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Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system *

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Abstract

This paper studies a problem of boundary observability for a coupled system of parabolic–hyperbolic type. First, we prove some Carleman estimates with singular weights for the heat and for the wave equations. Then we combine these results to obtain an observability result for the system. We conclude with a discussion about operators with constant coefficients.

1 Introduction

Let Ω be an open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let T be a positive number. Denote by $x = (x^0, \dots, x^n)$ the coordinates in $[0,T] \times \Omega$ and $\xi = (\xi_0, \dots, \xi_n)$ the corresponding Fourier variables. We also use the alternate notation $t = x^0$, $s = \xi_0$ and call t the "time" variable, while the other n coordinates are called the "space" variables, denoted by $x' = (x^1, \dots, x^n)$ and $\xi' = (\xi_1, \dots, \xi_n)$. By $\nu = (\nu_0, \dots, \nu_n)$ we denote the unit outer normal vector to $[0,T] \times \partial\Omega$.

This paper is devoted to the study of the coupled parabolic-hyperbolic system

$$\begin{aligned} P(x,D)w &= Q_1(x,D)\theta \quad \text{in }]0,T[\times\Omega \\ Q(x,D)\theta &= P_1(x,D)w \quad \text{in }]0,T[\times\Omega \\ w = \theta &= 0 \quad \text{on } [0,T]\times\partial\Omega\,, \end{aligned} \tag{1}$$

where $P(x,D)$ is a second order hyperbolic operator of the form

$$P(x,D) = \partial_t^2 - \sum_{i,j=1}^n \partial_i a^{ij}(x)\partial_j + a^0(x)\partial_t + \sum_{i=1}^n a^i(x)\partial_i + a(x)\,, \tag{2}$$

with

$$\sum_{i,j=1}^n a^{ij}\eta_i\eta_j \geq c_1|\eta|^2 \quad \forall \eta \in \mathbb{R}^n$$

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and $Q(x,D)$ is a second order parabolic operator of the form

$$Q(x,D)=\partial_t-\sum_{i,j=1}^{n}\partial_i b^{ij}(x)\partial_j+\sum_{i=1}^{n}b^i(x)\partial_i+b(x)\,,\tag{3}$$

with

$$\sum_{i,j=1}^{n}b^{ij}\eta_i\eta_j\geq c_2|\eta|^2\quad\forall\eta\in\mathbb{R}^n$$

for some positive constants c_1 and c_2.

The coupling terms $Q_1(x,D)$ and $P_1(x,D)$ are a second order operator (with respect to the "space" variables) and a first order operator (with respect all the variables) respectively[1]. We point out that equations of the form (1) arise in the study of linear thermo-elasticity (see e.g. [9] and references therein).

We are concerned with the following question. Knowing that the normal derivatives $\partial_\nu w$ and $\partial_\nu\theta$ are equal to 0 on $[0,T]\times\partial\Omega$ can we conclude that $w=\theta=0$, in $[0,T]\times\Omega$? More precisely, for any solution (w,θ) of problem (1), we want to prove an observability estimate of the form

$$\|(w,\theta)\|\leq C\|\partial_\nu(w,\theta)\|_\partial$$

where $\|\cdot\|$ and $\|\cdot\|_\partial$ are suitable interior and boundary Sobolev norms. In other words, this would say that the solution (w,θ) can be reconstructed in a stable fashion if one observes its conormal derivative on the boundary. Note that the initial data cannot be recovered stably due to the parabolic regularizing effect. However, the final data at time T can be obtained. By duality this implies an exact null controllability result for the adjoint problem.

The affirmative answer to the previous question (for sufficiently large T) is given using a–priori estimates of Carleman type. These estimates have been introduced in [2], and were intensively studied in [4] (for hyperbolic and elliptic operators) and in [6] (in the case of anisotropic operators). A general approach to Carleman estimates for boundary value problems was developed in [10, 12]. Carleman estimates for the initial boundary value problem for the heat equations have been independently obtained in [3, 5] and [13] while the analogous results in the case of hyperbolic equations were proved in [11]. We remark that other approaches to the observability problem for hyperbolic equations have been developed in [7] (using multipliers method) and in [1] (using microlocal analysis). For what concerns observability estimates for the heat equations, an observability result was proved in [8] using microlocal analysis and Carleman estimates for elliptic equations.

We start by deriving a Carleman estimate with singular weight for the heat equation. Next, we deduce a similar estimate for the wave equation. Finally, putting together the previous estimates we get an observability estimate for the coupled system, which, in particular, yields an affirmative answer to our

[1] Alternately one can take Q_1 to be first order and P_1 second order and obtain similar results

question. We point out that the last step will require a precise control on the constant that appear on the Carleman estimate for the heat equation. We conclude the paper discussing the case when P and Q have constant coefficients.

2 Notation and assumptions

We will use short notation

$$\partial_j = \frac{\partial}{\partial x^j}, \quad u_{i_1\ldots i_k} = \partial_{i_1}\ldots\partial_{i_k} u\,.$$

The symbols u' and u_t indicate the derivative of u with respect to t, ∇ represents the gradient with respect to all the variables while $\nabla' = (\partial_1, \ldots \partial_n)$ is the spatial gradient.

