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$C^{1,\alpha}$ convergence of minimizers of a Ginzburg-Landau functional *

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Abstract

In this article we study the minimizers of the functional

$$E_\varepsilon(u,G)=\frac{1}{p}\int_G|\nabla u|^p+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2,$$

on the class $W_g=\{v\in W^{1,p}(G,\mathbb{R}^2); v|_{\partial G}=g\}$, where $g:\partial G\to S^1$ is a smooth map with Brouwer degree zero, and p is greater than 2. In particular, we show that the minimizer converges to the p-harmonic map in $C^{1,\alpha}_{\rm loc}(G,\mathbb{R}^2)$ as ε approaches zero.

1 Introduction

Let $G\subset\mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G and g be a smooth map from ∂G into $S^1=\{x\in\mathbb{R}^2;|x|=1\}$. Consider the Ginzburg-Landau-type functional

$$E_\varepsilon(u,G)=\frac{1}{p}\int_G|\nabla u|^p+\frac{1}{4\varepsilon^p}\int_G(1-|u|^2)^2$$

with a small parameter $\varepsilon>0$. This functional has been studied in [1] for $p=2$, $d=\deg(g,\partial G)=0$, and in [2] for $p=2$, $d=\deg(g,\partial G)\neq 0$. Here $d=\deg(g,\partial G)$ denotes the Brouwer degree of the map g. For other related papers, we refer to [3]–[11].

In this paper we are concerned with the case $p>2$, $d=\deg(g,\partial G)=0$. It is easy to see that the functional $E_\varepsilon(u,G)$ achieves its minimum on

$$W_g=\{v\in W^{1,p}(G,\mathbb{R}^2):v|_{\partial G}=g\}$$

at a function u_ε and that

$$\lim_{\varepsilon\to 0}u_\varepsilon=u_p\quad\text{in }W^{1,p}(G,\mathbb{R}^2)\tag{1.1}$$

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where u_p is a p-harmonic map from G into S^1 with boundary value g [9]. Recall that $u \in W^{1,p}(G, S^1)$ is said to be a p-harmonic map on G, if u is a weak solution of the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p.$$

Under the condition $d = 0$, there exists exactly one p-harmonic map on G with the given boundary value g. However, there may be several minimizers of the functional. Let $\tilde{u}_\varepsilon$ be a minimizer that can be obtained as the limit of a subsequence of the minimizers u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u, G) = \frac{1}{p}\int_G (|\nabla u|^2 + \tau)^{p/2} + \frac{1}{4\varepsilon^p}\int_G (1 - |u|^2)^2, \quad (\tau > 0)$$

on W_g as $\tau_k \to 0$, namely

$$\lim_{\tau_k \to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon \quad \text{in } W^{1,p}(G, \mathbb{R}^2). \tag{1.2}$$

$\tilde{u}_\varepsilon$ is called the regularizable minimizer of $E_\varepsilon(u, G)$. Our main result reads as follows.

Theorem 1.1 *Assume that $p > 2$, $d = \deg(g, \partial G) = 0$. Let $\tilde{u}_\varepsilon$ be a regularizable minimizer of $E_\varepsilon(u, G)$. Then for some $\alpha \in (0, 1)$ we have*

$$\lim_{\varepsilon \to 0} \tilde{u}_\varepsilon = u_p \quad \textit{in } C^{1,\alpha}_{\text{loc}}(G, \mathbb{R}^2).$$

We shall prove a series of preliminary propositions in Sections 2, 3, and 4. Then we complete the proof of the main theorem in §5. In §6, we indicate how to extend our result to the higher dimensional case.

2 Convergence of $|u_\varepsilon^\tau|$

We start our argument with the following proposition.

Proposition 2.1

$$\lim_{\varepsilon, \tau \to 0} |u_\varepsilon^\tau| = 1 \quad \textit{in } C(\overline{G}, \mathbb{R}^2). \tag{2.1}$$

Proof. For $\tau \in (0, 1)$, we have

$$\begin{aligned} E_\varepsilon^\tau(u_\varepsilon, G) &\le E_\varepsilon^\tau(u_p, G) = \frac{1}{p}\int_G (|\nabla u_p|^2 + \tau)^{p/2} \\ &\le \frac{1}{p}\int_G (|\nabla u_p|^2 + 1)^{p/2} = C\,. \end{aligned}$$

Hence

$$\int_G |\nabla u_\varepsilon^\tau|^p \le \int_G (|\nabla u_\varepsilon^\tau|^2 + \tau)^{p/2} \le C, \tag{2.2}$$

$$\int_G (1 - |u_\varepsilon^\tau|^2)^2 \le C\varepsilon^p\,. \tag{2.3}$$

Here and below, we denote by C a universal constant which may take different values on different occasions. If necessary, we indicate explicitly its dependence.

¿From (2.3) it follows that there exists a subsequence $u_{\varepsilon_k}^{\tau_k}$ of u_ε^τ with $\varepsilon_k \to 0, \tau_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to\infty} |u_{\varepsilon_k}^{\tau_k}| = 1, \quad \text{a. e. in } G\,. \tag{2.4}$$

Inequality (2.2) combined with $|u_\varepsilon^\tau| \le 1$ (which follows from the maximum principle) means that $\|u_\varepsilon^\tau\|_{W^{1,p}(G,\mathbb{R}^2)} \le C$ which implies that there exist a function $u_* \in W^{1,p}(G,\mathbb{R}^2)$ and a subsequence of $u_{\varepsilon_k}^{\tau_k}$, supposed to be $u_{\varepsilon_k}^{\tau_k}$ itself without loss of generality, such that

$$\lim_{k\to\infty} u_{\varepsilon_k}^{\tau_k} = u_* \quad \text{in } C(\overline{G},\mathbb{R}^2)\,. \tag{2.5}$$

Combining (2.5) with (2.4) yields $|u_*| = 1$ in G and hence

$$\lim_{k\to\infty} |u_{\varepsilon_k}^{\tau_k}| = 1 \quad \text{in } C(\overline{G},\mathbb{R}^2)\,.$$

Since any subsequence of $|u_\varepsilon^\tau|$ contains a uniformly convergent subsequence and the limit is the same number 1, we may assert (2.1) which completes the proof.

Next, we prove some related facts about the asymptotic behaviour of $|u_\varepsilon^\tau|$, although Proposition 2.1 is enough for proving the next steps.

Proposition 2.2 *For all $q \in (1,p)$, there exist constants C, $\lambda > 0$, independent of ε such that*

$$\int_G |\nabla |u_\varepsilon^\tau||^q \le C\varepsilon^\lambda \tag{2.6}$$

for $\tau \in (0,1)$ and $\varepsilon \in (0,\eta)$ for some small $\eta > 0$.

