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# Colombeau's theory and shock wave solutions for systems of PDEs *

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**Abstract**

In this article we study the existence of shock wave solutions for systems of partial differential equations of hydrodynamics with viscosity in one space dimension in the context of Colombeau's theory of generalized functions. This study uses the equality in the strict sense and the association of generalized functions (that is the weak equality). The shock wave solutions are given in terms of generalized functions that have the classical Heaviside step function as macroscopic aspect. This means that solutions are sought in the form of sequences of regularizations to the Heaviside function that have to satisfy part of the equations in the strict sense and part of the equations in the sense of association.

## Introduction

Let $\tilde{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\}$. Fix $(\alpha, \beta)$ in $\mathbb{R}_+ \times \tilde{\mathbb{R}}_+$ with $\alpha < \beta$. Let $\nu$ be a function in $C^\infty\left((\mathbb{R}_+^*)^3; [\alpha, \beta[\right)$ satisfying some conditions to be introduced in §5. We consider two associated systems of hydrodynamic equations with viscosity $\nu$ in one space dimension. The system $(\tilde{S})$ consists of the equations

$$\begin{gathered}
\rho_t + (\rho u)_x \approx 0 \\
(\rho u)_t + (p + \rho u^2)_x \approx \{[\nu \circ (\rho, p, e) - \alpha] u_x\}_x \\
e_t + [(e + p)u]_x \approx \{[\nu \circ (\rho, p, e) - \alpha] u u_x\}_x \\
e \approx \lambda p + \tfrac{1}{2}\rho u^2 \,, \quad \lambda \in \mathbb{R}^*
\end{gathered}$$

and $(S)$ consists of the two last equations and

$$\begin{gathered}
\rho_t + (\rho u)_x = 0 \\
(\rho u)_t + (p + \rho u^2)_x = \{[\nu \circ (\rho, p, e) - \alpha] u_x\}_x
\end{gathered}$$

where $\rho$ is the density, $u$ the velocity, $p$ the pressure and $e$ the total energy. The symbol $\approx$ denotes the association relation in $\mathcal{G}_s(\mathbb{R}^2; \mathbb{R})$ (see §2). The purpose of this paper is to study the existence of shock wave solutions (see §5) for the systems $(\tilde{S})$ and $(S)$. More precisely, solutions with two constant states separated

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by a jump along a straight line. We know that, in theory of distributions, there exists a unique (Heaviside) distribution $\mathbb{Y}$ such that $\mathbb{Y}|_{\mathbb{R}^*_-} = 0$, $\mathbb{Y}|_{\mathbb{R}^*_+} = 1$ and whose derivative $\mathbb{Y}'$ is the Dirac distribution. The shock wave solutions for the system $(\tilde{S})$ are given in terms of generalized functions that have the function $\mathbb{Y}$ as macroscopic aspect (see [1] or [3]). These functions are called Heaviside generalized functions in $\mathbb{R}$. Also, recall that, for every Heaviside generalized function $H$ we have $H|_{\mathbb{R}^*_-} \approx 0$, $H|_{\mathbb{R}^*_+} \approx 1$ and the derivative $H'$ is associated with all generalized function that have the Dirac distribution as macroscopic aspect. The main results of this work are theorems 5.2 and 5.3. We will briefly describe the content of this paper. Generally speaking we can affirm that in the first four sections we collect the results to be used in the last one. In this work, unless otherwise stated, $E, F_1, \ldots, F_m$ and $G$ denote $\mathbb{K}-$ Banach spaces (where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$). $F$ denotes a $\mathbb{K}-$ Banach algebra (as a $\mathbb{K}-$ Banach space) and $\Omega$ (resp. $\Omega'$) denotes an open subset of $E$ (resp. $F$). In §1 we fix some basic definitions about the space of simplified generalized functions $\mathcal{G}_s(\Omega; F)$. In §2 we introduce the association relation in $\mathcal{G}_s(\Omega; \mathbb{K})$ (when $E = \mathbb{R}^n$) and we present some basic properties about the Heaviside generalized functions, although we omit well known proofs. In §3 we introduce the notion of composite function and we present a result about inverse multiplicative, for a certain class of generalized functions (adequate to the requirements of this work). This composition $\nu \circ (\rho, p, e)$ is "very delicate", it is a special case on composition of generalized functions. We need the inverse multiplicative when we study the system $(S)$. In §4 we discuss properties of functions of the form $\varphi \circ (a_1 H_1 + b_1, \ldots, a_m H_m + b_m)$ where $H_1, \ldots, H_m$ are Heaviside functions under certain conditions. The propositions 4.2, 4.3 and 4.4 are fundamental for the study of existence of shock wave solutions for the systems $(S)$ and $(\tilde{S})$. In §5 we study the initially proposed problem. We show that a necessary and sufficient condition in order that the system $(\tilde{S})$ has a solution $(\rho, u, p, e)$ is that the jump conditions and two relations of technical nature are held (see theorem 5.1). By using the proposition 4.3 and the theorem 5.1 we get a result on existence of solutions. According to which the system $(\tilde{S})$ has shock wave solutions if and only if the jump conditions hold (see theorem 5.2). Also we exhibit that the first two equations of the system $(S)$ can not have shock wave solutions, which takes us to a result on nonexistence of solutions (see theorem 5.3).

In [2] it was studied an analogous problem for systems when the viscosity function $\nu$ depends only on $\rho$. More precisely, $\nu$ is a strictly increasing function (under certain conditions) when the system is considered with all equations in the sense of association and $\nu(x) = x^2$ when we consider the equality in the first equation. This work constitutes a considerable advancement of the results contained in [2]. In the present paper an additional complication arises from the expression $\nu \circ (\rho, p, e)$, which requires the hard study of composition and inverse multiplicative of generalized functions in the sense of Colombeau's theory. A considerable amount of technical computations is developed to deal with the proposed problem. The basic references for Colombeau's theory are [1], [3], [4] and [5]. The general notations not mentioned in this work are those of [1].

# 1 The algebra $\mathcal{G}_s(\Omega; F)$

We denote by (the same symbol) $|\cdot|$ the norms in the considered spaces. The symbol $\mathcal{L}(F_1,\dots,F_m;G)$ denotes the space of continuous $m$ - linear mappings from product space $\mathbb{F}_m := F_1\times\cdots\times F_m$ into $G$ endowed with the norm

$$|\cdot|_m\colon A\in\mathcal{L}(F_1,\dots,F_m;G)\mapsto \sup_{\substack{|y_i|=1\\1\le i\le m}}|A(y_1,\dots,y_m)|\in\mathbb{R}_+\,.$$

If $F_1=\cdots=F_m=F$ this space is denoted by $\mathcal{L}(^mF;G)$ and $\mathcal{L}(^0F;G)=:G$. Let $\mathcal{E}_s[\Omega;F] := \{u\in F^{]0,1]\times\Omega}\mid u(\varepsilon,\cdot)\in C^\infty(\Omega;F)$ for all $\varepsilon\in]0,1]\}$ be. If $p\in\mathbb{N}$ and $x\in\Omega$ we set $u^{(p)}(\varepsilon,x):=[u(\varepsilon,\cdot)]^{(p)}(x)$. The notation $K\subset\subset\Omega$ means that $K$ is a compact subset of $\Omega$ and $|u^{(p)}(\varepsilon,\cdot)|_{p,K}:=\sup_{x\in K}|u^{(p)}(\varepsilon,x)|_p$. Let $\mathcal{E}_{s,M}[\Omega;F]$ denote the algebra of all $u\in\mathcal{E}_s[\Omega;F]$ such that for each $K\subset\subset\Omega$ and each $p\in\mathbb{N}$ there is $N\in\mathbb{N}$ such that $|u^{(p)}(\varepsilon,\cdot)|_{p,K}=o(\varepsilon^{-N})$ as $\varepsilon\downarrow 0$. By $\mathcal{N}_s[\Omega;F]$ we denote the set of all $u\in\mathcal{E}_s[\Omega;F]$ such that for each $K\subset\subset\Omega$ and each $(p,q)\in\mathbb{N}\times\mathbb{N}$ we have $|u^{(p)}(\varepsilon,\cdot)|_{p,K}=o(\varepsilon^q)$ as $\varepsilon\downarrow 0$. The *Colombeau algebra of generalized mappings on* $\Omega$ *with values in* $F$ is defined by

$$\mathcal{G}_s(\Omega;F):=\frac{\mathcal{E}_{s,M}[\Omega;F]}{\mathcal{N}_s[\Omega;F]}\,.$$

If $F=\mathbb{K}$ we write $\mathcal{G}_s(\Omega)$ instead of $\mathcal{G}_s(\Omega;\mathbb{K})$ and a similar notation is used for sets that generate (as well as for subsets of) $\mathcal{G}_s(\Omega;\mathbb{K})$. We indicate by $\mathcal{G}_{s,\ell b}(\Omega;F)$ the set of maps $f\in\mathcal{G}_s(\Omega;F)$ which have a representative $\widehat{f}$ such that for each $K\subset\subset\Omega$ there are $C>0$ and $\eta\in]0,1]$ satisfying $|\widehat{f}(\varepsilon,x)|\le C$ for all $(\varepsilon,x)$ in $]0,\eta[\times K$. If $f_i\in\mathcal{G}_s(\Omega;F_i), 1\le i\le m$, we denote by $(f_1,\dots,f_m)$ the class of

$$(\widehat{f}_1,\dots,\widehat{f}_m)\colon(\varepsilon,x)\in]0,1]\times\Omega\mapsto\Big(\widehat{f}_1(\varepsilon,x),\dots,\widehat{f}_m(\varepsilon,x)\Big)\in\mathbb{F}_m$$

where $\widehat{f}_i$ is an arbitrary representative of $f_i$. In this case, $f_1,\dots,f_m$ are called the *components* of $(f_1,\dots,f_m)$. We denote by $\mathcal{E}_{s,M}[\Omega;\Omega']$ the set of all $u$ in $\mathcal{E}_{s,M}[\Omega;F]$ such that $u(]0,1]\times\Omega)\subset\Omega'$. By $\mathcal{E}_{s,M,*}[\Omega;\Omega']$ we denote the set of all $u\in\mathcal{E}_{s,M}[\Omega;\Omega']$ such that for each $K\subset\subset\Omega$ there are $K'\subset\subset\Omega'$ and $\eta\in]0,1]$ such that $u(]0,\eta[\times K)\subset K'$. We indicate by $\mathcal{G}_{s,*}(\Omega;\Omega')$ the set of all elements of $\mathcal{G}_s(\Omega;F)$ which have at least a representative in $\mathcal{E}_{s,M,*}[\Omega;\Omega']$. If $(u,w)\in\mathcal{E}_s[\Omega;\Omega']\times\mathcal{E}_s[\Omega';G]$, let $w\circ u\in\mathcal{E}_s[\Omega;G]$ be defined by

$$(w\circ u)(\varepsilon,x):=w(\varepsilon,u(\varepsilon,x))\,,\quad((\varepsilon,x)\in]0,1]\times\Omega)\,.$$