By H^s we denote the classical Sobolev spaces, with norm $\|\cdot\|_s$ while $\|\cdot\|$ stands for the L^2 norm. In the Carleman estimates we use weighted Sobolev norms. Given a nonnegative function η define

$$\|u\|^2_{k,\eta} = \sum_{|\alpha|\le k} \int \eta^{2(k-|\alpha|)} |\partial^\alpha u|^2\, dx$$

and the corresponding parabolic (anisotropic) norm

$$\|u\|^2_{k,a,\eta} = \sum_{|\alpha|_a\le k} \int \eta^{2(k-|\alpha|_a)} |\partial^\alpha u|^2\, dx$$

where $\partial^\alpha = \partial_0^{\alpha_0}\ldots\partial_n^{\alpha_n}$ and $|\alpha|_a = 2\alpha_0 + \alpha_1 + \ldots + \alpha_n$. All these norms can also be defined in a bounded domain in a standard manner.

Denote by $p(x,\xi)$ and $q(x,\xi)$ the principal symbols of P and Q respectively. They are given by

$$p(x,\xi) = -\xi_0^2 + \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j$$

respectively

$$q(x,\xi) = i\xi_0 + \sum_{i,j=1}^n b^{ij}(x)\xi_i\xi_j$$

As usual, a covector N is called time–like if $p(N) < 0$ and space–like if $p(N) > 0$. We can split the time-like class into forward time-like and backward time-like with respect to a reference time-like vector field. This role is played naturally in our case by ∂_0. Consequently, we say that a time-like covector N is forward if $N_0 > 0$ and is backward if $N_0 < 0$. A hypersurface S is called space–like if $p(N) < 0$ and time–like if $p(N) > 0$, where $N \in N^*S$, the conormal bundle. If $p(N) = 0$ then the hypersurface is called characteristic.

Next we introduce the notion of pseudoconvexity. For general operators this is a bit more complicated, but here we give only the simpler form which is adapted to second order hyperbolic equations.

Definition 2.1 *We say that the level sets of a C^2 function ϕ are strongly pseudoconvex with respect to the operator P at x iff*

$$\{p,\{p,\phi\}\}(x,\xi)>0 \quad on \quad \{\xi\neq 0\,:\,p(x,\xi)=\{p,\phi\}(x,\xi)=0\} \tag{4}$$

and

$$\{\{p,\phi\},p(x,\nabla\phi)\}>0 \quad on \quad \{p(x,\nabla\phi)=0\} \tag{5}$$

Here by $\{\cdot,\cdot\}$ we have denoted the usual Poisson bracket

$$\{p,q\}=\sum_{j=0}^{n}p_{\xi_j}q_{x_j}-q_{\xi_j}p_{x_j}\,.$$

In order to derive our observability result we use the following assumptions

(H1) the coefficients of the principal part of P and Q are of class C^1 while the lower order and the coupling terms are measurable and bounded.

(H2) There exists a smooth[2] function ϕ in $]0,T[\times\Omega$ whose level sets are pseudoconvex with respect to $P(x,D)$ so that

(i) $\nabla'\phi(x)\neq 0$, for all $x\in]0,T[\times\Omega$.

(ii) $\nabla\phi$ is forward time–like at time 0.

(iii) $\nabla\phi$ is backward time–like at time T.

We remark that assumption $(H2)(i)$ ensures that the level sets of ϕ are pseudoconvex surfaces with respect to Q.

It is well–known that the finite speed of propagation of P implies that the "observability time" T must be large enough. Indeed, take $P_1=Q_1=0$, and consider the initial value problem for the wave equation with the Cauchy data localized in a small set away from $\partial\Omega$. Then, it is easy to see that, for small T, we get a solution of $Pu=0$ with u not identically zero and $\|u\|_\partial=0$. In other words, one cannot expect that observability holds true for every $T>0$.

More precisely, it was proved in [1] that one can have boundary observability iff any bicharacteristic ray hits the boundary at least once in a nondiffractive point. The bicharacteristic flow is not well defined if the coefficients are only C^1. However, if the coefficients are C^2 then the condition $(H2)$ guarantees that such a geometrical assumption is fulfilled. Indeed, let us suppose that γ be a null bicharacteristic ray goes from 0 to T without leaving $[0,T]\times\Omega$. Now, from assumptions $(H2)(ii)$ and (iii) it is easy to deduce that ϕ possesses a maximum point at some $t\in(0,T)$. On the other hand, the pseudoconvexity of ϕ implies that $\{p,\{p,\phi\}\}>0$. However, since $\{p,\{p,\phi\}\}(\gamma)$ is in fact the second derivative of ϕ along γ it cannot be strictly positive at a maximum point.

[2]In effect piecewise smooth and semiconvex is sufficient.

3 The heat equation

Consider the parabolic initial boundary value problem

$$\begin{aligned} Q(x,D)\theta(x) = f(x) \quad &\text{in }]0,T[\times\Omega\,,\\ \theta = 0 \quad &\text{on } [0,T]\times\partial\Omega\,,\\ \theta(0,x') = \theta_0(x') \quad &\text{in } \Omega\,, \end{aligned} \tag{6}$$

with $(f,\theta_0) \in L^2([0,T]\times\Omega)\times H_0^1(\Omega)$. Our goal is to prove a Carleman estimate of the form

$$\|e^{\tau\psi}\theta\| \leq \|e^{\tau\psi}f\| + \|e^{\tau\psi}\partial_\nu\theta\|_\partial\,,$$

with appropriate norms and a suitable function ψ, uniformly with respect to the large parameter τ. The obstruction to such an estimate is that no Sobolev norm of the initial data can be controlled by the right hand side. Hence the only hope is to consider a weight function ψ which approaches $-\infty$ at time 0. Estimates of this type have already been proved in ([3, 5] and [13]). The novelty here is that we give a direct proof and that we achieve a better control of the constants arising in the estimate.