Proof. As a minimizer of $E_\varepsilon^\tau(u,G)$, $u = u_\varepsilon^\tau$ satisfies the corresponding Euler equation

$$-\operatorname{div}(v^{(p-2)/2}\nabla u) = \tfrac{1}{\varepsilon^p} u(1-|u|^2) \quad \text{in } G \tag{2.7}$$

$$u|_{\partial G} = g\,, \tag{2.8}$$

where $v = |\nabla u|^2 + \tau$. Set $u = h(\cos\phi, \sin\phi)$ and $h = |u|$. Then

$$-\operatorname{div}(v^{(p-2)/2}h^2\nabla\phi) = 0 \tag{2.9}$$

$$-\operatorname{div}(v^{(p-2)/2}\nabla h) + h|\nabla\phi|^2 v^{(p-2)/2} = \tfrac{1}{\varepsilon^p} h(1-h^2)\,. \tag{2.10}$$

Fix $\beta \in (0,p/2)$ and set

$$S = \{x \in G; |h(x)| > 1-\varepsilon^\beta\}, \tilde{h} = \max(h, 1-\varepsilon^\beta).$$

Multiplying (2.8) with $h(1-\tilde{h})$, integrating over G and noticing that $\tilde{h}|_{\partial G} = 1$, we have

$$-\int_G v^{(p-2)/2} h\nabla h\nabla\tilde{h} + \int_G v^{(p-2)/2}|\nabla h|^2(1-\tilde{h}) + \int_G v^{(p-2)/2}h^2|\nabla\phi|^2(1-\tilde{h})$$
$$= \frac{1}{\varepsilon^p}\int_G h^2(1-h^2)(1-\tilde{h})$$

and thus we obtain

$$\int_G v^{(p-2)/2} h\nabla h\nabla\tilde{h} \le C\varepsilon^\beta \tag{2.11}$$

by using (2.2) and the facts $|\nabla u|^2 = |\nabla h|^2 + h^2|\nabla\phi|^2$ and $h = |u| \le 1$. Since $\tilde{h} = 1 - \varepsilon^\beta$ on $G \setminus S$, $\tilde{h} = h$ on S and $h > 1/2$ for $\varepsilon > 0$ small enough, (2.9) implies

$$\int_S v^{(p-2)/2}|\nabla h|^2 \le C\varepsilon^\beta$$

and hence

$$\int_S |\nabla h|^p \le C\varepsilon^\beta. \tag{2.12}$$

On the other hand, from the definition of S and (2.3), we have

$$C\,\mathrm{meas}(G \setminus S)\varepsilon^{2\beta} \le \int_{G\setminus S}(1-|u|^2)^2 \le C\varepsilon^p,$$

namely

$$\mathrm{meas}(G \setminus S) \le C\varepsilon^{p-2\beta}$$

and hence using (2.2) again we see that for any $q \in (1,p)$

$$\int_{G\setminus S}|\nabla h|^q \le \mathrm{meas}(G-S)^{1-q/p}(\int_G |\nabla h|^p)^{q/p} \le C\varepsilon^{(p-2\beta)(1-q/p)}$$

which and (2.10) imply the conclusion of Proposition 2.2.

Proposition 2.3 *There exists a constant C independent of $\varepsilon, \tau \in (0,1)$, such that*

$$\frac{1}{\varepsilon^p}\int_G (1-|u_\varepsilon^\tau|^2) \le C\,. \tag{2.13}$$

Proof. First we take the inner product of the both sides of (2.7) with u and then integrate over G

$$-\int_G \mathrm{div}(v^{(p-2)/2}\nabla u)u = \frac{1}{\varepsilon^p}\int_G |u|^2(1-|u|^2).$$

Integrating by parts and using (2.2) and the Holder inequality we obtain

$$\begin{aligned}
\frac{1}{\varepsilon^p}\int_G |u|^2(1-|u|^2) &\le \int_G v^{(p-2)/2}|\nabla u|^2 + \int_{\partial G} v^{(p-2)/2}|u_n||u| \\
&\le C + \int_{\partial G} v^{(p-2)/2}|u_n| \qquad (2.14)\\
&\le C + C\int_{\partial G} v^{(p-2)/2} + C\int_{\partial G} v^{(p-2)/2}|u_n|^2 \\
&\le C + C\int_{\partial G} v^{p/2}
\end{aligned}$$

where n denotes the unit outward normal to ∂G and u_n the derivative with respect to n.

To estimate $\int_{\partial G} v^{p/2}$, we choose a smooth vector field $\nu = (\nu_1, \nu_2)$ such that $\nu|_{\partial G} = n$, take the inner product of the both sides of (2.7) with $\nu \cdot \nabla u$ and integrate over G. Then we have

$$-\int_G \operatorname{div}(v^{(p-2)/2}\nabla u)(\nu \cdot \nabla u) = \frac{1}{2\varepsilon^p}\int_G (1-|u|^2)(\nu \cdot \nabla |u|^2).$$

Integrating by parts and noticing $|u|_{\partial G} = |g| = 1$ and

$$\int_G (1-|u|^2)(\nu \cdot \nabla |u|^2) = -\frac{1}{2}\int_G \nabla(1-|u|^2)^2 \cdot \nu = \frac{1}{2}\int_G (1-|u|^2)^2 \operatorname{div}\nu$$

yield

$$-\int_{\partial G} v^{(p-2)/2}|u_n|^2 + \int_G v^{(p-2)/2}\nabla u \cdot \nabla(\nu \cdot \nabla u) = \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2 \operatorname{div}\nu\,. \tag{2.15}$$

¿From the smoothness of ν and (2.2), (2.3) we have

$$\frac{1}{\varepsilon^p}\int_G (1-|u|^2)^2 |\operatorname{div}\nu| \le C \tag{2.16}$$

$$\begin{aligned} \int_G v^{(p-2)/2}\nabla u \nabla(\nu \cdot \nabla u) &\le C\int_G v^{(p-2)/2}|\nabla u|^2 + \frac{1}{2}\int_G v^{(p-2)/2}\nu \cdot \nabla v \\ &\le C + \frac{1}{p}\int_G \nu \cdot \nabla(v^{p/2}) \\ &\le C + \frac{1}{p}\int_G \operatorname{div}(\nu v^{p/2}) - \frac{1}{p}\int_G v^{p/2}\operatorname{div}\nu \\ &\le C + \frac{1}{p}\int_{\partial G} v^{p/2} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \int_{\partial G} v^{p/2} &= \int_{\partial G} v^{(p-2)/2}(|u_n|^2 + |g_t|^2 + \tau) \\ &\le \int_{\partial G} v^{(p-2)/2}|u_n|^2 + C\int_{\partial G} v^{(p-2)/2} \end{aligned} \tag{2.18}$$

where g_t denotes the derivative of g with respect to the tangent vector t to ∂G. Combining (2.13)-(2.16) we obtain

$$\int_{\partial G} v^{p/2} \le C\int_{\partial G} v^{(p-2)/2} + C + \frac{1}{p}\int_{\partial G} v^{p/2}$$

and derive

$$\int_{\partial G} v^{p/2} \le C \tag{2.19}$$

by using the Young inequality. Substituting (2.17) into (2.12) yields

$$\frac{1}{\varepsilon^p}\int_G |u|^2(1-|u|^2) \leq C$$

which together with (2.3) implies (2.11).

Using Proposition 2.2 and Proposition 2.3, we may obtain the following result which is similar to but stronger than the result in Proposition 2.1.