If $\dim F<+\infty$ and $(f,g)\in\mathcal{G}_{s,*}(\Omega;\Omega')\times\mathcal{G}_s(\Omega';G)$ we define the *composite function*

$$g\circ f:=\widehat{g}\circ\widehat{f}+\mathcal{N}_s[\Omega;G]$$

where $\widehat{f}\in\mathcal{E}_{s,M}[\Omega;\Omega']$ and $\widehat{g}$ are arbitrary representatives of $f$ and $g$, respectively.

Let $\mathcal{E}_{s,M}(F)$ be the set of all $\mu \in F^{]0,1]}$ such that there is $N \in \mathbb{N}$ satisfying $|\mu(\varepsilon)| = o(\varepsilon^{-N})$ as $\varepsilon \downarrow 0$ and let $\mathcal{N}_s(F)$ be the set of all functions $\mu \in \mathcal{E}_{s,M}(F)$ such that for each $q \in \mathbb{N}$ we have $|\mu(\varepsilon)| = o(\varepsilon^q)$ as $\varepsilon \downarrow 0$. The algebra of the *Colombeau generalized vectors in* $F$ is defined by

$$\bar{F}_s := \frac{\mathcal{E}_{s,M}(F)}{\mathcal{N}_s(F)} .$$

We can identify $F$ with a subspace of $\bar{F}_s$ and $\bar{F}_s$with a subspace of $\mathcal{G}_s(\Omega; F)$. The elements of the image of $\bar{F}_s$ in $\mathcal{G}_s(\Omega; F)$ are called *generalized constants*.

## 2 Association and Heaviside GFs

In this section will be considered the cases $E = \mathbb{R}^n$ and $F = \mathbb{K}$. We say that an element $f$ in $\mathcal{G}_s(\Omega)$ is *associated with* 0 (indicated by $f \approx 0$) if for some representative $\widehat{f}$ of $f$ we have $\widehat{f}(\varepsilon, \cdot) \to 0$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \downarrow 0$. We say that two elements $f$ and $g$ in $\mathcal{G}_s(\Omega)$ are associated with each other if $f - g \approx 0$.

The next result follows from dominated convergence theorem.

**Proposition 2.1** *Let $(\phi, f) \in \mathbb{K}^\Omega \times \mathcal{G}_{s,\ell b}(\Omega)$ be such that $\widehat{f}(\varepsilon, \cdot) \to \phi$ a.e. in $\Omega$ as $\varepsilon \downarrow 0$ for some representative $\widehat{f}$ of $f$. Then, $\widehat{f}(\varepsilon, \cdot) \to \phi$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \downarrow 0$. In particular, if $\phi \in C^\infty(\Omega)$ then $f \approx \phi$.*

We indicate by $\mathbb{Y}$ the classical Heaviside step function: $\mathbb{Y}(\lambda) = 0$ if $\lambda < 0$ and $\mathbb{Y}(\lambda) = 1$ if $\lambda > 0$. An element $H \in \mathcal{G}_s(\mathbb{R})$ is said to be a *Heaviside generalized function in* $\mathbb{R}$ if there is a representative $\widehat{H}$ of $H$ such that $\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \downarrow 0$. We indicate by $\mathcal{H}(\mathbb{R})$ the set of all Heaviside generalized functions in $\mathbb{R}$. We denote by $\mathcal{H}_p(\mathbb{R})$ the set of all elements $H$ in $\mathcal{G}_{s,\ell b}(\mathbb{R})$ which have a representative $\widehat{H}$ such that $\widehat{H}(\varepsilon, \cdot) \to \mathbb{Y}$ in $\mathbb{R}^*$ as $\varepsilon \downarrow 0$. We denote by

$$\Lambda := \{\varphi \in \mathcal{D}(\mathbb{R}) \mid \varphi \geq 0\,,\ \ \varphi(0) > 0\,,\ \ \mathrm{supp}(\varphi) \subset [-1, 1]\ \text{ and }\ \int \varphi = 1\}\,.$$

**Lemma 2.1** *There exists $u \in \mathcal{E}_{s,M}[\mathbb{R}]$ such that $0 \leq u(\varepsilon, \cdot) \leq 1$ in $\mathbb{R}$, $u(\varepsilon, \cdot) \equiv 1$ in $]-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}[$ and supp$[u(\varepsilon, \cdot)] \subset [-\frac{3\varepsilon}{4}, \frac{3\varepsilon}{4}]$, for all $\varepsilon \in ]0, 1]$. Then, if $v$ is defined by $v := 1$ in $]0, 1] \times \mathbb{R}_-$ and $v := u$ in $]0, 1] \times \mathbb{R}_+$, we have $0 \leq v(\varepsilon, \cdot) \leq 1$ in $\mathbb{R}$, supp$[v(\varepsilon, \cdot)] \subset ]-\infty, \varepsilon[$ and $v(\varepsilon, \cdot) \equiv 1$ in $]-\infty, \frac{\varepsilon}{4}[$, for all $\varepsilon \in ]0, 1]$.*

**Proof** If $\varphi \in \Lambda$ the function $u\colon (\varepsilon, x) \mapsto [\chi(\varepsilon, \cdot) * \widehat{\varphi}(\frac{\varepsilon}{4}, \cdot)](x)$ satisfies the required properties, where $\widehat{\varphi}\colon (\varepsilon, x) \mapsto \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $\chi(\varepsilon, \cdot)$ is the characteristic function of $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. $\square$

We say that a function $\widehat{H} \in \mathcal{E}_{s,M}[\mathbb{R}]$ *verifies the property* $(\mathcal{H}_r)$ if there is $\mu \in \mathbb{R}_+^{*\,]0,1]}$ such that $\lim_{\varepsilon\downarrow 0} \mu(\varepsilon) = 0$ and $\widehat{H}(\varepsilon, \lambda) = 0$ (resp. $\widehat{H}(\varepsilon, \lambda) = 1$) for $\lambda < -\mu(\varepsilon)$ (resp. $\lambda > \mu(\varepsilon)$), ($\varepsilon \in ]0, 1]$). We indicate by $\mathcal{H}_r(\mathbb{R})$ the set of all elements of $\mathcal{G}_{s,\ell b}(\mathbb{R})$ which have a representative verifying the property $(\mathcal{H}_r)$.

**Proposition 2.2** *Let $\mu \in \mathcal{E}_{s,M}(\mathbb{R})$ be such that $\mu \geq 1$ in $]0,1]$. Then, there exists $\widehat{V} \in \mathcal{E}_{s,M}[\mathbb{R};\mathbb{R}]$ verifying the following properties: $(\mathcal{H}_r)$, $0 \leq \widehat{V} \leq \mu$ in $\mathbb{R}\times]0,1]$, $\widehat{V}(\varepsilon,\cdot) \equiv \mu(\varepsilon)$ in $[-\varepsilon,\varepsilon]$ for all $\varepsilon \in ]0,1]$, and for each $K \subset\subset \mathbb{R}^*$ there is $\eta \in ]0,1]$ such that $0 \leq \widehat{V} \leq 1$ in $]0,\eta[\times K$.*

**Proof** If $\varphi \in \Lambda$ we consider the function $u\colon (\varepsilon,x) \mapsto [\chi(\varepsilon,\cdot) * \widehat{\varphi}(\frac{\varepsilon}{4},\cdot)](x)$ where $\widehat{\varphi}\colon (\varepsilon,x) \mapsto \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $\chi(\varepsilon,\cdot)$ is the characteristic function of $[-\frac{3\varepsilon}{2},\frac{3\varepsilon}{2}]$. The function $\widehat{V}: (\varepsilon,x) \mapsto 1 + v(\varepsilon,x)[\mu(\varepsilon)u(\varepsilon,x) - 1]$ satisfies the required properties (where $v$ is as in lemma 2.1). $\square$

**Proposition 2.3** *If $\varphi \in \Lambda$ and $\widehat{H}_\varphi\colon ]0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by*

$$\widehat{H}_\varphi(\varepsilon,\lambda) := \int_{-\infty}^{\lambda} \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right) dt\,, \qquad ((\varepsilon,\lambda) \in ]0,1] \times \mathbb{R})$$

*we have $0 \leq \widehat{H}_\varphi \leq 1$ in $]0,1] \times \mathbb{R}$ and $\widehat{H}_\varphi(\varepsilon,\lambda) = 0$ ( resp. $\widehat{H}_\varphi(\varepsilon,\lambda) = 1$) for $\lambda \leq -\varepsilon$ ( resp. $\lambda \geq \varepsilon$), for all $\varepsilon \in ]0,1]$. Furthermore, if $H_\varphi$ is the class of $\widehat{H}_\varphi$ then $H_\varphi \in \mathcal{H}_r(\mathbb{R})$.*

**Remark 2.1** Denoting by $\mathcal{H}_\Lambda(\mathbb{R}) := \{H_\varphi \in \mathcal{G}_s(\mathbb{R}) \mid \varphi \in \Lambda\}$, where $H_\varphi$ is defined as in previous proposition, we have

$$\mathcal{H}_\Lambda(\mathbb{R}) \subset \mathcal{H}_r(\mathbb{R}) \subset \mathcal{H}_p(\mathbb{R}) \subset \mathcal{H}(\mathbb{R})\,.$$

**Remark 2.2** If $H,K \in \mathcal{H}_p(\mathbb{R})$, we do not necessarily have $HK'|_{\mathbb{R}^*} \approx 0$. Indeed, if $v$ as in lemma 2.1, we consider $H$ and $K$ represented respectively by

$$\widehat{H}\colon (\varepsilon,x) \mapsto 1 + v(\varepsilon,x)\left[\varepsilon^{\frac{1}{4}}\cos(\frac{x}{\varepsilon}) - 1\right]\,,\ \widehat{K}\colon (\varepsilon,x) \mapsto 1 + v(\varepsilon,x)\left[\sqrt{\varepsilon}\mathrm{sen}(\frac{x}{\varepsilon}) - 1\right]\,.$$

The presented result as follows is a useful tool for the study of solvability of the systems $(S)$ and $(\tilde{S})$.