Thus, we introduce a C^2 function g defined as

$$g(t) = \begin{cases} \frac{1}{t} & \text{for } t \text{ near } 0\,,\\ \text{strictly decreasing} & \text{for } t\in[0,\delta]\,,\\ 1 & \text{for } t\in[\delta,T-\delta]\,,\\ \text{strictly increasing} & \text{for } t\in[T-\delta,T]\,,\\ \frac{1}{T-t} & \text{for } t \text{ near } T\,, \end{cases} \tag{7}$$

for some $\delta > 0$. Let $\phi(x)$ be a function such that

$$\nabla'\phi(x) \neq 0 \quad \text{for all } x\in[0,T]\times\Omega\,. \tag{8}$$

Define

$$\psi(x) := g(x_0)\big(e^{\lambda\phi(x)} - 2e^{\lambda\Phi}\big), \quad \Phi = \|\phi\|_{L^\infty((0,T)\times\Omega)}\,, \tag{9}$$

The weight function ψ thus defined approaches $-\infty$ at times $0,T$. The additional parameter λ is essential in order to obtain the control of the constants which enables us to handle arbitrarily large coefficients in the coupling terms.

Theorem 3.1 *Assume that* $(H1)$ *is fulfilled. Let* ψ *be given as in* (9) *with* ϕ *satisfying* (8). *Let* θ *be the solution of problem* (6). *Then there exists* λ_0 *so that for each* $\lambda > \lambda_0$ *there exists* $\tau(\lambda)$ *so that for* $\tau > \tau(\lambda)$ *the following estimate holds uniformly in* λ,τ*:*

$$C\lambda\|\tau_p^{-\frac{1}{2}}e^{\tau\psi}\theta\|^2_{2,\tau_p,a} \leq \|e^{\tau\psi}f\|^2 + \int_{[0,T]\times\partial\Omega}\tau_p e^{2\tau\psi}(\nu_i b^{ij}\phi_j)(\nu_i b^{ij}\nu_j)|\partial_\nu\theta|^2\,d\sigma\,, \tag{10}$$

Here $\tau_p = \lambda\tau g e^{\lambda\phi}$.

Proof: Since the estimate (10) is independent of the lower order terms of Q we can assume that

$$Q(x,D) = \partial_0 - \partial_i b^{ij} \partial_j \,,$$

where, for simplicity, we dropped the summation sign. As usual, with the substitution $v = e^{\tau\psi}\theta$, the estimate (10) reduces to

$$C\lambda \|\tau_p^{-\frac{1}{2}} v\|^2_{2,\tau_p,a} \leq \|Q_\tau v\|^2 + \int_{[0,T]\times\partial\Omega} \tau_p (\nu_i b^{ij} \phi_j)(\nu_i b^{ij} \nu_j) |\partial_\nu v|^2 \, d\sigma \,, \tag{11}$$

where Q_τ is the conjugated operator defined as

$$Q_\tau(x,D) := e^{\tau\psi} Q(x,D) e^{-\tau\psi} \,.$$

Note that the definition of v implies that

$$\begin{aligned} v(0,x') = v(T,x') = 0 \quad \text{in } \Omega \\ v = 0 \quad \text{on } \; [0,T] \times \partial\Omega \,. \end{aligned} \tag{12}$$

Split Q_τ into

$$Q_\tau(x,D) = Q^a_\tau(x,D) + Q^s_\tau(x,D)$$

where

$$Q^a_\tau(x,D) = \partial_t + \, \tau \left(\partial_i b^{ij}(x) \, \psi_j \, + \psi_i b^{ij}(x) \partial_j \right)$$

is the skew-symmetric part of Q_τ while

$$Q^s_\tau(x,D) = -\partial_i b^{ij}(x) \partial_j - \tau^2 \psi_i b^{ij}(x) \psi_j \, - \tau\psi_t$$

is the symmetric part of Q_τ. Since

$$\|Q_\tau v\|^2 = \|Q^s_\tau v\|^2 + \|Q^a_\tau v\|^2 + 2\langle Q^a_\tau v, Q^s_\tau v\rangle \,, \tag{13}$$

the crucial step of the proof will be to estimate from below $\langle Q^a_\tau v, Q^s_\tau v\rangle$ integrating by parts. It is easy to see that $\langle \partial_t v \,,\, Q^s_\tau v\rangle$ is a lower order term compared to the left hand side in (11) since

$$\begin{aligned} 2\langle \partial_t v \,,\, Q^s_\tau v\rangle &= \langle [Q^s_\tau, \partial_t] v, v\rangle \\ &= -\langle (\partial_t b^{ij}) \partial_i v, \partial_j v\rangle + \langle (\partial_t (\tau^2 \psi_i b^{ij}(x) \psi_j \, + \tau\psi_t) v, v\rangle \end{aligned}$$

therefore

$$|\langle \partial_t v \,,\, Q^s_\tau v\rangle| \leq c \|\nabla' v\|^2 + c_\lambda \tau^2 \|g^{\frac{3}{2}} v\|^2 \,,$$

for some positive constants c, c_λ. Similarly, for some $c_\lambda > 0$,

$$|\langle \, \tau(\partial_i b^{ij}(x) \psi_j + \psi_i b^{ij}(x) \partial_j) v, \tau\psi_t v\rangle| \leq c_\lambda \tau^2 \|g^{\frac{3}{2}} v\|^2 \,.$$

