligned} z(k) \leq& C + \sum_{s=0}^{k-2} \sum_{j=1}^{p} a_j(s) z(s - p_j)^{m_1} + \sum_{s=0}^{k-2} \sum_{i=1}^{q} b_i(s) z(s - q_i)^{m_2} \\ &+ \sum_{j=1}^{p} a_j(k-1) z(k-1-p_j)^{m_1} + \sum_{i=1}^{q} b_i(k-1) z(k-1-q_i)^{m_2}. \end{aligned}$$

By the induction assumption, we see that

$$\begin{aligned} z(k) \leq& C^{m^{k-1}} \prod_{s=0}^{k-2} [1 + d(s)] + \sum_{j=1}^{p} a_j(k-1) \{C^{m^{k-1-p_j}} \prod_{s=0}^{k-2-p_j} [1 + d(s)]\}^{m_1} \\ &+ \sum_{i=1}^{q} b_i(k-1) \{C^{m^{k-1-q_i}} \prod_{s=0}^{k-2-q_i} [1 + d(s)]\}^{m_2}. \end{aligned}$$

Moreover, since $C \le 1$, $m \le 1$, $m_i \ge m$, $i = 1, 2$, the following inequalities hold

$$
\begin{gathered}
C^{m^{k-1}} \le C^{m^k}, \quad [1+d(s)]^{m_i} \le [1+d(s)], \quad i = 1,2, \\
C^{m^{k-1-q_i}.m_2} \le C^{m^k}; C^{m^{k-1-p_j}.m_1} \le C^{m^k}, \quad j = 1,2,\dots,p, \quad i = 1,..,q, \\
\prod_{s=0}^{k-2-p_j} [1+d(s)]^{m_1} \le \prod_{s=0}^{k-2} [1+d(s)], \quad j = 1,2,\dots,p, \\
\prod_{s=0}^{k-2-q_i} [1+d(s)]^{m_2} \le \prod_{s=0}^{k-2} [1+d(s)], \quad i = 1,2,\dots,q\,.
\end{gathered}
$$

Therefore,

$$
z(k) \le C^{m^k} \prod_{s=0}^{k-2} [1+d(s)]\{1+d(k-1)\} = C^{m^k} \prod_{s=0}^{k-1} [1+d(s)]
$$

which implies that (7) holds for the step k.

b) Case $m_1 \le 1 < m_2$: It is easy to verify (8) for $k = 1$. Assume that (8) holds for the steps $1, 2, \dots, k-1$. Using (6) for the step k and by the induction assumption, we have

$$
\begin{aligned}
z(k) \le & C^{m_1^{k-1}} \prod_{s=0}^{k-2} [1+d(s)]^{m_2^{k-s-2}} \\
& + \sum_{j=1}^{p} a_j(k-1)\{C^{m_1^{k-1-p_j}} \prod_{s=0}^{k-2-p_j} [1+d(s)]^{m_2^{k-p_j-s-2}}\}^{m_1} \\
& + \sum_{i=1}^{q} b_i(k-1)\{C^{m_1^{k-1-q_i}} \prod_{s=0}^{k-2-q_i} [1+d(s)]^{m_2^{k-q_i-s-2}}\}^{m_2}\,.
\end{aligned}
$$

Similarly to Case a), we see that

$$
\begin{gathered}
C^{m_1.m_1^{k-p_j-1}} \le C^{m_1^k}, \quad C^{m_2.m_1^{k-q_i-1}} \le C^{m_1^k}, \quad j = 1,\dots,p, i = 1,\dots,q, \\
[1+d(s)]^{m_1.m_2^{k-s-2-p_j}} \le [1+d(s)]^{m_2^{k-s-2}}, \\
[1+d(s)]^{m_2.m_2^{k-s-2-q_i}} \le [1+d(s)]^{m_2^{k-s-2}}.
\end{gathered}
$$