**Proposition 2.4** *If $f,g \in \mathcal{G}_s(\mathbb{R}^n)$ and $S$ is a $C^\infty$ - diffeomorphism of $\mathbb{R}^n$ onto itself such that $J_S(x) > 0$ for all $x \in \mathbb{R}^n$ ($J_S$ denotes the jacobian of $S$) the following statements are held.*

**(a)** $(g \circ S)|_\Omega \approx 0$ *(resp. $(g \circ S)|_\Omega = 0$) if and only if $g|_{S(\Omega)} \approx 0$ (resp. $g|_{S(\Omega)} = 0$) and $(f \circ S^{-1})|_{S(\Omega)} \approx 0$ (resp. $(f \circ S^{-1})|_{S(\Omega)} = 0$) if and only if $f|_\Omega \approx 0$ (resp. $f|_\Omega = 0$).*

**(b)** *If $W$ is an open subset of $\mathbb{R}^n$ and $\pi\colon (\lambda,t) \in \mathbb{R}^n \times \mathbb{R}^m \mapsto \lambda \in \mathbb{R}^n$ then $(f \circ \pi)|_{W\times\mathbb{R}^m} \approx 0$ if and only if $f|_W \approx 0$ and $(f \circ \pi)|_{W\times\mathbb{R}^m} = 0$ if and only if $f|_W = 0$.*

**Proof** The result follows by a minor modification in the proof of the Proposition 2.11 in [2]. $\square$

Given $y \in C^\infty(\mathbb{R};\mathbb{R})$ consider the associated functions

$$y^* : (x,t) \in \mathbb{R}^2 \mapsto x - y(t) \in \mathbb{R} \text{ and } S : (x,t) \in \mathbb{R}^2 \mapsto (y^*(x,t),t) \in \mathbb{R}^2 . \quad (2.1)$$

We have $S(\Omega) = y^*(\Omega) \times \mathbb{R}$ for $\Omega = \Omega_-,\ \Omega_+,\ \Omega^*,\ \mathbb{R}^2$ where

$$\Omega_- := \{(x,t) \mid y^*(x,t) < 0\}\,,\ \ \Omega_+ := \{(x,t) \mid y^*(x,t) > 0\} \text{ and } \Omega^* := \Omega_- \cup \Omega_+ .$$

By using the proposition 2.4 we have the following result.

**Corollary 2.1** *If $\Omega$ is an open subset of $\mathbb{R}^2$ such that $S(\Omega) = y^*(\Omega) \times \mathbb{R}$ and $f \in \mathcal{G}_s(\mathbb{R})$ then, $(f \circ y^*)|_\Omega \approx 0$ if and only if $f|_{y^*(\Omega)} \approx 0$ and $(f \circ y^*)|_\Omega = 0$ if and only if $f|_{y^*(\Omega)} = 0$.*

The result below follows from definitions of generalized Heaviside functions and corollary 2.1.

**Corollary 2.2** *If $H \in \mathcal{H}(\mathbb{R})$ the following statements are held.*

**(a)** $(H \circ y^*)|_{\Omega_-} \approx 0\,,\ (H \circ y^*)|_{\Omega_+} \approx 1$ *and* $(H \circ y^*)_x|_{\Omega^*} \approx 0$.

**(b)** *If* $(\alpha_j, H_j) \in \mathbb{N}^* \times \mathcal{H}_p(\mathbb{R})$, $1 \le j \le m$, *then,* $(H_1^{\alpha_1} \cdots H_m^{\alpha_m}) \circ y^* \approx H \circ y^*$.

**Proposition 2.5** *Let $\Phi \in \mathcal{G}_s(\mathbb{R}^2)$ be such that $\Phi_x = 0$ and $\Phi|_\Omega \approx 0$ for some open subset $\Omega$ of $\mathbb{R}^2$ such that $S(\Omega) = y^*(\Omega) \times \mathbb{R}$. Then $\Phi \approx 0$.*

**Proof** Aiming to verify that $\Phi \circ S^{-1} \approx 0$, let $\widehat{\Phi}$ be a representative of $\Phi$ and we suppose that $\Omega \neq \mathbb{R}^2$. If $\psi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R})$, with $\operatorname{supp}(\varphi) \subset y^*(\Omega)$ and $\int \varphi = 1$, let $(\varphi_0, \psi_0) \in \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R}^2)$ be such that

$$\psi(\lambda,t) = \varphi(\lambda)\varphi_0(t) + \frac{\partial \psi_0}{\partial \lambda}(\lambda,t)\,.$$

Therefore, since $(\Phi \circ S^{-1})|_{S(\Omega)} \approx 0$ and $(\Phi \circ S^{-1})_\lambda = \Phi_x \circ S^{-1} = 0$, we have $< (\widehat{\Phi} \circ S^{-1})(\varepsilon, \cdot)\,,\ \psi > \to 0$ as $\varepsilon \downarrow 0$. $\square$

From previous proposition and corollary 2.2(a) follows the next result (see also [1], 6.3.1).

**Proposition 2.6** *Let $(f, H, a, b) \in \mathcal{G}_s(\mathbb{R}^2) \times \mathcal{H}(\mathbb{R}) \times \mathbb{R}^2$ be such that $f|_{\Omega^*} \approx 0$ and $a(H \circ y^*) + b(H \circ y^*)_x \approx f_x$. Then $a = b = 0$ and $f \approx 0$.*

# 3 Special cases of composition and invertibility

In this section we will introduce the notion of composite function and we will present a result about inverse multiplicative, for a certain class of generalized functions. Further information about these subjects can be found in [6]. In this part suppose that $F = \mathbb{R}^m$. Fix $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ in $(\tilde{\mathbb{R}}_+)^m$ with $\alpha < \beta$. We will use the notations

$$I_\alpha^\beta := \prod_{i=1}^m ]\alpha_i, \beta_i[\,, \quad [\alpha, \beta] := \prod_{i=1}^m [\alpha_i, \beta_i] \text{ and } ]\alpha, \beta] := \prod_{i=1}^m ]\alpha_i, \beta_i]$$

and we will consider $\Omega' = I_\alpha^\beta \subset (\tilde{\mathbb{R}}_+^*)^m$ (where $\tilde{\mathbb{R}}_+^* := \mathbb{R}_+^* \cup \{+\infty\}$).

Let $\mathcal{E}_{s,M,\oslash}[\Omega; I_\alpha^\beta]$ denote the set of all $u \in \mathcal{E}_{s,M}[\Omega; \mathbb{R}^m]$ such that for each $K \subset\subset \Omega$ there are $(\eta, a, b) \in ]0,1] \times I_\alpha^\beta \times I_\alpha^\beta$, $a < b$, and $\mu = (\mu_1, \dots, \mu_m)$ in $]\alpha, a]^{]0,1]}$ such that $[\varepsilon \mapsto (\mu_i(\varepsilon) - \alpha_i)^{-1}] \in \mathcal{E}_{s,M}(\mathbb{R})$ for all $i = 1, \dots, m$ and $u(\varepsilon, x) \in [\mu(\varepsilon), b]$ for all $(\varepsilon, x) \in ]0, \eta[ \times K$.

We indicate by $\mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta)$ the set of all elements of $\mathcal{G}_s(\Omega; \mathbb{R}^m)$ which have a representative in

$$\mathcal{E}_{s,M,\otimes}[\Omega; I_\alpha^\beta] := \mathcal{E}_{s,M}[\Omega; I_\alpha^\beta] \cap \mathcal{E}_{s,M,\oslash}[\Omega; I_\alpha^\beta]\,.$$

We denote by $\mathcal{E}_{s,QM}[I_\alpha^\beta; G]$ the set of all $w \in \mathcal{E}_s[I_\alpha^\beta; G]$ such that for each $(p, a, b) \in \mathbb{N} \times I_\alpha^\beta \times I_\alpha^\beta$, $a < b$, and each $\mu = (\mu_1, \dots, \mu_m) \in ]\alpha, a]^{]0,1]}$ such that $[\varepsilon \mapsto (\mu_i(\varepsilon) - \alpha_i)^{-1}] \in \mathcal{E}_{s,M}(\mathbb{R})$, $(1 \le i \le m)$, there are $N \in \mathbb{N}$, $C > 0$ and $\eta \in ]0,1]$ satisfying

$$\sup_{y \in [\mu(\varepsilon), b]} |w^{(p)}(\varepsilon, y)|_p \le C\varepsilon^{-N}\,, \quad (0 < \varepsilon < \eta)\,.$$

We define $C^\infty_{s,QM}[I_\alpha^\beta; G] := C^\infty(I_\alpha^\beta; G) \cap \mathcal{E}_{s,QM}[I_\alpha^\beta; G]$.