Proposition 2.4 *Uniformly for $\tau \in (0,1)$,*

$$\lim_{\varepsilon\to 0} |u_\varepsilon^\tau| = 1 \quad \textit{in } C(\overline{G},\mathbb{R}^2)\,.$$

Proof. From (2.6) and (2.11), we have

$$\int_G |\nabla|u_\varepsilon^\tau||^{(p+2)/2} \leq C\varepsilon^\lambda, \quad \forall\varepsilon\in(0,\eta), \tau\in(0,1)$$

$$\int_G (1-|u_\varepsilon^\tau|)^{(p+2)/2} \leq \int_G (1-|u_\varepsilon^\tau|) \leq \int_G (1-|u_\varepsilon^\tau|^2) \leq C\varepsilon^p, \quad \forall\varepsilon,\tau\in(0,1)$$

Thus

$$\|1-|u_\varepsilon^\tau|\|_{W^{1,(p+2)/2}(G,\mathbb{R}^2)} \leq C\varepsilon^\lambda, \quad \forall\varepsilon\in(0,\eta),\tau\in(0,1)$$

and hence by the embedding inequality, we obtain

$$\|1-|u_\varepsilon^\tau|\|_{C(G,\mathbb{R}^2)} \leq C\varepsilon^\lambda$$

which is a conclusion stronger than (2.18).

3 Estimate for $\|\nabla u_\varepsilon^\tau\|_{L^l_{\rm loc}}$

The main goal of this section is to establish uniform estimates for $\|\nabla u_\varepsilon^\tau\|_{L^l_{loc}}$.

Proposition 3.1 *There exists a constant C independent of $\varepsilon,\tau\in(0,\eta)$ for small $\eta>0$, such that*

$$\|\nabla u_\varepsilon^\tau\|_{L^l(K,\mathbb{R}^2)} \leq C = C(K,l) \tag{3.1}$$

where $K\subset G$ is an arbitrary compact subset and $l>1$.

Proof. Differentiate (2.7) with respect to x_j

$$-(v^{(p-2)/2}u_{x_i})_{x_ix_j} = \frac{1}{\varepsilon^p}(u(1-|u|^2))_{x_j}\,.$$

Here and in the sequel, double indices indicate summation.

Let $\zeta\in C_0^\infty(G,R)$ be a function such that $\zeta=1$ on K, $\zeta=0$ on $G\setminus\overline{G}_1, 0\leq \zeta\leq 1, |\nabla\zeta|\leq C$ on G, where $K\subset G_1$ and $G_1\subset\subset G$ be a sub-domain. Taking the the inner product of the both sides of (2.7) with $u_{x_j}\zeta^2$ and integrating over G, we obtain

$$\int_G (v^{(p-2)/2}u_{x_i})_{x_j}(\zeta^2 u_{x_j})_{x_i} = \frac{1}{\varepsilon^p}\int_G (1-|u|^2)\zeta^2|\nabla u|^2 - \frac{1}{2\varepsilon^p}\int_G \zeta^2(|u|^2)^2_{x_j}.$$

Summing up for $j=1,2$ and computing the term of the left side yield

$$\begin{aligned}&\int_G \zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2 + \frac{p-2}{4}\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2\\ \leq\ & \frac{1}{\varepsilon^p}\int_G (1-|u|^2)\zeta^2|\nabla u|^2 + 2|\int_G (v^{(p-2)/2}u_{x_i})_{x_j}u_{x_j}\zeta\zeta_{x_i}|\,. \end{aligned}\tag{3.2}$$

Applying Proposition 2.1 and the Young inequality, we derive from (2.7) that for any $\delta\in(0,1)$

$$\begin{aligned}&\frac{1}{\varepsilon^p}\int_G (1-|u|^2)\zeta^2|\nabla u|^2\\ \leq\ & \int_G |u|^{-1}|\nabla u|^2\zeta^2|\operatorname{div}(v^{(p-2)/2}\nabla u)| \qquad (3.3)\\ \leq\ & C\int_G \zeta^2 v(v^{(p-2)/2}|\Delta u| + \frac{p-2}{2}v^{(p-3)/2}|\nabla v|)\\ \leq\ & C(\delta)\int_G \zeta^2 v^{(p+2)/2} + \delta\int_G \zeta^2|\Delta u|^2 v^{(p-2)/2} + \delta\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2\\ \leq\ & C(\delta)\int_G \zeta^2 v^{(p+2)/2} + \delta\int_G \zeta^2\sum_{j=2}^{2}|\nabla u_{x_j}|^2 v^{(p-2)/2} + \delta\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2\,,\end{aligned}$$

where $\varepsilon,\tau\in(0,\eta)$ with $\eta>0$ small enough. Since

$$\begin{aligned}|(v^{(p-2)/2}u_{x_i})_{x_j}u_{x_j}\zeta\zeta_{x_i}| &= |\frac{1}{2}v^{(p-2)/2}v_{x_i} + \frac{p-2}{2}v^{(p-4)/2}v_{x_j}u_{x_i}u_{x_j}\zeta\zeta_{x_i}|\\ &\leq Cv^{(p-2)/2}\zeta|\nabla\zeta||\nabla v|\,,\end{aligned}$$

using the Young inequality again, for $\delta\in(0,1)$ we have

$$\begin{aligned}|\int_G (v^{(p-2)/2}u_{x_i})_{x_j}u_{x_j}\zeta\zeta_{x_i}| &\leq C\int_G v^{(p-2)/2}\zeta|\nabla\zeta||\nabla v| \qquad (3.4)\\ &\leq C(\int_G v^{(p-4)/2}|\nabla v|^2\zeta^2)^{1/2}(\int_G v^{p/2}|\nabla\zeta|^2)^{1/2}\\ &\leq \delta\int_G v^{(p-4)/2}|\nabla v|^2\zeta^2 + C(\delta)\int_G v^{p/2}|\nabla\zeta|^2\,.\end{aligned}$$

Substituting (3.3) and (3.4) into (3.2) and choosing δ small enough,yield

$$\begin{aligned}&\int_G \zeta^2 v^{(p-2)/2}\sum_{j=1}^{2}|\nabla u_{x_j}|^2 + \int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2\\ \leq\ & C\int_G \zeta^2 v^{(p+2)/2} + C\int_G v^{p/2}|\nabla\zeta|^2.\end{aligned}\tag{3.5}$$

Hence, by applying (2.2) to the last term, we obtain

$$\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2 \le C\int_G \zeta v^{(p+2)/2} + C. \tag{3.6}$$

Now we estimate $\int_G \zeta v^{(p+2)/2}$. To do this, we take $\phi = \zeta^{2/q}v^{(p+2)/2q}$ in the interpolation inequality

$$\|\phi\|_{L^q} \le C\|\nabla\phi\|_{L^1}^{\alpha}\|\phi\|_{L^1}^{1-\alpha}, \quad q\in(1,2), \alpha = 2(1-1/q).$$

Noticing that

$$|\nabla\phi| \le C\zeta^{2/q-1}|\nabla\zeta|v^{(p+2)/2q} + C\zeta^{2/q}v^{(p+2)/2q-1}|\nabla v|,$$

we have

$$\begin{aligned}\int_G \zeta^2 v^{(p+2)/2} \quad &\le \quad C(\int_G \zeta^{2/q}v^{(p+2)/2q})^{q(1-\alpha)} \\ &\quad\times(\int_G \zeta^{2/q-1}|\nabla\zeta|v^{(p+2)/2q} + \int_G \zeta^{2/q}v^{(p+2)/2q-1}|\nabla v|)^{q\alpha}\,. \end{aligned} \tag{3.7}$$