Therefore,

$$
\begin{aligned}
z(k) \le & C^{m_1^k} \prod_{s=0}^{k-2} [1+d(s)]^{m_2^{k-s-2}} + \sum_{j=1}^{p} a_j(k-1)\{C^{m_1^k} \prod_{s=0}^{k-2} [1+d(s)]^{m_2^{k-s-2}}\}^{m_1} \\
& + \sum_{i=1}^{q} b_j(k-1)\{C^{m_1^k} \prod_{s=0}^{k-2} [1+d(s)]^{m_2^{k-s-2}}\}^{m_2} \\
= & C^{m_1^k} \prod_{s=0}^{k-2} [1+d(s)]^{m_2^{k-s-2}} [1+d(k-1)] \\
= & C^{m_1^k} \prod_{s=0}^{k-1} [1+d(s)]^{m_2^{k-s-1}},
\end{aligned}
$$

which implies (8) for the step k.
(b) Case $m_1, m_2 > 1$: Using (6) for $k = 1$, we have

$$z(1) \le C + \sum_{j=1}^{p} a_j(0)C^{m_1} + \sum_{j=1}^{q} b_i(0)C^{m_2}$$

Since $C \le 1$, $m_i \ge m$, $C^{m_i} \le C^m$, $i = 1, 2$, we see that

$$z(1) \le C + d(0)C^m = C[1 + d(0)C^{m-1}],$$

where $m = \min\{m_1, m_2\}$. Applying Lemma 2.1 for $x = d(0)C^{m-1}$, $a = m - 1$, we obtain

$$[1 + d(0)C^{m-1}]^{m-1}[1 - (m-1)d(0)C^{m-1}] \le 1.$$

Therefore,

$$z(1) \le \frac{C}{\{1 - (m-1)d(0)C^{m-1}\}^{1/(m-1)}},$$

whenever $1 - (m-1)d(0)C^{m-1} > 0$, which implies (9) for $k = 1$. Suppose that the assertion holds for $1, 2, \dots, k-1$. We shall prove (9) for the step k, provided the condition (10). To see this, we consider the inequality at the step k, and by the induction assumptions we see that

$$z(k) \le D_{k-2} + \sum_{j=1}^{p} a_j(k-1)D^{m_1}_{k-2-p_j} + \sum_{i=1}^{q} b_i(k-1)D^{m_2}_{k-2-q_i},$$

where

$$D_l := \frac{C}{[1 - (m-1)C^{m-1}\sum_{s=0}^{l} d(s)]^{1/(m-1)}}\,.$$

Since $C^{m_i} \le C^m$, $i = 1, 2$, we have

$$[1 - (m-1)\sum_{s=0}^{k-2-p_j} d(s)]^{\frac{m_1}{m-1}} \ge [1 - (m-1)\sum_{s=0}^{k-2} d(s)]^{\frac{m}{m-1}}, \quad j = 1, \dots, p,$$
$$[1 - (m-1)\sum_{s=0}^{k-2-q_i} d(s)]^{\frac{m_2}{m-1}} \ge [1 - (m-1)\sum_{s=0}^{k-2} d(s)]^{\frac{m}{m-1}}, \quad i = 1, \dots, q,$$

and so

$$D^{m_1}_{k-2-p_i} \le D^m_{k-2}, i = 1, 2, \dots, p, \quad D^{m_2}_{k-2-q_j} \le D^m_{k-2}, j = 1, 2, \dots, q\,.$$

Therefore,

$$z(k) \le D_{k-2} + d(k-1)D^m_{k-2} = D_{k-2}[1 + d(k-1)D^{m-1}_{k-2}]\,.$$

Applying Lemma 2.1 for $x = d(k-1)D^{m-1}_{k-2}, a = (m-1)$, we obtain

$$z(k) \le \frac{D_{k-2}}{[1 - (m-1)d(k-1)D^{m-1}_{k-2}]^{1/m-1}},$$

whenever

$$1-(m-1)d(k-1)D_{k-2}^{m-1}>0\,. \tag{11}$$

It is easy to verify that the condition (11) is satisfied due to the condition (10). On the other hand, it is obvious that

$$\frac{D_{k-2}}{[1-(m-1)d(k-1)D_{k-2}^{m-1}]^{1/m-1}}=\frac{C}{[1-(m-1)C^{m-1}\sum_{s=0}^{k-1}d(s)]^{1/m-1}}\,,$$

and hence

$$z(k)\leq\frac{C}{[1-(m-1)C^{m-1}\sum_{s=0}^{k-1}d(s)]^{1/m-1}}\,,$$

whenever (10) holds. Then the present proof is complete

Theorem 2.1 has a corollary when $b_i(k)=0$, which will be used in obtaining asymptotic stability conditions of nonlinear system (3) in the next section.