**Proposition 3.1** *We have $w \in \mathcal{E}_{s,QM}[I_\alpha^\beta; G]$ if and only if for each $(\gamma, a, b)$ in $\mathbb{N}^m \times I_\alpha^\beta \times I_\alpha^\beta$, $a < b$, and for each $\mu = (\mu_1, \dots, \mu_m) \in ]\alpha, a]^{]0,1]}$ such that $[\varepsilon \mapsto (\mu_i(\varepsilon) - \alpha_i)^{-1}] \in \mathcal{E}_{s,M}(\mathbb{R})$, $(1 \le i \le m)$, there are $N \in \mathbb{N}$, $C > 0$ and $\eta \in ]0,1]$ satisfying*

$$\sup_{y \in [\mu(\varepsilon), b]} |\partial^\gamma w(\varepsilon, y)| \le C\varepsilon^{-N}\,, \quad (0 < \varepsilon < \eta)\,.$$

If $(f, \varphi) \in \mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta) \times C^\infty_{s,QM}[I_\alpha^\beta; G]$ we define the *composite function*

$$\varphi \circ f := \varphi \circ \widehat{f} + \mathcal{N}_s[\Omega; G]$$

where $\widehat{f} \in \mathcal{E}_{s,M}[\Omega; I_\alpha^\beta]$ is any representative of $f$.

To check the following result see [6], 3.6.

**Proposition 3.2** *If $\dim E < +\infty$ and $f \in \mathcal{G}_{s,\otimes}(\Omega; I_\alpha^\beta)$ then $f$ (resp. $f - \alpha$) has an inverse multiplicative and $\widehat{f}^{-1}$ (resp. $(\widehat{f} - \alpha)^{-1}$) is a representative of $f^{-1}$ (resp. $(f - \alpha)^{-1}$), for every $\widehat{f} \in \mathcal{E}_{s,M}[\Omega; I_\alpha^\beta]$ representative of $f$.*

# 4 On some shock wave functions

In this and the next section, if $\Omega$ is an open subset of $\mathbb{R}$ (or $\mathbb{R}^m$) we will write $\mathcal{G}_s(\Omega)$ instead of $\mathcal{G}_s(\Omega; \mathbb{R})$. A similar notation will be used for sets that generate (as well as for subsets of) $C^\infty(\Omega; \mathbb{R})$, $\mathcal{E}_{s,QM}[\Omega; \mathbb{R}]$ and $\mathcal{G}_s(\Omega; \mathbb{R})$.

**Hypothesis 4.1** (For propositions 4.1 and 4.4) Fix $(\alpha,\beta)$ in $\mathbb{R}_+\times\tilde{\mathbb{R}}_+$ with $\alpha<\beta$ and $(a,b)$ in $\mathbb{R}_+^*\times\mathbb{R}_+^*$ with $a<b$. Let $\Delta:=a-b$ and $\nu\in C^\infty(\mathbb{R}_+^*)$ be a strictly increasing function such that $\mathrm{Im}(\nu)=I_\alpha^\beta$. Let $\nu_r:=\nu(a)$, $\nu_\ell:=\nu(b)$, $\Delta\nu:=\nu_r-\nu_\ell$ and $\theta:=(\Delta\nu)^{-1}(\alpha-\nu_\ell)$.

**Proposition 4.1** *If* $\nu^{-1}\in\mathcal{E}_{s,QM}[I_\alpha^\beta]$ *then for each* $\mu\in[1,\theta[^{]0,1]}$ *such that* $[\varepsilon\mapsto(\Delta\nu\mu(\varepsilon)+\nu_\ell-\alpha)^{-1}]\in\mathcal{E}_{s,M}(\mathbb{R})$ *and* $\nu(a\varepsilon)\le\Delta\nu\mu(\varepsilon)+\nu_\ell$ *for all* $\varepsilon$ *in* $]0,1]$, *there exists a function* $\widehat{H}\in\mathcal{E}_{s,M}[\mathbb{R}]$ *verifying the following properties:* $(\mathcal{H}_r)$, (4.1), (4.2) *and* (4.3), *where*

$$\sup_{x\in\mathbb{R}}|\widehat{H}(\varepsilon,x)|\le\frac{a\varepsilon-b}{\Delta}\,,\quad(\varepsilon\in]0,1])\tag{4.1}$$

$$(\forall K\subset\subset\mathbb{R}^*)(\exists\eta\in]0,1])\mid\sup_{(\varepsilon,x)\in]0,\eta[\times K}|\widehat{H}(\varepsilon,x)|<-\frac{b}{\Delta}\tag{4.2}$$

$$\widehat{H}(\varepsilon,\cdot)\equiv\frac{\nu^{-1}\big(\Delta\nu\mu(\varepsilon)+\nu_\ell\big)-b}{\Delta}\ \ in\ \,[-\varepsilon,\varepsilon]\,,\quad(\varepsilon\in]0,1])\,.\tag{4.3}$$

*Furthermore, if* $H$ *is the class of* $\widehat{H}$ *we have* $H\in\mathcal{H}_r(\mathbb{R})\setminus\mathcal{H}_\Lambda(\mathbb{R})$.

**Proof** Consider $\widehat{H}:=\Delta^{-1}[\nu^{-1}\circ(\Delta\nu\widehat{V}+\nu_\ell)-b]$ (where $\widehat{V}$ is given in proposition 2.2) and $\chi\colon\varepsilon\mapsto\Delta\nu\mu(\varepsilon)+\nu_\ell$. Then $\chi\le\Delta\nu\widehat{V}+\nu_\ell\le\nu_\ell$ in $]0,1]\times\mathbb{R}$ and $[\varepsilon\mapsto(\chi(\varepsilon)-\alpha)^{-1}]\in\mathcal{E}_{s,M}(\mathbb{R})$. Therefore, $\Delta\nu\widehat{V}+\nu_\ell\in\mathcal{E}_{s,M,\otimes}[\mathbb{R};I_\alpha^\beta]$. If $V$ is the class of $\widehat{V}$ then $\nu^{-1}\circ(\Delta\nu\widehat{V}+\nu_\ell)$ is a representative of $\nu^{-1}\circ(\Delta\nu V+\nu_\ell)$. The function $\widehat{H}$ satisfies the required conditions. □

**Hypothesis 4.2** (For definition 4.1 and propositions 4.2 and 4.3) The following data are considered.

**(a)** Two elements $a=(a_1,\dots,a_m)$ and $b=(b_1,\dots,b_m)$ in $(\mathbb{R}_+^*)^m$ such that $a<b$. We define $\Delta_\imath:=a_\imath-b_\imath$ , $\imath=1,\dots,m$.

**(b)** Two elements $\alpha_*=(\alpha_1,\dots,\alpha_m)$ and $\beta_*=(\beta_1,\dots,\beta_m)$ in $(\tilde{\mathbb{R}}_+)^m$ with $\alpha_*<\beta_*$ and a function $\nu_*=(\nu_1,\dots,\nu_m)\in C^\infty(\mathbb{R}_+^*;\mathbb{R}^m)$ such that each component of $\nu_*$ is an increasing function. Let us suppose that $\{1,\dots,m\}$ is a reunion of two disjoint subsets $\mathbf{I}$ and $\mathbf{J}$ such that $\nu_\imath$ (resp. $\nu_\jmath$) is strictly increasing, with image $I_{\alpha_\imath}^{\beta_\imath}$ (resp. not strictly increasing, with image contained in $[\alpha_\jmath,\beta_\jmath[$) for each $\imath\in\mathbf{I}$ (resp. $\jmath\in\mathbf{J}$).

**Definition 4.1** If $\alpha_{(s)}:=\alpha_1+\dots+\alpha_m$ and $\beta_{(s)}:=\beta_1+\dots+\beta_m$, we indicate by $\nu_{(s)}$ the function $(y_1,\dots,y_m)\in(\mathbb{R}_+^*)^m\mapsto\sum_{\imath=1}^m\nu_\imath(y_\imath)\in I_{\alpha_{(s)}}^{\beta_{(s)}}$. If $\alpha_\jmath>0$ for each $\jmath\in\mathbf{J}$, $\alpha_{(\pi)}:=\alpha_1\dots\alpha_m$ and $\beta_{(\pi)}:=\beta_1\dots\beta_m$, we consider the function $\nu_{(\pi)}\colon(y_1,\dots,y_m)\in(\mathbb{R}_+^*)^m\mapsto\prod_{\imath=1}^m\nu_\imath(y_\imath)\in I_{\alpha_{(\pi)}}^{\beta_{(\pi)}}$.

For $\nu=\nu_{(s)}$, $\nu_{(\pi)}$ we will consider $\nu_r:=\nu(a)$, $\nu_\ell:=\nu(b)$ and $\Delta\nu:=\nu_r-\nu_\ell$. In propositions 4.2 and 4.3, $(\nu,\alpha,\beta)$ it indistinctly indicates $(\nu_{(s)},\alpha_{(s)},\beta_{(s)})$ or $(\nu_{(\pi)},\alpha_{(\pi)},\beta_{(\pi)})$. Let us recall that $\mathbb{Y}$ denotes the classical Heaviside function.

**Proposition 4.2** *Let $(H_1,\dots,H_m)$ in $\mathcal{G}_s(\mathbb{R};\mathbb{R}^m)$ be. Assume that each $H_\imath$, $1\le\imath\le m$, has a representative $\widehat{H}_\imath$ such that $(\widehat{H}_\imath,a_\imath,b_\imath,\Delta_\imath)$ verifies (4.1). Then the following assertions hold.*

**(a)** $\widehat{f}=(\widehat{f}_1,\dots,\widehat{f}_m):=(\Delta_1\widehat{H}_1+b_1,\dots,\Delta_m\widehat{H}_m+b_m)\in\mathcal{E}_{s,M,\otimes}[\mathbb{R};(\mathbb{R}_+^*)^m]$.