Since $p>2$, we can choose $q\in(1+2/p,2)$ and hence $\frac{p+2}{2q}<\frac{p}{2}$. Thus using (2.2) again,we derive that

$$\int_G \zeta^{2/q}v^{(p+2)/2q} \text{ and } \int_G \zeta^{2/q-1}|\nabla\zeta|v^{(p+2)/2q}$$

are bounded by

$$C\int_G v^{(p+2)/2q} \le C(\int_G v^{p/2})^{(p+2)/pq} \le C\,.$$

Substituting these inequalities into (3.7) gives

$$\begin{aligned}\int_G \zeta^2 v^{(p+2)/2} \quad &\le \quad C(1+\int_G \zeta^{2/q}v^{(p+2)/2q-1}|\nabla v|)^{q\alpha} \\ &\le \quad C[1+(\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2)^{1/2}(\int_G \zeta^{4/q-2}v^{(p+2)/q-p/2})^{1/2}]^{q\alpha} \\ &\le \quad C + C(\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2)^{q\alpha/2}(\int_G \zeta^{4/q-2}v^{(p+2)/q-p/2})^{q\alpha/2}\,. \end{aligned} \tag{3.8}$$

Here we have used the inequality

$$(a+b)^\lambda \le C(a^\lambda + b^\lambda), \quad (a,b\ge 0).$$

Since $q\in(1+\frac{2}{p},2)$, we have $\frac{q\alpha}{2}<1$, $\frac{p+2}{q}-\frac{p}{2}<\frac{p}{2}$. Thus, using the Holder inequality and (2.2), we obtain

$$\int_G \zeta^{4/q-2}v^{(p+2)/q-p/2} \le C(\int_G v^{p/2})^{2(p+2)/pq-1} \le C\,.$$

Hence from (3.8),we have for any $\delta \in (0,1)$

$$\begin{aligned}\int_G \zeta^2 v^{(p+2)/2} &\le C + C\Big(\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2\Big)^{q\alpha/2}\\ &\le C(\delta) + \delta \int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2\end{aligned}$$

since $\frac{q\alpha}{2} < 1$. Combining the last inequality with (3.6) we derive

$$\int_G \zeta^2 v^{(p-4)/2}|\nabla v|^2 \le C$$

or

$$\int_G \zeta^2 |\nabla w|^2 \le C$$

where $w = v^{p/4}$. Since (2.2) implies $\int_G \zeta^2 |w|^2 \le C$, we have $\zeta w \in W^{1,2}(G,R)$, and thus the embedding inequality gives

$$\int_G (\zeta w)^l \le C(l)$$

for any $l > 1$, which implies (3.1) since $\zeta = 1$ on K.

4 Estimate for $\|\nabla u_\varepsilon^\tau\|_{L^\infty_{\rm loc}}$

By means of the Moser iteration, from the estimate (3.1) we can further prove

Proposition 4.1 *There exists a constant C independent of $\varepsilon, \tau \in (0.\eta)$ for small $\eta > 0$ such that*

$$\|\nabla u_\varepsilon^\tau\|_{L^\infty(K,\mathbb{R}^2)} \le C = C(K) \tag{4.1}$$

where $K \subset G$ is an arbitrary compact subset.

Proof. Given any $x_0 \in G$. Let r be small enough and positive so that $B(x_0, 2r) \subset G$. Denote $Q_m = B(x_0, r_m), r_m = r + \frac{r}{2^m}$. Choose $\zeta_m \in C_0^\infty(Q_m, R)$ such that $\zeta_m = 1$ on $Q_{m+1}, |\nabla \zeta_m| \le Cr^{-1}2^m, (m = 1, 2, \ldots)$. Differentiate both sides of (2.7) with respect x_j, multiply by $\zeta_m^2 v^b u_{x_j}$ for $b \ge 1$ and integrate over Q_m. Then

$$\begin{aligned}&\int_{Q_m} (v^{(p-2)/2} u_{x_i})_{x_j} (\zeta_m^2 v^b u_{x_j})_{x_i}\\ &= \frac{1}{\varepsilon^p}\int_{Q_m} (1-|u|^2)\zeta_m^2 v^b |\nabla u|^2 - \frac{1}{2\varepsilon^p}\int_{Q_m} \zeta_m^2 v^b (|u|^2)^2_{x_j}\,.\end{aligned}$$

Similar to the derivation of (3.2) we can obtain

$$\begin{aligned} &\int_{Q_m} \zeta_m^2 v^{(p+2b-2)/2} \sum_j |\nabla u_{x_j}|^2 + \frac{p+2b-2}{4} \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \\ &\quad \le \frac{1}{\varepsilon^p} \int_{Q_m} (1-|u|^2) \zeta_m^2 v^{b+1} + \Big| \int_{Q_m} (v^{(p-2)/2} u_{x_i})_{x_j} v^b u_{x_j} \zeta_m \zeta_{m x_i} \Big| . \end{aligned} \tag{4.2}$$

Also for any $\delta \in (0,1)$, we have

$$\begin{aligned} &\frac{1}{\varepsilon^p} \int_{Q_m} (1-|u|^2) \zeta_m^2 v^{b+1} \\ &\le \int_{Q_m} |u|^{-1} \zeta_m^2 v^{b+1} |\operatorname{div}(v^{(p-2)/2} \nabla u)| \\ &\le C \int_{Q_m} \zeta_m^2 v^{(p+2b)/2} |\Delta u| + \frac{C(p+2b-2)}{2} \int_{Q_m} \zeta_m^2 v^{(p+2b-2)/2} |\nabla v| \qquad (4.3) \\ &\le C(\delta) \int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2} + \delta \int_{Q_m} \zeta_m^2 v^{(p+2b-2)/2} |\Delta u|^2 \\ &\quad + \frac{C(\delta)(p+2b-2)}{2} \int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2} \\ &\quad + \frac{\delta(p+2b-2)}{2} \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \end{aligned}$$

and

$$\begin{aligned} &\Big| \int_{Q_m} (v^{(p-2)/2} u_{x_i})_{x_j} v^b u_{x_j} \zeta_m \zeta_{m x_i} \Big| \\ &\le C \int_{Q_m} v^{(p+2b-2)/2} |\nabla v| \zeta_m |\nabla \zeta_m| \qquad (4.4) \\ &\le \delta \int_{Q_m} v^{(p+2b-1)/2} |\nabla v|^2 \zeta_m^2 + C(\delta) \int_{Q_m} v^{(p+2b)/2} |\nabla \zeta_m|^2 , \end{aligned}$$

where the constants C and $C(\delta)$ are independent of b, m, ε, τ. Combining (4.2) with (4.3)(4.4) and choosing δ small enough yield

$$\int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \le C \int_{Q_m} v^{(p+2b)/2} |\zeta_m|^2 + C \int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2}. \tag{4.5}$$