Corollary 2.1. *Let $z(k):\mathbb{Z}^+\to\mathbb{R}^+$. Assume that*

$$z(k)\leq C+\sum_{s=0}^{k-1}\sum_{j=1}^{p}a_j(s)z(s-p_j)^m\,,$$

where $m>0$; $p\geq 1$; $a_j(k):\mathbb{Z}^+\to\mathbb{R}^+$; $z(k)\leq C\leq 1$, $k=-p_p,\dots,0$.
(a) If $m\leq 1$, then

$$z(k)\leq C^{m^k}\prod_{s=0}^{k-1}[1+\sum_{j=1}^{p}a_j(s)].$$

(b) If $m>1$, then

$$z(k)\leq\frac{C}{\{1-(m-1)C^{m-1}\sum_{s=0}^{k-1}\sum_{j=1}^{p}a_j(s)\}^{1/(m-1)}},$$

whenever

$$1-(m-1)C^{m-1}\sum_{s=0}^{k-1}\sum_{j=1}^{p}a_j(s)>0\,.$$

3. Stability results

In this section we present sufficient conditions for the asymptotic stability of system (3) without controls. Let $x(k)$ be a solution of system (3) with the initial condition $x(k)=x_0$, $k=-r_r,\dots,0$, given by (4), (5), where for simplicity we assume that $k_0=0$. We first need the following lemma.

Lemma 3.1. *Assume that there exist numbers $K>0$, $w\in(0,1)$ such that*

$$\|G_s^k\|\leq Kw^{k-s},\quad\forall\,k>s\geq 0. \tag{12}$$

Then there is a number $K_1>0$ such that $\|P_k\|\leq K_1w^k$, $k\in\mathbb{Z}^+$.

Proof. Let

$$M=\max\{\|C_i(k)\|,\ k=0,1,\dots,r_i-1,\ i=2,\dots,r\}.$$

We have

$$\|P_k\| \le \|G_0^k\| + \sum_{s=0}^{r_2-1} \|G_{s+1}^k\|\|C_2(s)\| + \cdots + \sum_{s=0}^{r_r-1} \|G_{s+1}^k\|\|C_r(s)\| .$$

Since for all $i = 2, \dots, r$,

$$\begin{aligned}\sum_{s=0}^{r_j-1} \|G_{s+1}^k\|\|C_i(s)\| \le & MK \sum_{s=0}^{r_j-1} w^{k-s-1} = MKw^{k-r_j}(1 + \cdots + w^{r_j-1}) \\ \le & \frac{MKw^{k-r_j}}{1-w}, \quad j = 2, \dots, r .\end{aligned}$$

and since $w^{k-r_j} \le w^{k-r_r}$, we obtain

$$\|P_k\| \le Kw^k + MKw^{k-r_r}\frac{r-1}{1-w} \le Kw^k + \frac{MK(r-1)}{w^{r_r}(1-w)}w^k .$$

Therefore, $\|P_k\| \le K_1 w^k$, for all $k \in \mathbb{Z}^+$, where

$$K_1 = K + \frac{MK(r-1)}{w^{r_r}(1-w)}. \tag{13}$$

It is worth to note that condition (12) is a sufficient condition for the asymptotic stability of linear discrete-time delay systems of the form

$$x(k+1) = \sum_{s=1}^{k-1}\sum_{i=1}^{r} C_i(s)x(s-r_i), \quad k \in \mathbb{Z}^+, \tag{14}$$

since any solution $x(k)$ of linear system (14) with the initial condition $x(k) = x_0$, $k = -r_r, \dots, 0$, is given by $x(k) = P_k x_0$, where P_k is defined by (5).

Theorem 3.1. *Assume the condition (12) and suppose that*

$$\|g(k, x_1, \dots, x_r)\| \le \sum_{j=1}^{r} a_j(k)\|x_i\|^m,$$

where $m > 0$, $a_j(k) : \mathbb{Z}^+ \to \mathbb{R}^+$.
(i) If $m < 1$, and

$$\overline{\lim_{k\to\infty}} \sum_{j=1}^{r} \frac{a_j(k)}{w^{k(1-m)}} = 0,$$

then the system (3) is weakly asymptotically stable. If $m = 1$ then the system is uniformly asymptotically stable if $\overline{\lim}_{k\to\infty} \sum_{j=1}^{r} a_j(k) = 0$.
(ii) If $m > 1$, and

$$\sum_{k=0}^{\infty}\sum_{j=1}^{r} w^{k(m-1)}a_j(k) < +\infty,$$

then the system (3) is uniformly asymptotically stable.