*If $f=(f_1,\dots,f_m):=(\Delta_1H_1+b_1,\dots,\Delta_mH_m+b_m)$ and each component of $\nu_*$ belongs to $\mathcal{E}_{s,QM}[\mathbb{R}_+^*]$ we have furthermore,*

**(b)** $\nu\in\mathcal{E}_{s,QM}[(\mathbb{R}_+^*)^m]$ *(and hence $\nu\circ\widehat{f}$ is a representative of $\nu\circ f$)*

**(c)** *if $\widehat{H}_\imath(\varepsilon,\cdot)\to\mathbb{Y}$ in $\mathbb{R}^*$ as $\varepsilon\downarrow 0$, $(1\le\imath\le m)$, then $(\nu\circ\widehat{f})(\varepsilon,\cdot)\to\nu_\ell$ (resp. $\nu_r$) in $\mathbb{R}_-^*$ (resp. $\mathbb{R}_+^*$) as $\varepsilon\downarrow 0$ and $(\Delta\nu)^{-1}(\nu\circ f-\nu_\ell)\in\mathcal{H}_p(\mathbb{R})$*

**(d)** *if $1\in\mathbf{I}$ and $[\varepsilon\mapsto(\nu_1(r\varepsilon)-\alpha_1)^{-1}]\in\mathcal{E}_{s,M}(\mathbb{R})$ for each $r>0$ then, the function $\nu\circ\widehat{f}\in\mathcal{E}_{s,M,\otimes}[\mathbb{R};I_\alpha^\beta]$, $(\nu_1\circ f_1-\alpha_1)(\nu\circ f-\alpha)^{-1}\in\mathcal{G}_{s,\ell b}(\mathbb{R})$ and*

$$\left(\frac{\nu_1\circ\widehat{f}_1-\alpha_1}{\nu\circ\widehat{f}-\alpha}\right)(\varepsilon,\cdot)\to\begin{cases}\lambda_\ell & \text{in } \mathbb{R}_-^*\\ \lambda_r & \text{in } \mathbb{R}_+^*\end{cases}\quad \text{as } \varepsilon\downarrow 0 \tag{4.4}$$

*where $\lambda_\ell:=[\nu_1(b_1)-\alpha_1](\nu_\ell-\alpha)^{-1}$ and $\lambda_r:=[\nu_1(a_1)-\alpha_1](\nu_r-\alpha)^{-1}$. In the case of $(\nu,\alpha,\beta)=(\nu_{(\pi)},\alpha_{(\pi)},\beta_{(\pi)})$ we set the additional assumption $\alpha_\imath>0$ whenever $2\le\imath\le m$.*

**Proof** The statements (a), (b), (c) and (4.4) are clear. On the other hand, since $\nu_1(a_1\varepsilon)-\alpha_1\le\nu_{(s)}(a\varepsilon)-\alpha_{(s)}$ and $(\nu_1(a_1\varepsilon)-\alpha_1)\alpha_2\dots\alpha_m\le\nu_{(\pi)}(a\varepsilon)-\alpha_{(\pi)}$ we get $\nu\circ\widehat{f}\in\mathcal{E}_{s,M,\otimes}[\mathbb{R};I_\alpha^\beta]$. From the conditions $(\nu_1\circ\widehat{f}_1-\alpha_1)\le\nu_{(s)}\circ\widehat{f}-\alpha_{(s)}$ and $(\nu_1\circ\widehat{f}_1-\alpha_1)\alpha_2\dots\alpha_m\le\nu_{(\pi)}\circ\widehat{f}-\alpha_{(\pi)}$ in $]0,1]\times\mathbb{R}$ it follows that the element $(\nu_1\circ f_1-\alpha_1)(\nu\circ f-\alpha)^{-1}\in\mathcal{G}_{s,\ell b}(\mathbb{R})$. $\square$

**Proposition 4.3** *Suppose that $\nu_\imath\in\mathcal{E}_{s,QM}[\mathbb{R}_+^*]$ and $\nu_\imath^{-1}\in\mathcal{E}_{s,QM}[I_{\alpha_\imath}^{\beta_\imath}]$ for each $\imath\in\mathbf{I}$ and that $\nu_\jmath\equiv\alpha_\jmath$ in $\mathbb{R}_+^*$ for each $\jmath\in\mathbf{J}$. Then, there are $H_1,\dots,H_m$ in $\mathcal{H}_r(\mathbb{R})\setminus\mathcal{H}_\Lambda(\mathbb{R})$ such that*

$$\begin{array}{l}[\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_mH_m+b_m)-\alpha]\,H'\approx 0\\ [\nu\circ(\Delta_1H_1+b_1,\dots,\Delta_mH_m+b_m)-\alpha]\,HH'\approx 0\end{array}\qquad (H\in\mathcal{H}_\Lambda(\mathbb{R}))$$

*and each $H_\imath$, $1\le\imath\le m$, has a representative $\widehat{H}_\imath$ such that $(\widehat{H}_\imath,a_\imath,b_\imath,\Delta_\imath)$ (resp. $(\widehat{H}_\imath,b_\imath,\Delta_\imath)$) satisfies (4.1) (resp. (4.2)).*

**Proof** If $\mathbf{I}=\emptyset$ the result follows by using the proposition 2.2. We suppose that $\mathbf{I}\neq\emptyset$. For the sake of simplicity also we suppose that $\mathbf{I}=\{1,\dots,n\}(n<m)$ and that $(\nu,\alpha,\beta)=(\nu_{(s)},\alpha_{(s)},\beta_{(s)})$. Let us consider

$$\left(\nu_{(n)},\alpha_{(n)},\beta_{(n)}\right):=\left((y_1,\dots,y_n)\mapsto\sum_{\imath=1}^{n}\nu_\imath(y_\imath)\,,\ \alpha_1+\dots+\alpha_n\,,\ \beta_1+\dots+\beta_n\right).$$

We can choose $\eta\in]0,1]$ such that $\nu_\imath(a_\imath)\varepsilon+\nu_\imath(a_\imath\varepsilon)\le\nu_\imath(a_\imath)$ for all $(\varepsilon,\imath)$ in $]0,\eta[\times\{1,\dots,n\}$. For each $\imath=1,\dots,n$ fixed we consider $\Delta\nu_\imath:=\nu_\imath(a_\imath)-\nu_\imath(b_\imath)$ and $\mu_\imath(\varepsilon):=(\Delta\nu_\imath)^{-1}[\nu_\imath(a_\imath)\varepsilon+\nu_\imath(a_\imath\varepsilon)-\nu_\imath(b_\imath)]$ ( resp. $\mu_\imath(\varepsilon):=1$ ) for $0<\varepsilon<\eta$ (resp. for $\eta\le\varepsilon\le1$). By proposition 4.1 there is $\widehat{H}_\imath\in\mathcal{E}_{s,M}[\mathbb{R}]$ such that

$$\widehat{H}_\imath(\varepsilon,\cdot)\equiv\Delta_\imath^{-1}\left[\nu_\imath^{-1}\left(\Delta\nu_\imath\mu_\imath(\varepsilon)+\nu_\imath(b_\imath)\right)-b_\imath\right]\ \text{ in } [-\varepsilon,\varepsilon]\,,\ (\varepsilon\in]0,1]) \tag{4.5}$$

and the elements $\widehat{H}_\imath$ and $H_\imath:=\widehat{H}_\imath+\mathcal{N}_s[\mathbb{R}]$ satisfy the second statement. If $\mu(\varepsilon):=\sum_{\imath=1}^n[\mu_\imath(\varepsilon)\Delta\nu_\imath+\nu_\imath(b_\imath)]$ from (4.5) we get

$$\left(\nu_{(n)}\circ\widehat{f}\right)(\varepsilon,\cdot)\equiv\mu(\varepsilon)\ \text{ in }\ [-\varepsilon,\varepsilon]\,,\quad(\varepsilon\in]0,1]) \tag{4.6}$$

(where $\widehat{f}:=(\Delta_1\widehat{H}_1+b_1,\dots,\Delta_n\widehat{H}_n+b_n)$) and $\lim_{\varepsilon\downarrow0}\mu(\varepsilon)=\alpha_{(n)}$ (considering $\lim_{\varepsilon\downarrow0}\mu_\imath(\varepsilon)=\Delta_\imath^{-1}(\alpha_\imath-\nu_\imath(b_\imath))$, $1\le\imath\le n$). For fixed $\varphi\in\Lambda$, if $\psi\in\mathcal{D}(\mathbb{R})$ and $\lambda(\varepsilon):=\mu(\varepsilon)-\alpha_{(n)}$, from (4.6) we have (when $\varepsilon\downarrow0$)

$$\begin{aligned}&\left\langle[(\nu_{(n)}\circ\widehat{f})(\varepsilon,\cdot)-\alpha_{(n)}]\widehat{\varphi}(\varepsilon,\cdot)\,,\psi\right\rangle=\lambda(\varepsilon)\left\langle\widehat{\varphi}(\varepsilon,\cdot)\,,\psi\right\rangle\to0\ \text{ and}\\ &\left\langle[(\nu_{(n)}\circ\widehat{f})(\varepsilon,\cdot)-\alpha_{(n)}](\widehat{H}_\varphi\widehat{H}'_\varphi)(\varepsilon,\cdot)\,,\psi\right\rangle=\tfrac12\lambda(\varepsilon)\left\langle(\widehat{H}^2_\varphi)'(\varepsilon,\cdot)\,,\psi\right\rangle\to0\end{aligned}$$

where $\widehat{\varphi}\colon(\varepsilon,x)\mapsto\varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $\widehat{H}_\varphi\colon(\varepsilon,\lambda)\mapsto\int_{-\infty}^{\lambda}\widehat{\varphi}(\varepsilon,x)dx$. Therefore

$$\begin{aligned}&\left[\nu_{(n)}\circ(\Delta_1H_1+b_1,\dots,\Delta_nH_n+b_n)-\alpha\right]H'_\varphi\approx0\ \text{ and}\\ &\left[\nu_{(n)}\circ(\Delta_1H_1+b_1,\dots,\Delta_nH_n+b_n)-\alpha\right]H_\varphi H'_\varphi\approx0\,.\end{aligned} \tag{4.7}$$