To estimate $\int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2}$, we take

$$\phi = \zeta_m^{2/q} v^{(p+2b+2)/2q}$$

in the interpolation inequality (3.6) and observe that

$$|\nabla \phi| \le \frac{2}{q} \zeta_m^{2/q-1} |\nabla \zeta_m| v^{(p+2b+2)/2q} + \frac{p+2b+2}{2q} \zeta_m^{2/q} v^{(p+2b+2)/2q-1} |\nabla v|.$$

Then we obtain

$$\begin{aligned}
&\int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2} \\
&\leq C\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q}\Big)^{q(1-\alpha)} \Big(\frac{2}{q}\int_{Q_m} \zeta_m^{2/q-1}|\nabla\zeta_m| v^{(p+2b+2)/2q} \\
&\quad + \frac{p+2b+2}{2q}\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q-1}|\nabla v|\Big)^{q\alpha} \\
&\leq C\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q}\Big)^{q(1-\alpha)} \Big(\frac{2}{q}\Big)^{q\alpha}\Big(\int_{Q_m} \zeta_m^{2/q-1}|\nabla\zeta_m| v^{(p+2b+2)/2q}\Big)^{q\alpha} \\
&\quad + \Big(\frac{p+2b+2}{2q}\Big)^{q\alpha}\Big(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q-1}|\nabla v|\Big)^{q\alpha}.
\end{aligned} \tag{4.6}$$

Now we estimate all integrals on the right-hand side of (4.6). Choose r small enough so that $\operatorname{meas}(Q_m) \leq 1$. In computing we need to notice that $q \in (1+\frac{2}{p}, 2)$, which implies $q > 1 + \frac{2}{p+2b}$ or $\frac{p+2b}{2q} < \frac{p+2b}{2}$. We have

$$\begin{aligned}
&\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q} \\
&\leq \int_{Q_m} v^{(p+2b+2)/2q} \\
&\leq (\operatorname{meas}(Q_m))^{1-(p+2b+2)/(q(p+2b))}\Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))} \\
&\leq \Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))},
\end{aligned}$$

$$\begin{aligned}
\int_{Q_m} \zeta_m^{2/q-1}|\nabla\zeta_m| v^{(p+2b+2)/2q} &\leq \frac{2^m}{r}\int_{Q_m} v^{(p+2b+2)/2q} \\
&\leq \frac{2^m}{r}\Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q-1}|\nabla v| \\
&\leq \Big(\int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2}|\nabla v|^2\Big)^{1/2}\Big(\int_{Q_m} \zeta_m^{4/q-2} v^{(p+2b+2)/q-(p+2b)/2}\Big)^{1/2} \\
&\leq \Big(\int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2}|\nabla v|^2\Big)^{1/2}\Big(\int_{Q_m} v^{(p+2b)/2}\Big)^{(p+2b+2)/(q(p+2b))-1/2}.
\end{aligned}$$

Combining these inequalities with (4.5) and (4.6) yields

$$I_1 \leq C\Big[\Big(\frac{2^m}{r}\Big)^2 I_2 + \Big(\frac{2^m}{r}\Big)^{q\alpha} I_2^{1+2/(p+2b)} + \Big(\frac{p+2b+2}{2q}\Big)^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/(p+2b)-q\alpha/2}\Big], \tag{4.7}$$

where

$$I_1 = \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2, \text{ and } I_2 = \int_{Q_m} v^{(p+2b)/2}.$$

Let

$$p + 2b = s^m, \quad w = v^{(p+2b)/4} = v^{s^m/4}$$

with $s > 2$ to be determined later. Then (4.7) becomes

$$I_1 \le C[(\frac{2^m}{r} I_2 + (\frac{2^m}{r})^{q\alpha} I_2^{1+2/S^m} + (\frac{s^m+2}{2q})^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/s^m - q\alpha/2}].$$

The Young inequality applied to the last term on the right-hand side yields

$$\begin{aligned} & C(\frac{s^m+2}{2q})^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/(s^m)-q\alpha/2} \\ \le\ & \delta I_1 + C(\delta)[(\frac{s^m+2}{2q})^{q\alpha} I_2^{1+2/(s^m)-q\alpha/2}]^{2/(2-q\alpha)} \\ =\ & \delta I_1 + C(\delta)(\frac{s^m+2}{2q})^{2q\alpha/(2-q\alpha)} I_2^{2(1+2/(s^m)-q\alpha/2)/(2-q\alpha)}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} I_1 \ \le\ & C(\delta)[(\frac{2^m}{r})^2 I_2 + (\frac{2^m}{r})^{q\alpha} I_2^{1+2/(s^m)} \\ & +(\frac{s^m+2}{2q})^{2q\alpha/(2-q\alpha)} I_2^{2(1+2/(s^m)-q\alpha/2)/(2-q\alpha)}]. \end{aligned} \tag{4.8}$$

By the embedding theorem, we have for any $s > 1$

$$\begin{aligned} \int_{Q_m} (\zeta_m w)^{2s} \ \le\ & C(s)[\int_{Q_m} (\zeta_m w)^2 + \int_{Q_m} |\nabla(\zeta_m w)|^2]^s \\ \le\ & C(s)[\int_{Q_m} (\zeta_m w)^2 + \int_{Q_m} |\nabla\zeta_m|^2 w^2 + \int_{Q_m} \zeta_m^2 |\nabla w|^2]^s \\ \le\ & C(s)[(1 + (\frac{2^m}{r})^2) I_2 + (\frac{s^m}{4})^2 I_1]^s, \end{aligned}$$

which by using (4.8), turns out to be

$$\begin{aligned} \int_{Q_m} (\zeta_m w)^{2s} \ \le\ & C(s)[(1 + (\frac{2^m}{r})^2 + (\frac{s^m}{4})^2 (\frac{2^m}{r})^2) I_2 + (\frac{s^m}{4})^2 (\frac{2^m}{r})^{q\alpha} I_2^{1+\frac{2}{s^m}} \\ & +(\frac{s^m}{4})^2 (\frac{s^m+2}{2q})^{\frac{2q\alpha}{2-q\alpha}} I_2^{(1+\frac{2}{s^m}-\frac{q\alpha}{2})\frac{2}{2-q\alpha}}]^s. \end{aligned} \tag{4.9}$$

If there exists a subsequence of positive integers $\{m_i\}$ with $m_i \to \infty$ such that

$$I_2 = \int_{Q_{m_i}} v^{s^{m_i}/2} < 1,$$

then as $m_i \to \infty$,

$$\|v\|_{L^\infty(Q_\infty, R)} \le C(r). \tag{4.10}$$

Otherwise, there must be a positive integer m_0 such that

$$I_2 = \int_{Q_m} v^{s^m/2} \geq 1\,,\ \forall m \geq m_0\,.$$

Since

$$(1+\frac{2}{s^m}-\frac{q\alpha}{2})\frac{2}{2-q\alpha} = 1+\frac{2}{s^m}\frac{1}{2-q} > 1+\frac{2}{s^m} > 1\,,$$

the exponent of the last term in (4.9) is higher than those of the other terms. Now comparing the coefficients of the terms in (4.9), we have

$$(\frac{s^m}{r})^2 \geq 1\,, \quad (\frac{2^m}{r})^2 \geq (\frac{2^m}{r})^{q\alpha}$$

and, if we choose $s > 2q(\frac{2}{r})^{\frac{2(q-1)}{2-q}}$, $r \leq 1$, then

$$(\frac{s^m+2}{2q})^{\frac{2q\alpha}{2-q\alpha}} = (\frac{s^m+2}{2q})^{\frac{2(q-1)}{2-q}} \geq (\frac{2^m}{r})^2.$$