Proof. By Lemma 3.1 we obtain that

$$\|P_k\| \le K_1 w^k, \quad k \in \mathbb{Z}^+, \tag{15}$$

where K_1 is defined by (13).
(i) Case $m < 1$: Let $x(k)$ be any solution of the system (3) given by (4). Taking (12) and (15) into account, the following estimate holds

$$\|x(k)\| \le K_1 w^k \|x_0\| + \sum_{s=0}^{k-1} K w^{k-s-1} \sum_{j=1}^{r} a_j(s) \|x(s-r_j)\|^m, \quad k \in \mathbb{Z}^+.$$

Multiplying both sides of the above inequality with w^{-k} and setting

$$z(k) = w^{-k}\|x(k)\|, \quad \bar{a}_j(k) = K w^{k(m-1)-1-mr_j} a_j(k),$$

we obtain

$$z(k) \le K_1 \|x_0\| + \sum_{s=0}^{k-1} \sum_{j=1}^{r} \bar{a}_j(s) z(s-r_j)^m. \tag{16}$$

Let $\delta > 0$ be a chosen number such that $\|x(0)\| < \delta$ and $K_1\|x(0)\| \le 1$. By Corollary 2.1, we have

$$z(k) \le C^{m^k} \prod_{s=0}^{k-1} [1 + \sum_{j=1}^{r} \bar{a}_j(s)], \quad k \in \mathbb{Z}^+,$$

where $C = K_1\|x_0\|$. Therefore,

$$\begin{aligned} \|x(k)\| \le& C^{m^k} w^k \prod_{s=0}^{k-1} [1 + K w^{s(m-1)-1-mr_j} \sum_{j=1}^{r} a_j(s)] \\ \le& (K_1\|x_0\|)^{m^k} \prod_{s=0}^{k-1} [w + K w^{s(m-1)-mr_j} \sum_{j=1}^{r} a_j(s)]. \end{aligned}$$

On the other hand, by the assumption (i), there are numbers $N > 0$, $l \in (0, 1-w)$ such that

$$K w^{k(m-1)-mr_j} \sum_{j=1}^{r} a_j(k) \le l < 1 - w, \quad \forall\, k \ge N\,.$$

Consequently, for all $k \ge N$ we have

$$K_1 w^{k(m-1)-mr_j} \sum_{j=1}^{r} a_j(k) + w < l + w = v < 1,$$

and hence there exists a number $M > 0$, such that for all $k \ge N$ we have

$$\|x(k)\| \le M v^{k-N},$$

which implies that $\lim_{k\to\infty}\|x(k)\| = 0$, whenever $\|x(0)\| < \delta$, i.e., the zero solution is weakly asymptotically stable. For the case $m = 1$, as before, we obtain the following estimate

$$\|x(k)\| \le Cw^k \prod_{s=0}^{k-1}[1 + K\sum_{j=1}^{p}(s)], \quad k \in \mathbb{Z}^+.$$

Therefore,

$$\|x(k)\| \le K_1\|x_0\|v^k, \quad k \in \mathbb{Z}^+$$

which implies uniform asymptotic stability of the system.

(ii) Case $m > 1$. By the same arguments used in case (i) we have arrived at the inequality (16), where $C := K_1\|x_0\|$, $\bar{a}_j(k) = Kw^{k(m-1)-1-mr_j}a_j(k)$, $m > 1$. Using Corollary 2.1 again, we have

$$z(k) \le \frac{C}{\{1-(m-1)C^{m-1}\sum_{s=0}^{k-1}\sum_{j=1}^{r}\bar{a}_j(s)\}^{\frac{1}{m-1}}},$$

whenever

$$1-(m-1)C^{m-1}\sum_{s=0}^{k-1}\sum_{i=1}^{r}\bar{a}_i(s) > 0, \quad k \in \mathbb{Z}^+. \tag{17}$$

Let $l \in (0, 1)$ be an arbitrary number. We shall show that the condition (16) holds for all x_0 satisfying

$$\|x_0\| \le \{\frac{l}{(m-1)K_1^{m-1}\gamma}\}^{\frac{1}{m-1}} := R,$$

where $\gamma := \sum_{k=0}^{\infty}\sum_{j=1}^{r}\bar{a}_j(k)$, due to the assumption (ii), is finite. Indeed, for all that x_0, we have

$$(m-1)K_1^{m-1}\|x_0\|^{m-1}\sum_{s=0}^{k-1}\sum_{j=1}^{r}\bar{a}_j(s) \le (m-1)K_1^{m-1}\gamma\|x_0\|^{m-1} \le l,$$

and we obtain

$$1-(m-1)K_1^{m-1}\|x_0\|^{m-1}\sum_{s=0}^{k-1}\sum_{i=1}^{r}\bar{a}_i(s) \ge 1-l > 0\,,$$

as desired. Therefore,

$$\|x(k)\| \le K_2w^k\|x_0\|, \quad k \in \mathbb{Z}^+\,,$$

where

$$K_2 = \frac{K_1}{(1-l)^{1/m-1}}.$$

The last inequality shows that for any $\epsilon > 0$, we can choose a suitable number $0 < \delta < \min\{R, \epsilon/K_2\}$ and a number $N > 0$ such that $\|x(k)\| < \epsilon$, for all $k > N$, whenever $\|x_0\| < \delta$, which implies the uniform asymptotic stability of the zero solution of system (3). The proof is complete.