Finally, by choosing (for instance) $H_\jmath:=H_1$, for $n+1\le\jmath\le m$, from (4.7) we get the required relations. □

**Proposition 4.4** *Let $\widehat{H}\in\mathcal{E}_{s,M}[\mathbb{R}]$ be such that $(\widehat{H},a,b,\Delta)$ verifies (4.1) and let $f$ be the class of $\widehat{f}:=\Delta\widehat{H}+b$. If $\nu\in\mathcal{E}_{s,QM}[\mathbb{R}_+^*]$ then, for each $n\in\mathbb{N}$, $n\ge2$, there is a strictly increasing function $\varphi\in C^\infty_{s,QM}[\mathbb{R}_+^*]$ such that*

$$(\varphi\circ f)'=(\nu\circ f-\alpha)f^{-n}f'\,. \tag{4.8}$$

*Furthermore, if $\widehat{H}(\varepsilon,\cdot)\to\mathbb{Y}$ in $\mathbb{R}^*$ as $\varepsilon\downarrow0$ and if $(\widehat{H},b,\Delta)$ verifies (4.2) we have $(\varphi\circ f)|_{\mathbb{R}_-^*}\approx\varphi(b)$ and $(\varphi\circ f)|_{\mathbb{R}_+^*}\approx\varphi(a)$.*

**Proof** If $\varphi\colon y\in\mathbb{R}_+^*\mapsto\int_{y_0}^y(\nu(t)-\alpha)t^{-n}dt\in\mathbb{R}$ we have $\varphi':y\mapsto(\nu(y)-\alpha)y^{-n}$, where $y_0>0$. If $A,B\in\mathbb{R}_+^*$, $A<B$, and if $y\in[A,B]$ we get $|\varphi(y)|\le|\varphi(A)|$ or $|\varphi(y)|\le|\varphi(B)|$. Then, $\varphi\in C^\infty_{s,QM}[\mathbb{R}_+^*]$ (see proposition 3.1) and the equality $(\varphi\circ f)'=(\varphi'\circ f)f'$ implies (4.8). On the other hand, for fixed $K\subset\subset\mathbb{R}^*$, let $\eta\in]0,1]$ be verifying (4.2). Choosing

$$\widehat{C}:=\sup_{]0,\eta[\times K}|\widehat{H}|\,,\ C_0:=\max\{1,\widehat{C}\}\,,\ C_1:=b+\Delta C_0\ \text{ and }\ C_2:=b-\Delta C_0$$

we get $\sup_{x\in K}|(\varphi\circ\widehat{f})(\varepsilon,x)|\le\max\{|\varphi(C_1)|,|\varphi(C_2)|\}$ for all $\varepsilon\in]0,\eta[$. Hence, the conditions $(\varphi\circ\widehat{f})(\varepsilon,\cdot)\to\varphi(b)$ in $\mathbb{R}_-^*$ and $(\varphi\circ\widehat{f})(\varepsilon,\cdot)\to\varphi(a)$ in $\mathbb{R}_+^*$ as $\varepsilon\downarrow0$ imply the two last relations (see proposition 2.1). □

# 5 Systems of equations from hydrodynamics

In the remaining of this section we will assume the following hypothesis.

**Hypothesis 5.1** Fix $\lambda$ in $\mathbb{R}^*$. The data of hypothesis 4.2(b) are considered for $m=3$: $\alpha_* = (\alpha_1,\ \alpha_2,\ \alpha_3)$, $\beta_* = (\beta_1, \beta_2, \beta_3)$, $\nu_* = (\nu_1, \nu_2, \nu_3)$, etc., being each component of $\nu_*$ an element of $\mathcal{E}_{s,QM}[\mathbb{R}^*_+]$. We indicate by $(\nu, \alpha, \beta)$ either $(\nu_{(s)}, \alpha_{(s)}, \beta_{(s)})$ or $(\nu_{(\pi)}, \alpha_{(\pi)}, \beta_{(\pi)})$ (see definition 4.1).

We consider two systems. The system $(\tilde{S})$ consists of the equations

$$\rho_t + (\rho u)_x \approx 0 \tag{5.1}$$

$$(\rho u)_t + (p + \rho u^2)_x \approx \{[\nu \circ (\rho, p, e) - \alpha] u_x\}_x \tag{5.2}$$

$$e_t + [(e+p)u]_x \approx \{[\nu \circ (\rho, p, e) - \alpha] u u_x\}_x \tag{5.3}$$

$$e \approx \lambda p + \tfrac{1}{2}\rho u^2 \tag{5.4}$$

and $(S)$ consists of the equations (5.5), (5.6), (5.3) and (5.4), where

$$\rho_t + (\rho u)_x = 0 \tag{5.5}$$

$$(\rho u)_t + (p + \rho u^2)_x = \{[\nu \circ (\rho, p, e) - \alpha] u_x\}_x\,. \tag{5.6}$$

**Definition 5.1** A *shock wave solution* for the system $(\tilde{S})$ (or $(S)$) is any element $(\rho, u, p, e)$ in $\mathcal{G}_s(\mathbb{R}^2; \mathbb{R}^4)$ so that the component functions are given by

$$\begin{array}{llll} (s_1) & \rho = \Delta\rho H_\rho \circ y^* + \rho_\ell & (s_2) & u = \Delta u H_u \circ y^* + u_\ell \\ (s_3) & p = \Delta p H_p \circ y^* + p_\ell & (s_4) & e = \Delta e H_e \circ y^* + e_\ell \end{array}$$

which are solutions of $(\tilde{S})$ (or $(S)$) and satisfy the following assumptions:

$(A_1)$ $y^*$ is the function associated to $y{:}\, t \in \mathbb{R} \mapsto ct \in \mathbb{R}$ (see (2.1)), for some $c \in \mathbb{R}$ (that is $y^*(x,t) := x - ct$).

$(A_2)$ Let $\tau$ denote indistinctly $\rho, u, p$ or $e$. There are $(\tau_r, \tau_\ell) \in \mathbb{R}^2$ such that $0 < \tau_r < \tau_\ell$ and we define $\Delta\tau := \tau_r - \tau_\ell$.

$(A_3)$ $H_\rho,\ H_u,\ H_p,\ H_e \in \mathcal{H}_p(\mathbb{R})$.

$(A_4)$ Each $H_\tau$ $(\tau = \rho,\ u,\ p,\ e)$ has a representative $\widehat{H}_\tau$ such that

(a) $(\widehat{H}_\tau, \tau_r, \tau_\ell, \Delta\tau)$ satisfies (4.1)

(b) $(\widehat{H}_\tau, \tau_\ell, \Delta\tau)$ satisfies (4.2).

$(A_5)$ $(HH'_u)|_{\mathbb{R}^*} \approx 0$ (or equivalently $[(H \circ y^*)u_x]|_{\Omega^*} \approx 0$) for every $H \in \mathcal{H}_p(\mathbb{R})$.

By introducing the generalized functions:

$$\begin{array}{ll} \rho_* := \Delta\rho H_\rho + \rho_\ell & u_* := \Delta u H_u + u_\ell \\ p_* := \Delta p H_p + p_\ell & e_* := \Delta e H_e + e_\ell \end{array} \tag{5.7}$$

the component functions of $(\rho, u, p, e)$ might be written in the following way

$$\rho = \rho_* \circ y^*\,, \quad u = u_* \circ y^*\,, \quad p = p_* \circ y^*\,, \quad e = e_* \circ y^*\,.$$

In what follows we assume the following hypothesis.

**Hypothesis 5.2** Let $(c, \tau_r, \tau_\ell, H_\tau) \in \mathbb{R}^3 \times \mathcal{H}_p(\mathbb{R})$ be such that $0 < \tau_r < \tau_\ell$ for each $\tau = \rho$, $u$, $p$, $e$. Under these conditions it will be considered:

- the elements $y^*$ and $(\Delta\rho, \Delta u, \Delta p, \Delta e)$ introduced in previous definition
- the data $a, b \in \mathbb{R}^3$ of hypothesis 4.2(a) are defined by $a := (\rho_r, p_r, e_r)$ and $b := (\rho_\ell, p_\ell, e_\ell)$
- the element $(\rho_*, u_*, p_*, e_*)$ whose components are defined in (5.7)
- the generalized function $(\rho, u, p, e)$ such that their components are, respectively, defined by $(s_1)$, $(s_2)$, $(s_3)$ and $(s_4)$.

**Jump conditions** We will also consider the presented formulas as it follows, called *jump conditions* of the hydrodynamic equations, which are relations among the elements $c$, $(\rho_r, u_r, p_r, e_r)$ and $(\rho_\ell, u_\ell, p_\ell, e_\ell)$ above fixed.

$$
\begin{array}{ll}
(j_1)\ \ c = \Delta u(1 + \frac{\rho_\ell}{\Delta\rho}) + u_\ell & (j_2)\ \ \frac{\Delta p}{(\Delta u)^2} = \rho_\ell(1 + \frac{\rho_\ell}{\Delta\rho}) \\
(j_3)\ \ \Delta p(1 + \frac{u_\ell}{\Delta u}) = \rho_\ell \frac{\Delta e}{\Delta\rho} - e_\ell - p_\ell & (j_4)\ \ e_k = \lambda p_k + \frac{1}{2}\rho_k(u_k)^2\,,\ k = r, l\,.
\end{array}
$$

## Solvability of the system $(\tilde{S})$

We will give preliminary results that will allow us to study the solvability.