Then it remains to estimate

$$-\tau \, \langle (\partial_i b^{ij} \psi_j + \psi_i b^{ij} \partial_j \,) v, \, (\partial_k \, b^{kl} \partial_l + \tau^2 \psi_k b^{kl} \psi_l) v\rangle \,.$$

Compute first the leading zero order terms. Integrating by parts, we get

$$-\tau \langle(\partial_i b^{ij}\psi_j + \psi_i b^{ij}\partial_j)v\,,\, \tau^2\psi_k b^{kl}\psi_l v\rangle = \tau^3\,\langle b^{ij}\psi_j v,\, (\partial_i\psi_k b^{kl}\psi_l)v\rangle\,.$$

Since $\psi_k = \lambda g(x_0)\phi_k e^{\lambda\phi(x)}$, the highest order terms occur when the derivative falls on the exponential. Hence we get

$$-\tau \langle(\partial_i b^{ij}\psi_j + \psi_i b^{ij}\partial_j)v\,,\, \tau^2\psi_k b^{kl}\psi_l v\rangle = 2\lambda\langle\tau_p^3\big(\phi_i b^{ij}\phi_j\big)^2 v\,,\, v\rangle + R \tag{14}$$

where R stands for lower order terms,

$$R = O(\|\tau_p^{-\frac{1}{2}} v\|^2_{2,\tau_p,a}).$$

Now compute the leading first order terms integrating by parts the expression

$$-\tau\,\langle(\partial_i b^{ij}\,\psi_j + \psi_i\, b^{ij}\,\partial_j)v\,,\, \partial_k b^{kl}\partial_l v\rangle\,.$$

Since one operator is selfadjoint and the other is skew-adjoint, we need to compute their commutator. This might seem complicated at first; however, it simplifies considerably once we observe that every time a derivative falls on the coefficients b^{ij} we produce a lower order term. Furthermore, since $\psi_k = \lambda g(x_0)\phi_k e^{\lambda\phi(x)}$, only the derivatives falling on the exponential contribute to the leading order terms, since they are the only ones to contain an additional factor of λ. Taking these observations into account, it is not difficult to conclude that

$$\begin{aligned} &-\tau\,\langle(\partial_i b^{ij}\,\psi_j + \psi_i\, b^{ij}\,\partial_j)v,\, \partial_k b^{kl}\partial_l v\rangle \\ &= \tau\int_{[0,T]\times\partial\Omega} \psi_i b^{ij}\nu_j v_k b^{kl} v_l - 2\psi_i b^{ij} v_j \nu_k b^{kl} v_l d\sigma \\ &\quad +2\lambda^2\tau\langle g e^{\lambda\phi}\phi_i b^{ij}\partial_j v, \phi_k b^{kl}\partial_l v\rangle + R\,. \end{aligned} \tag{15}$$

Hence, if we put together (14) and (15) and use the homogeneous Dirichlet boundary condition for the boundary term then we get

$$\begin{aligned} &4\lambda\int_{[0,T]\times\Omega} \tau_p^3|v|^2 + \tau_p|\phi_i b^{ij} v_j|^2\,dx \\ &\leq\ 2\langle Q_\tau^a v, Q_\tau^s v\rangle + 2\int_{[0,T]\times\partial\Omega} \tau_p(\nu_i b^{ij}\phi_j)(\nu_i b^{ij}\nu_j)|\partial_\nu\theta|^2\,d\sigma + R \end{aligned}$$

Substituting this in (13) we obtain

$$\begin{aligned} &4\lambda\int_{[0,T]\times\Omega} \tau_p^3|v|^2 + \tau_p|\phi_i b^{ij} v_j|^2\,dx + \|Q_\tau^s v\|^2 + \|Q_\tau^a v\|^2 \\ &\leq\ \|Q_\tau v\|^2 + 2\int_{[0,T]\times\partial\Omega} \tau_p(\nu_i b^{ij}\phi_j)(\nu_i b^{ij}\nu_j)|\partial_\nu\theta|^2\,d\sigma + R\,. \end{aligned}$$

In the first term on the left we already control the appropriate weighted L^2 norm of v. The corresponding norm of $\partial_t v$ is easily obtained from the second and the fourth term, while the weighted L^2 norms of $\partial' v$, respectively ${\partial'}^2 v$ are obtained in an elliptic fashion from the first and the third term to obtain

$$C\lambda \|\tau_p^{-\frac{1}{2}} v\|^2_{2,\tau_p,a} \leq \|Q_\tau v\|^2 + 2\int_{[0,T]\times\partial\Omega} \tau_p (\nu_i b^{ij}\phi_j)(\nu_i b^{ij}\nu_j)|\partial_\nu \theta|^2\, d\sigma + R\,.$$

Now the lower order terms in R are negligible (i.e. much smaller than the left hand side) for sufficiently large λ, τ and we obtain (11). $\diamondsuit$

4 The wave equation

In this section we study the initial boundary value problem

$$\begin{gathered} P(x,D)w(x) = f(x) \quad \text{in }]0,T[\times\Omega\,, \\ w = 0 \quad \text{on } [0,T]\times\partial\Omega\,, \\ (w(0,x'), w'(0,x')) = (w_0(x'), w_1(x')) \quad \text{in } \Omega\,, \end{gathered} \tag{16}$$

with $(f, w_0, w_1) \in L^2([0,T]\times\Omega)\times H^1_0(\Omega)\times L^2(\Omega)$. Our goal is to show that some weighted H^1 norm of w is estimated by the sum of the L^2 boundary norm of the normal derivative of w with the L^2 norm of f. More precisely we will prove the wave equation analogue of Theorem 3.1 of the previous section. Carleman estimates for the wave equation are well-known. Here the additional difficulty comes from the singular weight that we plug into the estimate.