4. Stabilizability results

We first consider the nonlinear control system (1), where $B_i(k) = 0$,

$$x(k+1) = \sum_{j=1}^{p} A_j(k)x(k-p_j) + f_{p,q}(k, x_k, u_k), \quad k \in \mathbb{Z}^+. \tag{18}$$

In the sequel we assume that $\exists a_j(k), b_i(k) : \mathbb{Z}^+ \to \mathbb{R}^+$ such that

$$\|f(k, x_1, \dots, x_p, u_1, \dots, u_r)\| \le \sum_{j=1}^{p} a_j(k)\|x_i\|^{m_1} + \sum_{i=1}^{q} b_i(k)\|u_i\|^{m_2}, \tag{19}$$

where $m_1, m_2 > 0$, $p, q \ge 1$. Associated with the condition (12) we consider the condition

$$\exists K > 0,\ w \in (0,1) : \ \|G_s^k\| \le Kw^{\sum_{i=s}^{k-1} m_2^i}, \tag{20}$$

Let

$$m_2(k,s) = \sum_{t=s}^{k-1} m_2^t\,,$$

$$\begin{aligned} l_{p_j}(k) &= w^{m_1.m_2(k-p_j,0)-m_2(k+1,0)}, \\ l_{q_i}(k) &= w^{m_2.m_2(k-q_i,0)-m_2(k+1,0)}\,. \end{aligned} \tag{21}$$

Theorem 4.1. *Assume that the conditions (12) and (19) are satisfied. Moreover, suppose that there are $(n \times m)$ matrices $D(k)$, $k \ge -q_q$, such that*
(i) if $m_1, m_2 \le 1$, and

$$\overline{\lim_{k\to\infty}}\Big[\sum_{j=1}^{p} \frac{a_j(k)}{w^{k(1-m_1)}} + \sum_{i=1}^{q} \frac{b_i(k)\|D(k-q_i)\|^{m_2}}{w^{k(1-m_2)}}\Big] = 0\,, \tag{22}$$

then the system (18) is weakly stabilizable.
(ii) If $m_1 \le 1 < m_2$, and we assume the condition (20) instead of (12), then the system (18) is weakly stabilizable whenever

$$\overline{\lim_{k\to\infty}}\Big[\sum_{j=1}^{p} w^{l_{p_j}(k)} a_j(k) + \sum_{i=1}^{q} w^{l_{q_i}(k)} b_i(k)\|D(k-q_i)\|^{m_2}\Big] = 0\,. \tag{23}$$

(iii) If $m_1, m_2 > 1$, and

$$\sum_{k=0}^{\infty}\Big\{\sum_{j=1}^{p} w^{k(m_1-1)} a_j(k) + \sum_{i=1}^{q} w^{k(m_2-1)} b_i(k)\|D(k-q_i)\|^{m_2}\Big\} < +\infty\,, \tag{24}$$

then the system (18) is stabilizable by feedback control $u(k) = D(k)x(k)$.

Proof. Taking $\delta > 0$ such that $\|x_0\| < \delta$, $K_1\|x_0\| \le 1$, where K_1 is defined by (13) and, as in the proof of Theorem 3.1, we arrived at the estimate

$$\|x(k)\| \le K_1 w^k \|x_0\| + \sum_{s=0}^{k-1} K w^{k-s-1}\Big\{\sum_{j=1}^{p} a_j(s)\|x(s-p_j)\|^{m_1} + \sum_{i=1}^{q} b_i(s)\|u(s-q_i)\|^{m_2}\Big\}.$$

Setting $u(k) = D(k)x(k)$, $k \geq -q_q$, and by (19), and multiplying by w^{-k} we obtain

$$\begin{aligned} w^{-k}\|x(k)\| \leq & K_1\|x_0\| + \sum_{s=0}^{k-1} Kw^{-s-1}\{\sum_{j=1}^{p} a_j(s)\|x(s-p_j)\|^{m_1} + \\ & + \sum_{i=1}^{q} Kw^{-s-1} b_i(s)\|D(s-q_i)\|^{m_2}\|x(s-q_i)\|^{m_2}\}\,. \end{aligned}$$