Therefore, the coefficient of the last term in (4.9) is greater than those of the other terms. Hence we have

$$\int_{Q_m} (\zeta_m w)^{2s} \leq C[(\frac{s^m}{4})^2(\frac{s^m+2}{2q})^{\frac{2(q-1)}{2-q}} I_2^{1+\frac{2}{s^m}\frac{1}{2-q}}]^s$$

or

$$\int_{Q_{m+1}} v^{s^{m+1}/2} \leq (C_0 C_1^m)^s (\int_{Q_m} v^{s^m/2})^{(1+C_2/S^m)s} \tag{4.11}$$

with some constant $C_0 > 0$, $C_2 = \frac{2}{2-q}$, $C_1 = s^{(2+\frac{2(q-1)}{2-q})s}$. Using an iteration proposition which will be stated and proved later, we also reach estimate (4.10). Thus the proof of Proposition 4.1 is complete.

Proposition 4.2 *Let $Q_m (m = 1, 2, \ldots) \subset G$ be a sequence of bounded, open subsets such that $Q_{m+1} \subset Q_m$. If for any $l \geq 1$, $v \in L^l(Q_1, R)$ and there exist constants $\lambda, C_0, C_1, C_2 > 0$, $s > 1$, $\lambda s \geq 1$, such that*

$$\int_{Q_{m+1}} |v|^{\lambda s^{m+1}}\,dx \leq (C_0 C_1^m)^s (\int_{Q_m} |v|^{\lambda s^m}\,dx)^{(1+C_2/s^m)s},$$

for $m = 1, 2, \ldots$, then

$$\|v\|_{L^\infty(Q_\infty,R)} \leq C_0^{A_1} C_1^{A_2} (\int_{Q_{n_0}} |v|^{\lambda s^{n_0}}\,dx)^{A_3/(\lambda s^{n_0})},$$

where A_1, A_2, A_3 are constants depending only on s, C_2, and n_0 is an arbitrary nonnegative integer.

Proof. From (4.12) by iteration, we obtain

$$\int_{Q_{m+1}} |v|^{\lambda s^{m+1}}\,dx \le C_0^{X_m} C_1^{Y_m} (\int_{Q_{n_0}} |v|^{\lambda s^{n_0}}\,dx)^{s^{m-n_0+1} Z_m} \tag{4.12}$$

where

$$\begin{aligned} X_m &= s + s^2\lambda_m + \ldots + s^{m-n_0+1}\lambda_m\lambda_{m-1}\ldots\lambda_{n_0+1} \\ Y_m &= ms + (m-1)s^2\lambda_m + \ldots + n_0 s^{m-n_0+1}\lambda_m\lambda_{m-1}\ldots\lambda_{n_0+1} \\ Z_m &= \lambda_{n_0}\ldots\lambda_{m-1}\lambda_m \end{aligned}$$

with $\lambda_m = 1 + C_2/s^m$. Since $\lambda_j \ge 1$ for $j = n_0 - 1, \ldots, m-1, m, \ldots$, Z_m is an increasing sequence. Noticing that $\ln(1+x) \le x$ for $x > 0$, we have

$$\begin{aligned} lnZ_m &= ln\lambda_{n_0} + \ldots + ln\lambda_{m-1} + ln\lambda_m \\ &\le C_2[(\frac{1}{s})^{n_0} + \ldots + (\frac{1}{s})^{m-1} + (\frac{1}{s})^m] \\ &\le C_2\frac{(1/s)^{n_0}}{1-1/s} = \gamma \end{aligned}$$

or $Z_m \le e^\gamma$. Hence $\lim_{m\to\infty} Z_m = A_3$ exists. Clearly, we also have

$$\begin{aligned} X_m &\le e^\gamma[s + s^2 + \ldots + s^{m-n_0+1}] \\ Y_m &\le e^\gamma[ms + (m-1)s^2 + \ldots + n_0 s^{m-n_0+1}]\,. \end{aligned}$$

From which it is easily seen that the following two limits exist:
$\lim_{m\to\infty} s^{-(m+1)}X_m = A_1$ and $\lim_{m\to\infty} s^{-(m+1)}Y_m = A_2$. Taking the $1/\lambda s^{m+1}$ power on the both sides of (4.14), letting $m \to \infty$, and noticing that

$$\|v\|_{L^\infty(Q_\infty,R)} = \lim_{m\to\infty}(\int_{Q_{m+1}} |v|^{\lambda s^{m+1}}\,dx)^{1/\lambda s^{m+1}},$$

we obtain (4.13).

5 Completion of the proof

Once the locally uniform estimate $\|\nabla u_\varepsilon^\tau\|_{L^\infty(K)}$ is established, it is not difficult to prove the following proposition.

Proposition 5.1 *Let $\psi_\varepsilon^\tau = \frac{1}{\varepsilon^p}(1 - |u_\varepsilon^\tau|^2)$. Then there exists a constant C independent of $\varepsilon, \tau \in (0, \eta)$ with $\eta > 0$ small enough, such that*

$$\|\psi_\varepsilon^\tau\|_{L^\infty(K,R)} \le C = C(K),$$

where $K \subset G$ is an arbitrary compact subset.

Proof. Take the inner product of both sides of (2.7) with u,

$$-\operatorname{div}(v^{(p-2)/2}\nabla u)u = \frac{1}{\varepsilon^p}|u|^2(1-|u|^2) = |u|^2\psi$$

where $u = u_\varepsilon^\tau, \psi = \psi_\varepsilon^\tau$. This and

$$\nabla\psi = -\frac{2}{\varepsilon^p}u\cdot\nabla u$$
$$-\operatorname{div}(v^{(p-2)/2}\nabla u)u = -\operatorname{div}(v^{(p-2)/2}u\cdot\nabla u) + v^{(p-2)/2}|\nabla u|^2$$

give

$$|u|^2\psi = v^{(p-2)/2}|\nabla u|^2 + \frac{\varepsilon^p}{2}\operatorname{div}(v^{(p-2)/2}\nabla\psi).$$

Using Proposition 2.1 we obtain

$$\frac{1}{2}\psi \le v^{(p-2)/2}|\nabla u|^2 + \frac{\varepsilon^p}{2}\operatorname{div}(v^{(p-2)/2}\nabla\psi),\ \forall\varepsilon,\tau\in(0,\eta).$$

Since at the point where ψ achieves its maximum, $\nabla\psi = 0$, $\Delta\psi \le 0$, and

$$\operatorname{div}(v^{(p-2)/2}\nabla\psi) = v^{(p-2)/2}\Delta\psi + \frac{p-2}{2}v^{(p-4)/2}\nabla v\nabla\psi \le 0\,,$$

we derive (5.1) from (5.2) by using Proposition 4.1.