Due to the assumption (H2) we can find a small $\delta > 0$ so that $\nabla\phi$ is forward time-like in $[0,\delta]$ and backward time-like in $[T-\delta, T]$. Corresponding to this δ we consider the same weight function ψ as in the previous section.

Theorem 4.1 *Assume that* $(H1)$ *and* $(H2)$ *are fulfilled and let* ψ *be as in* (9). *Let* w *be the solution of equation* (16). *Then there exists* λ_0 *so that for each* $\lambda > \lambda_0$ *there exists* $\tau(\lambda)$ *so that for* $\tau > \tau(\lambda)$ *the following estimate holds uniformly in* λ, τ.

$$C\|\tau_h^{\frac{1}{2}} e^{\tau\psi} w\|^2_{1,\tau_h} \leq \|e^{\tau\psi} f\|^2 + \int_{[\delta,T-\delta]\times\partial\Omega} \tau_h e^{2\tau\psi} (\nu_i a^{ij}\phi_j)(\nu_i a^{ij}\nu_j)|\partial_\nu w|^2\, d\sigma\,, \tag{17}$$

Here $\tau_h = \tau(\lambda g e^{\lambda\phi} + |g_t| e^{\lambda\Phi})$ *has size comparable to* $\tau\nabla\psi$.

Proof: Since lower order terms in P do not affect the estimate (17) we can assume that P takes the form

$$P(x,D) = \partial_0^2 - \partial_i a^{ij}\partial_j\,.$$

With the substitution $v = e^{\tau\psi} w$, the estimate (17) reduces to

$$C\|\tau_h^{\frac{1}{2}} v\|^2_{1,\tau_h} \leq \|P_\tau v\|^2 + \int_{[\delta,T-\delta]\times\partial\Omega} \tau_h (\nu_i a^{ij}\phi_j)(\nu_i a^{ij}\nu_j)|\partial_\nu v|^2\, d\sigma\,,$$

where $P_\tau(x,D)$ is the conjugated operator defined as

$$P_\tau(x,D) = e^{\tau\psi} P(x,D) e^{-\tau\psi} .$$

Decompose $P_\tau(x,D)$ as follows

$$P_\tau(x,D) = P_\tau^s(x,D) + P_\tau^a(x,D) ,$$

where

$$P_\tau^a(x,D) = -\tau[\partial_0 \psi_t + \psi_t \partial_t - \psi_i a^{ij} \partial_j - \partial_i a^{ij} \psi_j]$$

is the skew–symmetric part of $P_\tau(x,D)$ and

$$P_\tau^s(x,D) = P(x,D) + \tau^2(\psi_t^2 - \psi_j a^{jk} \psi_j \psi_k)$$

is the symmetric part. We divide the time interval in three regions. First, we prove an estimate in $[0,\delta]$; the similar estimate holds in $[T-\delta,T]$. Next we estimate w in the remaining region $[\delta, T-\delta]$. Finally, we add together the three estimates to get (17).

Step 1: For weighted norms on a fixed time slice we introduce the notation

$$\|v\|^2_{1,\tau_h,t} = \int_\Omega |\nabla v(t,x')|^2 + \tau_h^2 v^2(t,x')\, dx' , \tag{18}$$

We claim that, for small δ and sufficiently large λ and τ the following estimate holds in $[0,\delta]\times\Omega$:

$$C\|\tau_h v\|^2_{1,\tau_h} \leq \|P_\tau(x,D)v\|^2 + \|\tau_h^{\frac{1}{2}} v\|^2_{1,\tau_h,\delta} \tag{19}$$

for some constant $C > 0$. To prove this we estimate from below the scalar product of $P_\tau(x,D)v$ with $\tau_h(\tau\psi_t - \partial_t)v$,

$$\langle P_\tau(x,D)v, \tau_h(\tau\psi_t v - v_t)\rangle = \langle (P_\tau^s(x,D) + P_\tau^a(x,D))v, \tau_h(\tau\psi_t - \partial_t)v\rangle$$

The terms involving a selfadjoint and a skew-adjoint operator can be integrated by parts to get

$$|\langle P_\tau^s v, -\tau_h v_t\rangle + \langle P_\tau^a v, -\tau\tau_h \psi_t v\rangle| \leq \tau^{-1} c_\lambda \|\tau_h v\|^2_{1,\tau_h} + c\|\tau_h^{\frac{1}{2}} v\|^2_{1,\tau_h,\delta}$$

for some constants $c_\lambda, c > 0$ (we get no contribution on the lateral boundary due to the homogeneous Dirichlet boundary condition and no contribution at time 0 since v vanishes there of infinite order). The rest is a quadratic form