Let

$$\begin{aligned} \bar{a}_j(k) &= Kw^{k(m_1-1)-1-m_1p_j} a_j(k), \\ \bar{b}_i(k) &= Kw^{s(m_2-1)-1-m_2q_i} b_i(k)\|D(k-q_i)\|^{m_2}, \\ z(k) &= w^{-k}\|x(k)\|, \quad C = K_1\|x_0\|, \\ d(s) &= \sum_{j=1}^{p} \bar{a}_j(s) + \sum_{i=1}^{q} \bar{b}_i(s)\,. \end{aligned}$$

We have

$$\|z(k)\| \leq C + \sum_{s=0}^{k-1}\sum_{j=1}^{p} \bar{a}_j(s)\|z(s-p_j)\|^{m_1} + \sum_{s=0}^{k-1}\sum_{i=1}^{q} \bar{b}_i(s)\|z(s-q_i)\|^{m_2}. \tag{25}$$

(i) Case $m_1, m_2 \leq 1$: Applying Theorem 2.1.(a) to the inequality (25), we have

$$\|z(k)\| \leq C^{m^k} \prod_{s=0}^{k-1}[1 + d(s)]\,,$$

and hence

$$\begin{aligned} \|x(k)\| \leq & (K_1\|x_0\|)^{m^k} \prod_{s=0}^{k-1}\{w + \sum_{j=1}^{p} Kw^{s(m_1-1)-m_1p_j} a_j(s) \\ & \sum_{i=1}^{q} Kw^{s(m_2-1)-m_2q_i} b_i(s)\|D(s-q_i)\|^{m_2}\}\,. \end{aligned}$$

Since $w < 1$ and by the assumption (i), there exist a number $l < 1 - w$ and an integer $N > 0$ such that for all $s \geq N$

$$\sum_{j=1}^{p} Kw^{s(m_1-1)-m_1p_j} a_j(s) + \sum_{i=1}^{q} Kw^{s(m_2-1)-m_2q_i} b_i(s)\|D(s-q_i)\|^{m_2}] \leq l.$$

Therefore, there is a number $M > 0$ such that for all $k \geq N$, we have

$$\|x(k)\| \leq Mv^{k-N}, \quad k \in \mathbb{Z}^+,$$

where $l + w = v < 1$, which means that the system is weakly stabilizable by the feedback control $u(k) = D(k)x(k)$.

(ii) Case $m_1 \le 1 < m_2$: Using the assumption (20) and by the same arguments that used in the proof of Lemma 3.1 we can find some number $K_2 > 0$ such that

$$\|P_k\| \le K_2 w^{m_2(k,0)}, \quad k \in \mathbb{Z}^+.$$

Let

$$\begin{aligned}
\bar{a}_j(k) &= K a_j(k) w^{m_1.m_2(k-p_j,0)-m_2(k+1,0)},\\
\bar{b}_i(k) &= K b_i(k) w^{m_2.m_2(k-q_i,0)-m_2(k+1,0)},\\
z(k) &= w^{-m_2(k,0)}\|x(k)\|, \quad C = K_2\|x_0\|,\\
\bar{d}(s) &= \sum_{j=1}^{p} \bar{a}_j(s) + \sum_{i=1}^{q} \bar{b}_i(s)\,.
\end{aligned}$$

Similarly, we obtain the estimate (25) and hence, applying Theorem 2.1 to the case $m_1 \le 1 < m_2$, we have

$$\|z(k)\| \le C^{m_1^k} \prod_{s=0}^{k-1} [1+\bar{d}(s)]^{m_2^{k-s-1}}.$$

Therefore,

$$\|x(k)\| \le (K_2\|x_0\|)^{m_1^k} \prod_{s=0}^{k-1} \{w + \sum_{j=1}^{p} K a_j(s) w^{l_{p_j}(s)+1} + \sum_{i=1}^{q} K w^{l_{q_i}(s)+1}\}^{m_2^{k-s-1}},$$

where $l_{p_j}(s)$, $l_{q_i}(s)$ are defined by (21). Then the proof is complete as in Case (i) above.