**Lemma 5.1** *If $(j_1)$ holds, we have the following properties.*

**(a)** *Let $a_1, \dots, a_6 \in \mathbb{R}$ be defined by*

$$
\begin{array}{lll}
a_1 := u_\ell(u_\ell - c)\Delta\rho & a_2 := \rho_\ell(2u_\ell - c)\Delta u & a_3 := \Delta p \\
a_4 := (2u_\ell - c)\Delta\rho\Delta u & a_5 := \rho_\ell(\Delta u)^2 & a_6 := \Delta\rho(\Delta u)^2\,.
\end{array}
$$

*If $a := a_1 + \dots + a_6$ then, $(j_2)$ holds if and only if $a = 0$.*

**(b)** *Let $b_1, \dots, b_5 \in \mathbb{R}$ be defined by*

$$
\begin{array}{lll}
b_1 := (e_\ell + p_\ell)\Delta u & b_2 := u_\ell\Delta p & b_3 := (u_\ell - c)\Delta e \\
b_4 := \Delta u\Delta p & b_5 := \Delta u\Delta e\,. &
\end{array}
$$

*If $b := b_1 + \dots + b_5$ then, $(j_3)$ holds if and only if $b = 0$.*

From corollaries 2.1, 2.2 and proposition 4.2(c) we obtain the following result.

**Proposition 5.1** *If $(H_\rho, H_p, H_e)$ and $H_u$ satisfy the hypothesis $(A_4)(a)$ and $(A_5)$ respectively, then $[\nu\circ(\rho, p, e) - \alpha]u_x|_{\Omega^*} \approx 0$ and $[\nu\circ(\rho, p, e) - \alpha]uu_x|_{\Omega^*} \approx 0$.*

A proof of the following result may be found in [2] (3.2 and 3.5).

**Proposition 5.2** **(a)** *$(\rho, u)$ is a solution of (5.1) if and only if $(j_1)$ holds.*

**(b)** *$(\rho, u, p, e)$ is a solution of (5.4) if and only if $(j_4)$ holds.*

**Proposition 5.3** *Suppose that* $(H_\rho, H_p, H_e)$ *and* $H_u$ *satisfy* $(A_4)(a)$ *and* $(A_5)$ *respectively. Then the following statements hold.*

**(a)** *The generalized function* $(\rho, u, p, e)$ *is a solution of the equations* (5.1) *and* (5.2) *if and only if the formulas* $(j_1)$, $(j_2)$ *and* $[\nu \circ (\rho, p, e) - \alpha]u_x \approx 0$ *hold.*

**(b)** $(\rho, u, p, e)$ *is a solution of* (5.1), (5.2) *and* (5.3) *if and only if the formulas* $(j_1) - (j_3)$, $[\nu \circ (\rho, p, e) - \alpha]u_x \approx 0$ *and* $[\nu \circ (\rho, p, e) - \alpha]uu_x \approx 0$ *hold.*

**Proof** (a) (See proposition 5.2(a)). From $(\rho u)_t = (-c\rho u)_x$ it follows that

$$(\rho u)_t + (p + \rho u^2)_x = [p + \rho u(u - c)]_x = [p_* + \rho_* u_*(u_* - c)]' \circ y^* . \tag{5.8}$$

On the other hand, from $(s_1)$, $(s_2)$, $(s_3)$, $(s_4)$ and (5.7), we have

$$p + \rho u(u - c) = a_0 + \big(a_1 H_\rho + a_2 H_u + a_3 H_p + a_4 H_\rho H_u + a_5 H_u^2 + a_6 H_\rho H_u^2\big) \circ y^*$$

where $a_0 := p_\ell + \rho_\ell u_\ell(u_\ell - c)$ and

$$\begin{array}{lll} a_1 := u_\ell(u_\ell - c)\Delta\rho & a_2 := \rho_\ell(2u_\ell - c)\Delta u & a_3 := \Delta p \\ a_4 := (2u_\ell - c)\Delta\rho\Delta u & a_5 := \rho_\ell(\Delta u)^2 & a_6 := \Delta\rho(\Delta u)^2 . \end{array}$$

Since $H \circ y^* \approx H_\rho \circ y^*$ for $H = H_u$, $H_p$, $H_\rho H_u$, $H_u^2$, $H_\rho H_u^2$ (see corollary 2.2(b)) the previous equality implies that $p + \rho u(u - c) \approx a_0 + a(H_\rho \circ y^*)$, where $a := a_1 + \cdots + a_6$, and therefore we have

$$[p + \rho u(u - c)]_x \approx a(H_\rho \circ y^*)_x . \tag{5.9}$$

Suppose that $(\rho, u, p, e)$ is a solution of (5.2). Hence by (5.8) and (5.9) we have $a(H_\rho \circ y^*)_x \approx \{[\nu \circ (\rho, p, e) - \alpha]u_x\}_x$. Which implies that (see propositions 2.6 and 5.1) $a = 0$ and $[\nu \circ (\rho, p, e) - \alpha]u_x \approx 0$. Since $(j_1)$ holds, from lemma 5.1(a), the condition $(j_2)$ holds too. The necessary condition is immediate.
(b). One can write

$$e_t + [(e + p)u]_x = [(e + p)u - ce]_x . \tag{5.10}$$

From $(s_1)$ - $(s_4)$ and (5.7), we have

$$(e + p)u - ce = b_0 + (b_1 H_u + b_2 H_p + b_3 H_e + b_4 H_u H_p + b_5 H_u H_e) \circ y^*$$

where the real numbers $b_0, \ldots, b_5$ are defined by

$$\begin{array}{lll} b_0 := (e_\ell + p_\ell)u_\ell - ce_\ell & b_1 := (e_\ell + p_\ell)\Delta u & b_2 := u_\ell \Delta p \\ b_3 := (u_\ell - c)\Delta e & b_4 := \Delta u \Delta p & b_5 := \Delta u \Delta e . \end{array}$$

The previous equality implies the relation $[(e + p)u - ce]_x \approx b(H_u \circ y^*)_x$, where $b := b_1 + \cdots + b_5$. Using this condition (together with (5.10)), by a minor modification in the proof of (a), it follows (b). □

We can summarize propositions 5.2(b) and 5.3(b) in the following result.

**Theorem 5.1** *If $(H_\rho, H_p, H_e)$ and $H_u$ satisfy $(A_4)(a)$ and $(A_5)$ respectively, then $(\rho, u, p, e)$ is a solution of the system $(\tilde{S})$ if and only if the formulas $(j_1)-(j_4)$, $[\nu \circ (\rho, p, e) - \alpha]u_x \approx 0$ and $[\nu \circ (\rho, p, e) - \alpha]uu_x \approx 0$ hold.*

**Corollary 5.1** *With the hypothesis of theorem 5.1, $(\rho, u, p, e)$ is a solution of the system $(\tilde{S})$ if and only if the jump conditions $(j_1)-(j_4)$ and the relations*

$$VK' \approx \frac{\alpha - \nu_\ell}{\Delta\nu}\delta \quad \text{and} \quad VKK' \approx \frac{1}{2}\frac{\alpha - \nu_\ell}{\Delta\nu}\delta$$

*hold, where $K := H_u$, $\delta := K'$ and $V := (\Delta\nu)^{-1}[\nu \circ (\rho_*, p_*, E_*) - \nu_\ell]$. In this case $VKK' \approx \frac{1}{2}VK'$.*

The following theorem generalizes strongly, via association, the classical result without viscosity (see [3], chapter 3 and [4], chapters 4 and 5, for instance). Let us recall that the set $\mathcal{E}_{s,QM}[I_{\alpha_i}^{\beta_i}]$ was defined in §3. Let us recall also that $\{1, 2, 3\} = \mathbf{I} \cup \mathbf{J}$ and that each component of $\nu_* = (\nu_1, \nu_2, \nu_3)$ is an element of $\mathcal{E}_{s,QM}[\mathbb{R}_+^*]$ (see hypotheses 4.2, 5.1 and 5.2).

**Theorem 5.2** *Suppose that $\nu_i^{-1} \in \mathcal{E}_{s,QM}[I_{\alpha_i}^{\beta_i}]$ for each $i \in \mathbf{I}$ and $\nu_j \equiv \alpha_j$ in $\mathbb{R}_+^*$ for each $j \in \mathbf{J}$. Then, there are $H_\rho, H_u, H_p$ and $H_e$ in $\mathcal{G}_s(\mathbb{R})$ satisfying the assumptions $(A_3)$, $(A_4)$ and $(A_5)$ of definition 5.1 such that if $\rho, u, p$ and $e$ are given by $(s_1), (s_2), (s_3)$ and $(s_4)$ respectively, then $(\rho, u, p, e)$ is a solution of the system*

$$\begin{gathered}
\rho_t + (\rho u)_x \approx 0 \\
(\rho u)_t + (p + \rho u^2)_x \approx \{[\nu \circ (\rho, p, e) - \alpha]u_x\}_x \\
e_t + [(e + p)u]_x \approx \{[\nu \circ (\rho, p, e) - \alpha]uu_x\}_x \\
e \approx \lambda p + \tfrac{1}{2}\rho u^2
\end{gathered}$$

*if and only if the jump conditions $(j_i), 1 \le i \le 4$, hold.*

**Proof** By proposition 4.3, there are $H_\rho, H_p$ and $H_e$ in $\mathcal{H}_r(\mathbb{R}) \setminus \mathcal{H}_\Lambda(\mathbb{R})$ satisfying $(A_4)$ and the relations

$$[\nu \circ (\rho_*, p_*, e_*) - \alpha]H' \approx 0 \text{ and } [\nu \circ (\rho_*, p_*, e_*) - \alpha]HH' \approx 0\,, \quad (H \in \mathcal{H}_\Lambda(\mathbb{R}))\,.$$

Choosing $H_u \in \mathcal{H}_\Lambda(\mathbb{R})$, from above relations it follows respectively

$$[\nu \circ (\rho, p, e) - \alpha]u_x \approx 0 \quad \text{and} \quad [\nu \circ (\rho, p, e) - \alpha](H_u \circ y^*)u_x \approx 0$$

which together with the equality $uu_x = \Delta u(H_u \circ y^*)u_x + u_\ell u_x$ imply that $[\nu \circ (\rho, p, e) - \alpha]uu_x \approx 0$. The result follows from theorem 5.1. □

**Remark 5.1** If $(\rho, u, p, e)$ and $(\rho_0, u_0, p_0, e_0)$ are two of any shock wave solutions of $(\tilde{S})$ then $(\rho, u, p, e) \approx (\rho_0, u_0, p_0, e_0)$ (according to corollary 2.2(b)).