$$\begin{aligned} Q(v,v) &\\ = \;& \langle P_\tau^s v, \tau_h \tau\psi_t v\rangle - \langle P_\tau^a v, \tau_h v_t\rangle \\ = \;& -\langle \tau\psi_t \partial_t v, \tau_h \partial_t v\rangle + \langle \tau\psi_t \partial_i v, \tau_h a^{ij} \partial_j v\rangle + \tau^2 \langle \tau\psi_t v, \tau_h(\psi_t^2 - \psi_i a^{ij} \psi_j)v\rangle \\ & -2\tau\langle \psi_i a^{ij} \partial_j v, \tau_h v_t\rangle + \text{lower order terms} \end{aligned}$$

with the symbol $q(x,\xi) = \tau_h(p^s_\tau \tau\psi_t + p^a_\tau i\xi_0)$. This is positive because $\nabla\psi$ is forward time-like in $[0,\delta]$. Indeed,

$$\begin{aligned}
\tau_h^{-1}q(x,\xi) &\geq \tau\psi_t\Big(a^{jk}\xi_j\xi_k - \xi_0^2 + c_1\tau_h^2\Big) - i\xi_0\Big(2\tau i\xi_0\psi_t - 2\tau i a^{jk}\psi_j\xi_k\Big) \\
&= \tau\psi_t(\xi_0^2 + a^{jk}\xi_j\xi_k + c_1\tau_h^2) - 2\tau\xi_0\psi_j a^{jk}\xi_k \\
&\geq \tau\psi_t(\xi_0^2 + a^{jk}\xi_j\xi_k + c_1\tau_h^2) - \tau\sqrt{a^{jk}\psi_j\psi_k}\Big(\xi_0^2 + a^{jk}\xi_j\xi_k\Big) \\
&\geq \tau(\psi_t - \sqrt{a^{jk}\psi_j\psi_k})(\xi_0^2 + a^{jk}\xi_j\xi_k + c_1\tau_h^2) \\
&\geq c\tau_h(\xi_0^2 + a^{jk}\xi_j\xi_k + \tau_h^2).
\end{aligned}$$

Putting together the two pieces of information above we get

$$c\|\tau_h v\|_{1,\tau_h} \leq \langle P_\tau(x,D)v\,,\,\tau_h(\tau\psi_t - \partial_t)v\rangle + \|\tau_h^{\frac12} v\|_{1,\tau_h,\delta} \tag{20}$$

for sufficiently large τ. This implies (19).

Step 2: Let us now consider the behaviour of w for t in $[\delta, T-\delta]$. We claim that the following estimate holds in $[\delta, T-\delta]\times\Omega$:

$$\begin{aligned}
&C_2(\|\tau_h^{\frac12}v\|^2_{1,\tau_h} + \|\tau_h^{\frac12}v\|^2_{1,\tau_h,\delta} + \|\tau_h^{\frac12}v\|^2_{1,\tau_h,T-\delta}) \\
&\quad\leq\; C_3\|P_\tau(x,D)v\|^2 + 2\tau\int_{[\delta,T-\delta]\times\partial\Omega} \tau_h(a^{kl}\nu_k\nu_l)(a^{kl}\phi_k\nu_l)|\partial_\nu v|^2
\end{aligned}$$

for some positive constants C_2, C_3.

First we observe that it suffices to prove that there exists a smooth function χ such that for large enough τ,

$$\begin{aligned}
&C(\|\tau_h^{\frac12}v\|^2_{1,\tau_h} + \|\tau_h^{\frac12}v\|^2_{1,\tau_h,\delta} + \|\tau_h^{\frac12}v\|^2_{1,\tau_h,T-\delta} + \|P^a_\tau v\|^2) \\
&\quad\leq\; \langle P_\tau v, (2P^a_\tau + \tau_h\chi)v\rangle + 2\int_{[\delta,T-\delta]\times\partial\Omega} \tau_h(a^{kl}\nu_k\nu_l)(a^{kl}\phi_k\nu_l)|\partial_\nu v|^2\, d\sigma
\end{aligned} \tag{21}$$

We decompose $P_\tau = P^s_\tau + P^a_\tau$. The term $\langle P^a_\tau v, \tau_h\chi v\rangle$ is a lower order term; indeed, integration by parts yields

$$|\langle P^a_\tau v, \tau_h h v\rangle| \leq \lambda\|\tau_h v\|^2 + \|\tau_h v(\delta)\|^2 + \|\tau_h v(T-\delta)\|^2$$

Then it suffices to show that

$$\begin{aligned}
&C(\|\tau_h^{\frac12}v\|^2_{1,\tau_h} + \|\tau_h^{\frac12}v\|^2_{1,\tau_h,\delta} + \|\tau_h^{\frac12}v\|^2_{1,\tau_h,T-\delta}) \\
&\quad\leq\; \langle P^s_\tau v, (2P^a_\tau + \tau_h\chi)v\rangle + \|\tau_h^{-\frac12}P^a_\tau v\|^2 \\
&\qquad +2\int_{[\delta,T-\delta]\times\partial\Omega} \tau_h(a^{kl}\nu_k\nu_l)(a^{kl}\phi_k\nu_l)|\partial_\nu v|^2\, d\sigma
\end{aligned} \tag{22}$$

Now we integrate by parts in the inner product above and use the homogeneous Dirichlet boundary condition exactly as in the parabolic case. Then the right hand side in (22) reduces to the integral of an algebraic quadratic form in v, ∇v, i.e.