(iii) Case $m_1, m_2 > 1$: Taking (25) into account and applying Theorem 2.1 for $m > 1$, we have

$$z(k) \le \frac{C}{\{1-(m-1)C^{m-1}\sum_{s=0}^{k-1} d(s)\}^{1/(m-1)}},$$

whenever $1-(m-1)C^{m-1}\sum_{s=0}^{k-1} d(s) > 0$. Therefore, by the same arguments that used in the proof of Theorem 3.1 for the case $m > 1$, there exist numbers $\delta > 0$, $K_2 > 0$ such that

$$\|x(k)\| \le K_2\|x_0\| w^k, \quad \forall k \in \mathbb{Z}^+,$$

whenever $\|x_0\| < \delta$. This inequality implies the stabilizability of the system with the feedback control $u(k) = D(k)x(k)$.

We are now in position to give sufficient conditions for the stabilizability of nonlinear control system (1) based on Theorem 4.1. Let $D(k)$, $k \ge -q_q$ be arbitrary $(n \times m)$ matrices. Let $M = \{p_j, q_i :\ j = 2,3,\dots,p,\ i = 2,3,\dots,q\}$. For $r_1 = 0 = p_1 = q_1$, we set $C_1(k) = A_1(k) + B_1(k)D(k)$. Let $r_2 = \min M$, then there is some $j_1 \in \{2,\dots,p\}$ or $i_1 \in \{2,\dots,q\}$, such that $r_2 = p_{j_1}$ or $r_2 = q_{i_1}$. Without loss of generality we assume that $r_2 = p_{j_1}$ and we then set $C_2(k) = A_{j_1}(k)$. Denoting $M_{-1} = M \setminus j_1$, we choose $r_3 = \min M_{-1}$. Then there is some $i_2 \in \{2,\dots,p\} \setminus i_1$ or

$j_2 \in \{2, \ldots, q\}$ such that $r_3 = q_{i_2}$ or $r_3 = p_{j_2}$. Without loss of generality, we assume that $r_3 = q_{i_2}$. We set $C_3(k) = B_{i_2}(k)D(k - q_{i_2})$. Continuing the process, we can define the sequence $r_1, r_2, \ldots, r_r$ where $r = p + q - 1$ and $C_i(k)$, $i = 1, 2, \ldots, r$ are matrices. The system (1) with feedback control $u(k) = D(k)x(k)$, $k \geq -r_r$ is then reduced to the system (18) of the form

$$x(k+1) = \sum_{j=1}^{r} C_j(k)x(k - r_j) + f_{p,q}(k, x_k, D(k)x_k).$$

Let H_s^k be the transition matrix of the above system defined by

$$\begin{gathered} H_s^{k+1} = \sum_{j=1}^{r} C_j(k) H_s^{k-r_j}, \\ H_k^k = I, \\ H_s^k = 0 \text{ for } k < s\,. \end{gathered}$$

In the sequel we need the following assumptions

$$\exists K > 0,\ w \in (0,1) : \|H_s^k\| \leq K w^{k-s}, \quad \forall k > s \geq 0\,, \tag{26}$$

$$\exists K > 0,\ m_2 > 1,\ w \in (0,1) : \|H_s^k\| \leq K w^{m_2(k,s)}, \tag{27}$$

The theorem below is proved using the same arguments as in Theorem 4.1.

Theorem 4.2. *Assume that (19), (26) are satisfied. Suppose that there are $(n \times m)$ matrices $D(k)$, $k \geq -q_q$, such that if $m_1, m_2 \geq 1$ and (22) holds then the system (1) is weakly stabilizable. If $m_1 \leq 1 < m_2$, assume the condition (23) and the condition (27) instead of (26), then the system (1) is weakly stabilizable. If $m_1, m_2 > 1$ and we assume (24), then the system (1) is stabilizable by the feedback control $u(k) = D(k)x(k)$.*

Theorem 4.2 has a corollary which gives stabilizability conditions of nonlinear control system (1) via the stabilizability of its linear control system

$$x(k+1) = L_{p,q}(x_k, u_k), \quad k \in \mathbb{Z}^+. \tag{28}$$

Corollary 4.1. *Assume that (19),(26) are satisfied. Suppose that the linear control system (28) is stabilizable by some feedback control*

$$u(k) = D(k)x(k), \quad k \geq -q_q,$$

satisfying one of the conditions (22) - (24), then the nonlinear control system (1) is stabilizable by the same feedback control.