## The non - solvability of the system $(S)$

The concept of shock wave solution for the equations (5.5) and (5.6) is introduced in an analogous way at definition 5.1. We suppose that $1 \in \mathbf{I}$.

**Proposition 5.4** *$(\rho, u)$ is a solution of (5.5) if and only if there is a generalized constant $z \in \mathcal{G}_s(\mathbb{R}^2)$ such that $\rho(u-c) = z$. In this case, $z \approx \rho_\ell(u_\ell - c)$.*

**Proof** From $\rho_t = -c\rho'_* \circ y^* = (-c\rho)_x$ it follows that

$$\rho_t + (\rho u)_x = [\rho(u-c)]_x = [\rho_*(u_* - c)]' \circ y^* .$$

Since $[\rho_*(u_* - c)]' = 0$ if and only if there is a generalized constant $z$ in $\mathcal{G}_s(\mathbb{R})$ such that $\rho_*(u_* - c) = z$, the first statement holds. On the other hand, from the following formulas

$$\rho(u-c) = [(u_\ell - c)\Delta\rho H_\rho + \rho_\ell \Delta u H_u + \Delta\rho\Delta u H_\rho H_u] \circ y^* + \rho_\ell(u_\ell - c)$$

and $H \circ y^* \approx H_\rho \circ y^*$, for $H = H_u,\ H_\rho H_u$, we get

$$\rho(u-c) \approx \Delta\rho \left[ u_\ell - c + \left(1 + \frac{\rho_\ell}{\Delta\rho}\right) \Delta u \right] H_\rho \circ y^* + \rho_\ell(u_\ell - c) . \tag{5.11}$$

Since $(\rho, u)$ is a solution of (5.1), $(j_1)$ holds and hence the conditions $\rho(u-c) = z$ and (5.11) imply the other statement. $\square$

**Proposition 5.5** *Suppose that $(H_\rho, H_p, H_e)$ and $H_u$ satisfy $(A_4)(a)$ and $(A_5)$ respectively. Then $(\rho, u, p, e)$ is a solution of (5.6) if and only if there is a generalized constant $z \in \mathcal{G}_s(\mathbb{R}^2)$ such that $[\nu \circ (\rho, p, e) - \alpha]u_x = p + \rho u(u-c) - z$. In this case, for some representative $\widehat{g}$ of $g := p + \rho u(u-c) - z$ we have $\widehat{g}(\varepsilon, \cdot) \to 0$ in $\Omega^*$ as $\varepsilon \downarrow 0$.*

**Proof** From $(\rho u)_t + (p + \rho u^2)_x = [p_* + \rho_* u_*(u_* - c)]' \circ y^*$ (see (5.8)) and

$$\{[\nu \circ (\rho, p, e) - \alpha]u_x\}_x = \{[\nu \circ (\rho_*, p_*, e_*) - \alpha]u'_*\}' \circ y^*$$

by the corollary 2.1 it follows that $(\rho, u, p, e)$ is a solution of (5.6) if and only if $[p_* + \rho_* u_*(u_* - c)]' = \{[\nu \circ (\rho_*, p_*, e_*) - \alpha]u'_*\}'$, which implies the first statement. On the other hand using the notations introduced in the proof of proposition 5.3(a) we have $p + \rho u(u-c) = a_0 + f$ where

$$f := (a_1 H_\rho + a_2 H_u + a_3 H_p + a_4 H_\rho H_u + a_5 H_u^2 + a_6 H_\rho H_u^2) \circ y^* .$$

If $\widehat{f} := (a_1 \widehat{H}_\rho + a_2 \widehat{H}_u + a_3 \widehat{H}_p + a_4 \widehat{H}_\rho \widehat{H}_u + a_5 \widehat{H}_u^2 + a_6 \widehat{H}_\rho \widehat{H}_u^2) \circ y^*$ we get $\widehat{f}(\varepsilon, \cdot) \to 0$ (resp. $a$) in $\Omega_-$ (resp. in $\Omega_+$) as $\varepsilon \downarrow 0$, where $\widehat{H}_\tau$ is any representative of $H_\tau$ ($\tau = \rho,\ u,\ p$). By using proposition 2.1 we get $f|_{\Omega_-} \approx 0$ and $f|_{\Omega_+} \approx a$. Therefore, using the formula $[\nu \circ (\rho, p, e) - \alpha]u_x = a_0 + f - z$ and proposition 5.1 it follows that $z \approx a_0$ and $a = 0$. Hence $\widehat{g} := \widehat{f} + a_0 - \widehat{z}$ satisfies the second statement (where $\widehat{z}$ is any representative of $z$). $\square$

**Proposition 5.6** *The equations* (5.5) *and* $(\nu_1 \circ \rho - \alpha_1)u_x \approx 0$ *have no shock wave solutions.*

**Proof** Otherwise (see proposition 5.4) let $z \in \mathcal{G}_s(\mathbb{R}^2)$ be a generalized constant such that $\rho(u-c) = z$ and $a := \rho_\ell(u_\ell - c) \approx z$. By proposition 3.2, $\rho$ has an inverse multiplicative. From $(\nu_1 \circ \rho - \alpha_1)u_x = -z(\nu_1 \circ \rho - \alpha_1)\rho^{-2}\rho_x$ we get $z(\nu_1 \circ \rho - \alpha_1)\rho^{-2}\rho_x \approx 0$. Let $\varphi \in C^\infty_{s,QM}[\mathbb{R}^*_+]$ be a strictly increasing function such that $(\varphi\circ\rho)_x = (\nu_1\circ\rho-\alpha_1)\rho^{-2}\rho_x$ , $(\varphi\circ\rho)|_{\Omega_-} \approx \varphi(\rho_\ell)$ and $(\varphi\circ\rho)|_{\Omega_+} \approx \varphi(\rho_r)$ (see proposition 4.4 and corollary 2.1). Let $\Phi \in \mathcal{G}_s(\mathbb{R}^2)$ be such that $\Phi_x = 0$ and $z\varphi \circ \rho \approx \Phi$ (see [1], 6.3.1(d)). By restriction to $\Omega_-$ we get $\Phi \approx a\varphi(\rho_\ell)$ (see proposition 2.5). Hence $z\varphi \circ \rho \approx a\varphi(\rho_\ell)$. By restriction to $\Omega_+$ we get $a\varphi(\rho_r) = a\varphi(\rho_\ell)$. Being $a < 0$ (see proposition 5.2(a)) we have $\varphi(\rho_r) = \varphi(\rho_\ell)$. This is a contradiction in view of the assumption $(A_2)$ of definition 5.1. □

Suppose that for each $r > 0$ we have $[\varepsilon \mapsto (\nu_1(r\varepsilon) - \alpha_1)^{-1}] \in \mathcal{E}_{s,M}(\mathbb{R})$. In the next two results, if $(\nu, \alpha, \beta) = (\nu_{(\pi)}, \alpha_{(\pi)}, \beta_{(\pi)})$, we will suppose that $\alpha_2 > 0$ and $\alpha_3 > 0$.

**Proposition 5.7** *The equations* (5.5) *and* (5.6) *have no shock wave solutions.*

**Proof** Assume that the equations (5.5) and (5.6) have a shock wave solution $(\rho, u, p, e)$. Setting $f_* := (\rho_*, p_*, e_*)$, $\widehat{f}_* := (\widehat{\rho}_*, \widehat{p}_*, \widehat{e}_*)$ and $\widehat{\tau} := \widehat{\tau}_* \circ y^*$, where $\widehat{\tau}_* := \Delta\tau\widehat{H}_\tau + \tau_\ell$ for each $\tau = \rho, p, e$, we have $f := f_* \circ y^* = (\rho, p, e)$ and $\widehat{f} := \widehat{f}_* \circ y^* = (\widehat{\rho}, \widehat{p}, \widehat{e})$. Using the proposition 4.2(d), applied to $f_*$ and $\widehat{f}_*$, we get (see proposition 3.2 also): $(\nu_1 \circ \rho - \alpha_1)(\nu \circ f - \alpha)^{-1} \in \mathcal{G}_{s,\ell b}(\mathbb{R}^2)$ and

$$\left(\frac{\nu_1 \circ \widehat{\rho} - \alpha_1}{\nu \circ \widehat{f} - \alpha}\right)(\varepsilon, \cdot) \to \begin{cases} \lambda_\ell & \text{in } \Omega_- \\ \lambda_r & \text{in } \Omega_+ \end{cases} \qquad \text{as } \varepsilon \downarrow 0$$

where $\lambda_k := [\nu_1(\rho_k) - \alpha_1](\nu_k - \alpha)^{-1}$, $k = r, \ell$. Let $g \in \mathcal{G}_{s,\ell b}(\mathbb{R}^2)$ be such that $(\nu \circ f - \alpha)u_x = g$ and $\widehat{g}(\varepsilon, \cdot) \to 0$ in $\Omega^*$ as $\varepsilon \downarrow 0$ for some representative $\widehat{g}$ of $g$ (see proposition 5.5). Hence, $(\nu_1 \circ \rho - \alpha_1)u_x = (\nu_1 \circ \rho - \alpha_1)(\nu \circ f - \alpha)^{-1}g \approx 0$ (see proposition 2.1) which can not occur (see proposition 5.6). □

From proposition 5.7 we get the following result.

**Theorem 5.3** *The system*

$$\begin{gathered} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (p + \rho u^2)_x = \{[\nu \circ (\rho, p, e) - \alpha]u_x\}_x \\ e_t + [(e+p)u]_x \approx \{[\nu \circ (\rho, p, e) - \alpha]uu_x\}_x \\ e \approx \lambda p + \tfrac{1}{2}\rho u^2 \end{gathered}$$

*has no shock wave solutions.*

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