To complete the proof of Theorem 1.1, we apply a theorem in [12] (Page 244 Line 19–23). Now according to Proposition 5.1 the right hand side of (2.7) is bounded on every compact subset $K \subset G$ uniformly in $\varepsilon,\tau\in(0,\eta)$. Thus applying the theorem in [12] (Page 244) yields

$$\|u_\varepsilon^\tau\|_{C^{1,\beta}(K)} \le C = C(K) \tag{5.1}$$

for some $\beta\in(0,1)$, where the constant C does not depend on $\varepsilon,\tau\in(0,\eta)$. From this it follows that there exist a function u_* and a subsequence $u_{\varepsilon_k}^{\tau_k}(\varepsilon_k,\tau_k\to 0$, as $k\to\infty)$ of u_ε^τ, such that

$$\lim_{k\to\infty} u_{\varepsilon_k}^{\tau_k} = u_*, \quad \text{ in } C^{1,\alpha}(K,\mathbb{R}^2), \alpha\in(0,\beta)\,.$$

By an argument similar to that in the proof of (1.1) and (1.2), we obtain

$$\lim_{\varepsilon,\tau\to 0} u_\varepsilon^\tau = u_p, \quad \text{in } W^{1,p}(K,\mathbb{R}^2).$$

Certainly $u_* = u_p$. From the fact that any subsequence of u_ε^τ contains a subsequence convergent in $C^{1,\alpha}(K,\mathbb{R}^2)$ and the limit is the same function u_p, we may assert

$$\lim_{\varepsilon,\tau\to 0} u_\varepsilon^\tau = u_p, \quad \text{ in } C^{1,\alpha}(K,\mathbb{R}^2). \tag{5.2}$$

On the other hand, for any $\varepsilon\in(0,\eta)$, as a regularizable minimizer of $E_\varepsilon(u,G)$, $\tilde{u}_\varepsilon$ is the limit of some subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ in $W^{1,p}(G,\mathbb{R}^2)$. For large k, $u_\varepsilon^{\tau_k}$

satisfies (5.3) and hence it contains a subsequence, for simplicity we suppose it is $u_\varepsilon^{\tau_k}$ itself, such that

$$\lim_{k\to\infty} u_\varepsilon^{\tau_k} = w, \quad \text{in } C^{1,\alpha}(K,\mathbb{R}^2)$$

where the function w must be $\tilde u_\varepsilon$. Combining this with (5.4) we finally obtain

$$\lim_{\varepsilon\to 0} \tilde u_\varepsilon = u_p, \quad \text{in } C^{1,\alpha}(K,\mathbb{R}^2)$$

and complete the proof of Theorem 1.1.

Remark Using Proposition 2.4 instead of Proposition 2.1 we may also prove our theorem. In this way, we may obtain (3.1), (4.1) and (5.1) for $\varepsilon \in (0,\eta), \tau \in (0,1)$ instead of those for $\varepsilon, \tau \in (0,\eta)$. The remainder of the proof is just the same as above.

6 Extension of the argument

Our argument can be extended to the higher dimensional case. Let $n > 2$ $G \subset \mathbb{R}^n$ be a bounded and simply connected domain with smooth boundary ∂G, and $g : \partial G \to S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ be a smooth map with $d = \deg(g, \partial G) = 0$. Consider the functional

$$E_\varepsilon(u,G) = \frac{1}{p}\int_G |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2, \quad (\varepsilon > 0)$$

and its regularized functional

$$E_\varepsilon^\tau(u,G) = \frac{1}{p}\int_G (|\nabla u|^2+\tau)^{p/2} + \frac{1}{4\varepsilon^p}\int_G (1-|u|^2)^2, \quad (\varepsilon,\tau > 0)$$

on

$$W_g = \{v \in W^{1,p}(G,\mathbb{R}^n); v|_{\partial G} = g\}.$$

Similar to the case $n = 2$, we may prove that if $p > 1$, then $E_\varepsilon(u,G)$ and $E_\varepsilon^\tau(u,G)$ achieve their minimum on W_g by some u_ε and u_ε^τ; u_ε and u_ε^τ satisfy

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2), \quad \text{in } G$$

and

$$-\operatorname{div}(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2), \quad \text{in } G$$

respectively where $v = |\nabla u|^2 + \tau$, and

$$|u_\varepsilon|, |u_\varepsilon^\tau| \le 1, \quad \text{in } G.$$

It can also be proved that if $p > 1$, then there exists a subsequence u_{ε_k} of u_ε with $\varepsilon_k \to 0$ such that

$$\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_p, \quad \text{in } W^{1,p}(G,\mathbb{R}^n)$$

where u_p is a p-harmonic map with boundary value g. However, differ to the case $n = 2$, here we can only prove the convergence for a subsequence because of the lack of uniqueness result for p-harmonic map with given boundary value.

Similarly, for some subsequence we have $u_\varepsilon^{\tau_k}(\tau_k \to 0)$ of u_ε^τ

$$\lim_{\tau_k \to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon, \qquad in W^{1,p}(G, \mathbb{R}^n)$$

and the limit $\tilde{u}_\varepsilon$ is a minimizer of $E_\varepsilon(u, G)$, called regularizable minimizer. The main result is the following

Theorem 6.1 *Assume that $p > 2n - 2$ and $d = \deg(g, \partial G) = 0$. Let $\tilde{u}_\varepsilon$ be a regularizable minimizer of $E_\varepsilon(u, G)$. Then there exists a subsequence $\tilde{u}_{\varepsilon_k}$ of $\tilde{u}_\varepsilon$ with $\varepsilon_k \to 0$ such that for some $\alpha \in (0, 1)$*

$$\lim_{\varepsilon_k \to 0} \tilde{u}_{\varepsilon_k} = u_p, \quad in\ C^{1,\alpha}_{\mathrm{loc}}(G, \mathbb{R}^n).$$

The proof is similar to the case $n = 2$. First we have

$$\lim_{\varepsilon, \tau \to 0} |u_\varepsilon^\tau| = 1 \quad \text{in } C(\overline{G}, \mathbb{R}^n)$$

and also

$$\lim_{\varepsilon \to 0} |u_\varepsilon^\tau| = 1 \quad \text{in } C(\overline{G}, \mathbb{R}^n)$$

uniformly for $\tau \in (0, 1)$.

Next we prove

$$\|\nabla u_\varepsilon^\tau\|_{L^l(K, \mathbb{R}^n)} \le C = C(K, l)$$

where $K \subset G$ is an arbitrary compact subset and $l > 1$.