Example 4.1. Consider a control system in $\mathbb{R}^2$ of the form

$$\begin{aligned} x_1(k+1) &= \frac{1}{k+2}x_1(k) + 2^{-k}u^{1/3}(k), \quad k \in \mathbb{Z}^+ \\ x_2(k+1) &= -x_2(k) + \frac{1}{2^{k+2}}x_2(k-2) + ku(k) + 2^{-k}x_2^{1/3}(k-2), \end{aligned} \tag{29}$$

where $x_1(k), x_2(k), u(k) \in R$. The system (29) is of the form of control system (1), where

$$A_1(k) = \begin{pmatrix} \frac{1}{k+2} & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2(k) = \begin{pmatrix} 0 & 0 \\ 0 & 1/2^{k+2} \end{pmatrix}, \quad B(k) = [0, k]^T,$$
$$f(k, x(k), x(k-2), u(k)) = [2^{-k} u^{1/3}(k), 2^{-k} x_2^{1/3}(k-2)]^T.$$

We have $m_1 = 1/3$, $m_2 = 1/3$, $p = 2$, $p_2 = 2$, $q = 1$, and

$$\|f(k, x(k), x(k-2), u(k))\| \le 2^{-k} \|x_2(k-2)\|^{1/3} + 2^{-k} \|u(k)\|^{1/3}.$$

For the feedback control $u(k) = D(k)x(k)$ with $D(k) = (0, 1/k)$, we have

$$C_1(k) = A_1(k) + B_1(k)D(k) = \begin{pmatrix} \frac{1}{k+2} & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2(k) = A_2(k).$$

Therefore, it is easy to verify that the transition matrix G_s^k of the system (29) satisfies (13), where $K = 1$, $w = 1/2$. Also conditions (20), (22) of Theorem 4.2, where $a_1(k) = 0$, $a_2(k) = 2^{-k}$, $b_1(k) = 2^{-k}$, $m_1 = m_2 = 1/3$, hold for the above feedback control. Then system (29) is weakly stabilizable.

Example 4.2. Consider the control system

$$\begin{aligned} x_1(k+1) &= \frac{1}{k+2} x_1(k) + k^3 u^2(k), \quad k \in \mathbb{Z}^+ \\ x_2(k+1) &= -x_2(k) + \frac{1}{2^{k+2}} x_2(k-2) + ku(k) + \sin\frac{k\pi}{3} x_2^3(k-2)\,. \end{aligned} \tag{30}$$

Then we have $m_1 = 3$, $m_2 = 2$, $p = 2$, $p_2 = 2$, $q = 1$, and

$$f(k, x(k), x(k-2), u(k)) = [k^3 u^2(k), \sin\frac{k\pi}{3} x_2^3(k-2)]^T.$$

Therefore,

$$\|f(k, x(k), x(k-2), u(k))\| \le \|\sin\frac{k\pi}{3}\| \|x_2(k-2)\|^3 + k^3 \|u(k)\|^2.$$

Let us consider the same feedback control as in Example 4.1. It is easy to verify that (24) with $a_1(k) = 0$, $a_2(k) = \sin\frac{k\pi}{3}$, $b_1(k) = k^3$ holds for the above feedback control. Then system (30) is stabilizable.

Example 4.3. Consider the following control system

$$\begin{aligned} x_1(k+1) &= \frac{1}{2^{2^k}} x_1(k) + 2^{-k} x_2^{1/3}(k-2), \quad k \in \mathbb{Z}^+ \\ x_2(k+1) &= -x_2(k) + 2^{-\sum_{i=0}^k 2^i} x_2(k-2) + \sin\frac{k\pi}{3} u^2(k) + ku(k). \end{aligned} \tag{31}$$

Then we have $m_1 = 1/3$, $m_2 = 2$, $p = 2$, $p_2 = 2$, $q = 1$, and

$$A_1(k) = \begin{pmatrix} \frac{1}{2^{2^k}} & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2(k) = \begin{pmatrix} 0 & 0 \\ 0 & 2^{-\sum_{i=0}^k 2^i} \end{pmatrix}, \quad B(k) = [0, k]^T,$$
$$f(k, x(k), x(k-2), u(k)) = [2^{-k} x_2^{1/3}(k-2), \sin\frac{k\pi}{3} u^2(k)]^T.$$

Therefore,

$$\|f(k,x(k),x(k-2),u(k))\| \leq \|\sin\frac{k\pi}{3}\|\|u(k)\|^2 + 2^{-k}\|x_2(k-2)\|^{1/3}.$$

Let us consider the feedback control $D(k) = (0, 1/k)$. It is easy to verify that (27) holds for $K = 2$, $w = 1/2$, and the condition (23) with $a_1(k) = 0$, $a_2(k) = 2^{-k}$, $b_1(k) = \sin\frac{k\pi}{3}$ holds. Then system (31) is stabilizable.

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