$$\begin{aligned}
&\langle P^s_\tau v, (2P^a_\tau + \tau_h\chi)v\rangle + \|\tau_h^{-\frac12}P^a_\tau v\|^2 + 2\int_{[\delta,T-\delta]\times\partial\Omega}\tau_h(a^{kl}\nu_k\nu_l)(a^{kl}\phi_k\nu_l)|\partial_\nu v|^2\,d\sigma\\
&= \int_{\{T-\delta\}\times\Omega} G_1(\nabla v,\tau_h v)dx' - \int_{\{\delta\}\times\Omega} G_1(\nabla v,\tau_h v)dx'\\
&+ \int_{[\delta,T-\delta]\times\Omega} G(\nabla v,\tau_h v)dx + \int_{[\delta,T-\delta]\times\partial\Omega}\tau_h(a^{kl}\nu_k\nu_l)(a^{kl}\phi_k\nu_l)|\partial_\nu v|^2\,d\sigma.
\end{aligned}$$

Here G is an interior quadratic form and G_1 is a boundary quadratic form. The quadratic form G,

$$(\nabla v,\tau_h v) \to G(\nabla v,\tau_h v) = g^{ij}(x)\partial_i v\partial_j v + g^i(x)\partial_i v\,\tau_h v + g(x)\tau_h^2 v^2$$

has its symbol given by

$$g(x,\xi,\tau) = \frac{1}{i}\{p^s_\tau\,,\,p^a_\tau\} + \frac{1}{\tau_h}|p^a_\tau|^2 + \tau_h p^s_\tau\chi\,.$$

On the other hand for G_1 we get the symbol

$$g_1(x,\xi,\tau) = -p^s_\tau\tau\psi_t - p^a_\tau i\xi_0$$

At time δ, $\nabla\psi$ is forward time-like, which implies that the symbol of G_1 is negative,

$$g_1(x,\xi,\tau) \le -c\tau_h(\xi^2+\tau_h^2)$$

At time $T-\delta$, $\nabla\psi$ is backwards time-like, and the symbol of G_1 is positive.

Hence, in order to get (21) it suffices to choose the function χ independent of λ,τ so that the symbol $g(x,\xi,\tau)$ is a positive quadratic form in (ξ,τ). Computing g explicitly yields

$$\begin{aligned}
g(x,\xi,\tau) &= \tau_h\Big((1+\lambda)|\{p,\phi\}|^2 + \chi p(x,\xi) + \{p,\{p,\phi\}\}\Big)\\
&\quad+\tau_h^3\Big(4\lambda p^2(\nabla\phi) - \chi p(\nabla\phi) - \{p(\nabla\phi),\{p,\phi\}\}\Big)
\end{aligned}$$

If we use the second part of the pseudoconvexity condition we see that the coefficient of τ_h^3 is positive for sufficiently large λ. It remains to look at the coefficient of τ_h, namely

$$(1+\lambda)|\{p,\phi\}|^2 + \chi p(x,\xi) + \{p,\{p,\phi\}\}$$

The pseudoconvexity condition states that

$$\{p,\{p,\phi\}\}(x,\xi) > 0 \quad \text{whenever} \quad p(x,\xi) = \{p,\phi\} = 0,\ \xi \ne 0\,. \tag{23}$$

The following Lemma shows the way we use this condition.

Lemma 4.2 *Assume that (23) above holds. Then there exists $\lambda > 0$ and a smooth function χ such that*

$$0 < (1+\lambda)|\{p,\phi\}|^2 + \chi p(x,\xi) + \{p,\{p,\phi\}\} \tag{24}$$

Proof of Lemma 4.2: Note first that it suffices to prove (24) for fixed x; these local versions of (24) can then be put together using a suitable partition of unit. According to (23) if λ is large enough then we have that

$$q(x,\xi) = (1+\lambda)|\{p,\phi\}|^2 + \{p,\{p,\phi\}\} > 0 \quad \text{whenever} \\ p(x,\xi) = 0,\, \xi \neq 0 \ .$$

Look now at the zero set Z_σ for

$$q(x,\xi) + \sigma p(x,\xi).$$

If σ is small enough the Z_σ is contained in $\{p(x,\xi) > 0\}$, while if σ is large enough then Z_σ is contained in $\{p(x,\xi) < 0\}$. Then there are two possibilities.

a) There exists some σ such that $Z_\sigma = \emptyset$. Then the conclusion of the lemma follows.

b) There exists some σ such that Z_σ intersects both $\{p(x,\xi) > 0\}$ and $\{p(x,\xi) < 0\}$.

Since Z_σ cannot intersect $\{p(x,\xi) = 0\}$, it follows that it is projectively disconnected. But this is impossible, for the zero set of a quadratic form in $\mathbb{R}^n$ is always projectively connected. Then, we deduce that (21) holds.

Step 3: Add up the estimates in the two cases. $\diamondsuit$

5 The observability estimate

In this section we prove the observability result for the system (1).