For this purpose, we proceed as in section 3: first differentiate (6.1) with respect to x_j, take the inner product of the both sides with $\zeta^2 v^b u_{x_j} (b \ge 0)$, where $\zeta \in C_0^\infty(G, R)$ with $0 \le \zeta \le 1$, and integrate over G. Then as in (3.5) and in (4.5), we obtain

$$\int_G \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \le C \int_G \zeta^2 v^{(p+2b+2)/2} + C \int_G v^{(p+2b)/2} |\nabla \zeta|^2 . \tag{6.1}$$

Using the interpolation inequality

$$\|\phi\|_{L^q} \le C \|\nabla \phi\|_{L^1}^\alpha \|\phi\|_{L^1}^{1-\alpha}, \quad q \in (1, n/(n-1)),\ \alpha = n(q-1)/q$$

for $\phi = \zeta^{2/q} v^{(p+2b+2)/2q}$ to estimate the last term of (6.3) yields

$$\begin{aligned} &\int_G \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \\ \le\ & C \int_G |\nabla \zeta|^2 v^{(p+2b)/2} \\ & + C\Big(\int_G \zeta^{2/q} v^{(p+2b+2)/2q}\Big)^{q(1-\beta)} \Big(\int_G \zeta^{2/q-1} |\nabla \zeta| v^{(p+2b+2)/2q}\Big)^{q\beta} \\ & + C\Big(\int_G \zeta^{2/q} v^{(p+2b+2)/2q}\Big)^{\lambda_1} \Big(\int_G \zeta^{4/q-2} v^{(p+2b+2)/q-(p+2b)/2}\Big)^{\lambda_2}, \end{aligned}$$

where the constants λ_1, $\lambda_2 > 0$ depend on p, q, b, α only. Set $w = v^{(p+2b)/4}$. Since $p > 2n-2$, we may choose $q \in (1-2/p, n/(n-1))$ such that

$$\frac{p+2b+2}{2q} < \frac{p+2b}{2} \text{ and } \frac{p+2b+2}{q} - \frac{p+2b}{2} < \frac{p+2b}{2}\,.$$

Then use the Holder inequality to obtain

$$\int_G \zeta^2|\nabla w|^2 \le C\int_G |\nabla\zeta|^2 w^2 + C(\int_G w^2)^\lambda[1+(\int_G |\nabla\zeta|^\lambda)^\lambda]\,,$$

where the constants C and $\lambda > 0$ are independent of ε and τ.

Now we choose ζ such that $\zeta = 1$ on G_1, where G_1, G_0 are sub-domains of G satisfying $K \subset G_1 \subset\subset G_0 \subset\subset G$, $\zeta = 0$ on $G\setminus\overline{G}_0$, $|\nabla\zeta| \le C$ on G and $b = 0$. From

$$E^\tau_\varepsilon(u^\tau_\varepsilon, G) \le E^\tau_\varepsilon(u_p, G) \le C$$

we have

$$\int_G \zeta^2|\nabla w|^2 \le C$$

and hence $\|\zeta w\|_{L^2(G,\mathbb{R}^n)} \le C$. By the embedding theorem,

$$\|\zeta w\|_{L^r(G,\mathbb{R}^n)} = (\int_G (\zeta w)^r)^{1/r} \le C\|\zeta w\|_{L^2(G,\mathbb{R}^n)} \le C\,, \tag{6.2}$$

where $r \le \frac{2n}{n-2}$. Clearly $r = 2 + \frac{8}{np} \le \frac{2n}{n-2}$. Choosing $r = 2 + \frac{8}{np}$ in (6.5) and noticing that $\zeta = 1$ on G_1 we see that $\nabla u \in L^{s_1}(G_1)$and

$$\int_{G_1} |\nabla u|^{s_1} \le C\,, \tag{6.3}$$

where $s_1 = p + \frac{4}{n}$. In the present case $n > 2$, we can not derive (6.2) directly by using the embedding theorem once. To prove (6.2) we choose G_2, a sub-domain of G_1, such that $K \subset G_2 \subset\subset G_1$ and $\zeta = 1$ on $G_2, \zeta = 0$ on $G\setminus\overline{G}_1, |\nabla\zeta| \le C$ on G. Set $b = \frac{2}{n}, w = v^{(p+4/n)/4}$. Then from (6.6)

$$\int_{G_1} w^2 = \int_{G_1} v^{(p+4/n)/2} = \int_{G_1} |\nabla u|^{s_1} \le C\,,$$

and from (6.4),

$$\int_{G_1} \zeta^2|\nabla w|^2 \le C$$

Thus $\|\zeta w\|_{L^2(G,\mathbb{R}^n)} \le C$. Applying the embedding theorem to ζw, we obtain

$$\int_{G_2} |\nabla u|^{s_2} \le C$$

where

$$s_2 = s_1 + \frac{4(n+2)}{n^2} = p + \frac{4}{n} + \frac{4(n+2)}{n^2} = p + \frac{8}{n} + \frac{8}{n^2}$$

For a given $l > 1$, proceeding inductively, we find an s_i for some i such that $s_i > l$ and

$$\int_{G_i} |\nabla u|^{s_i} \leq C\,,$$

where G_i is a sub-domain of G_{i-1} such that $K \subset G_i \subset\subset G_{i-1} \subset\subset \ldots \subset\subset G$. Thus (6.2) is proved.

The remainder of the proof is just the same as in sections 4 and 5. However, we are not able to establish the convergence for the full $\tilde{u}_\varepsilon$ because of the lack of uniqueness result for the p-harmonic map with the given boundary value.

References

[1] F. Bethuel, H. Brezis, F.Helein: *Asymptotics for the minimization of a Ginzburg-Landau functional,* Calc. Var. PDE., **1** (1993).123-148.

[2] F. Bethuel, H. Brezis, F. Helein: *Ginzburg-Landau Vortices,* Birkhauser. 1994.

[3] S.J. Ding, Z.H. Liu: *On the zeroes and asymptotic behaviour of minimizers to the Ginzburg-Landau functional with variable coefficient*, J. Partial Diff. Eqns., **10** No.1. (1997).45-64.

[4] R. Hardt, F.H. Lin: *Singularities for p-energy minimizing unit vector fields on planner domains,* Cal. Var. PDE., **3** (1995), 311-341.

[5] M.C. Hong: *Asymptotic behavior for minimizers of a Ginzburg-Landau type functional in higher dimensions associated with n-harmonic maps,* Adv. in Diff. Eqns., **1** (1996), 611-634.

[6] M.C. Hong: *On a problem of Bethuel, Brezis and Helein concerning the Ginzburg-Landau functional,* C. R. Acad. Sic. Paris, **320** (1995), 679-684.

[7] M.C. Hong: *Two estimates concerning asymptotics of the minimizations of a Ginzburg-Landau functional,* J. Austral. Math. Soc. Ser. A, **62** No.1, (1997), 128-140.

[8] F.H. Lin: *Solutions of Ginzburg-Landau equations and critical points of the renormalized energy,* Ann. Inst. P. Poincare Anal. Nonlineare, **12** No.5, (1995), 599-622.

[9] Y.T. Lei: *Asymptotic behavior of minimizers for a functional* Acta. Scien. Natur. Univer. Jilin., **1** (1999), 1-6.

[10] M. Struwe: *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2-dimensions,* Diff. and Int. Eqns., **7** (1994), 1613-1624.

[11] M. Struwe: *An aysmptotic estimate for the Ginzburg-Landau model,* C.R.Acad. Sci. Paris., **317** (1993), 677-680.

[12] P. Tolksdorf: *Everywhere regularity for some quasilinear systems with a lack of ellipticity,* Anna. Math. Pura. Appl., **134** (1983), 241-266.

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