Theorem 5.1 *Assume that $(H1)$ and $(H2)$ are fulfilled and let ψ be as in* (9). *Then there exists λ_0 so that for each $\lambda > \lambda_0$ there exists $\tau(\lambda)$ so that for $\tau > \tau(\lambda)$ the following estimate holds uniformly in λ, τ for all solutions (w,θ) of the system (1)*

$$\begin{aligned} &\|e^{\tau\psi} w\|^2_{1,\tau_h} + \|e^{\tau\psi}\tau_p^{-\frac{1}{2}}\theta\|^2_{2,a,\tau_p} \\ &\leq \ C\Big(\int_{[\delta,T-\delta]\times\partial\Omega} e^{2\tau\psi}(\nu_i a^{ij}\phi_j)(\nu_i a^{ij}\nu_j)|\partial_\nu w|^2 \, d\sigma \\ &\quad + \int_{[0,T]\times\partial\Omega} \tau_p e^{2\tau\psi}(\nu_i b^{ij}\phi_j)(\nu_i b^{ij}\nu_j)|\partial_\nu \theta|^2 \, d\sigma\Big) \end{aligned} \tag{25}$$

This estimate shows that solutions to (1) can be reconstructed in a stable manner if one observes their normal derivative on the boundary. As one can see from the above estimate, this observation needs not be on the entire boundary; it suffices to observe the hyperbolic, respectively the parabolic equation in the regions Γ_h, respectively Γ_p given by

$$\Gamma_h = \{x \in \partial\Omega;\ \phi_i a^{ij}\nu_j > 0\}, \quad \Gamma_p = \{x \in \partial\Omega;\ \phi_i b^{ij}\nu_j > 0\},$$

Combining the straightforward energy estimates with (25) we obtain the following consequence.

Corollary 5.2 *Assume that* $(H1)$ *and* $(H2)$ *are fulfilled. Then for all solutions* (w, θ) *of the system (1) we have*

$$\|\nabla w(T)\|_{L^2(\Omega)} + \|\nabla'\theta(T)\|_{L^2(\Omega)} \le c(\|\partial_\nu w\|_{L^2(\Gamma_h)} + \|\partial_\nu \theta\|_{L^2(\Gamma_p)})$$

Proof of Theorem 5.1: Applying Theorem 4.1 to $\tau_h^{-1/2} w$ we deduce that

$$\begin{aligned} \|e^{\tau\psi} w\|^2_{1,\tau_h} \quad &\le \quad c_1 \|e^{\tau\psi}([P, \tau_h^{-\frac{1}{2}}] + \tau_h^{-\frac{1}{2}} P) w\|^2 \\ &\quad + c_2 \int_{[\delta, T-\delta]\times\partial\Omega} e^{2\tau\psi}(\nu_i a^{ij}\phi_j)(\nu_i a^{ij}\nu_j)|\partial_\nu w|^2 \, d\sigma \end{aligned}$$

For large enough τ the commutator is small compared to the left hand side therefore we get

$$\|e^{\tau\psi} w\|^2_{1,\tau_h} \le \|e^{\tau\psi}\tau_h^{-1/2} Q_1\theta\|^2 + \int_{[\delta, T-\delta]\times\partial\Omega} e^{2\tau\psi}(\nu_i a^{ij}\phi_j)(\nu_i a^{ij}\nu_j)|\partial_\nu w|^2 \, d\sigma. \tag{26}$$

On the other hand, Theorem 3.1 implies that

$$\begin{aligned} &\lambda \|e^{\tau\psi}\tau_p^{-1/2}\theta\|^2_{2,a,\tau_p} \\ &\quad \le \quad c_1 \|e^{\tau\psi} P_2 w\|^2 \ + c_2 \int_{[0,T]\times\partial\Omega} \tau_p e^{2\tau\psi}(\nu_i b^{ij}\phi_j)(\nu_i b^{ij}\nu_j)|\partial_\nu \theta|^2 \, d\sigma \end{aligned} \tag{27}$$

Then, the conclusion follows by adding (26) and (27) provided that λ is sufficiently large. The fact that $\tau_h \ge \tau_p$ is essential. ♢

6 The constant coefficient case

In this section we consider the following system of linear thermoelasticity

$$\begin{aligned} w_{tt} - \Delta w + \alpha\Delta\theta = 0 \quad &\text{in }]0, T[\times\Omega \\ \theta_t - \nu\Delta\theta + \beta w_t = 0 \quad &\text{in }]0, T[\times\Omega \\ w = \theta = 0 \qquad &\text{on } [0, T]\times\partial\Omega\,, \end{aligned} \tag{28}$$

where the coupling parameters α, β and the the viscosity ν are assumed to be positive constants. For simplicity, let us suppose that $0 \notin \Omega$[3] and define

$$\psi(x) = g(x_0)\{e^{\lambda\phi(x)} - 2e^{\lambda\Phi}\} \tag{29}$$

with $\phi(x) = |x'|^2 - c(t - \frac{T}{2})^2$, and g defined as in formula (7).

Theorem 6.1 *Let (w, θ) be a solution of* (28) *and let ψ be as in (29). Assume that, for some $c \in]0,1[$,*

$$cT > 2 \sup_{x' \in \Omega} |x'| \, . \tag{30}$$

Then there exists λ_0 so that for each $\lambda > \lambda_0$ there exists $\tau(\lambda)$ so that for $\tau > \tau(\lambda)$ the following estimate holds uniformly in λ, τ

$$\|e^{\tau\psi}w\|^2_{1,\tau_h} + \|e^{\tau\psi}\tau_p^{-\frac{1}{2}}\theta\|^2_{2,a,\tau_p} \leq C \int_\Gamma e^{2\tau\psi}(|\partial_\nu w|^2 + \tau_p|\partial_\nu\theta|^2)d\sigma \tag{31}$$

where $\Gamma = \{x \in [0,T] \times \partial\Omega \, : \, \partial_\nu\phi(x) > 0\}$.

Proof: It suffices to show that assumption $(H2)$ is fulfilled and to use Theorem 5.1. First, we observe that the function ϕ defined above satisfies $(H2)(i)$. Moreover, since

$$-p(\nabla\phi) = 4c^2\Big(t - \frac{T}{2}\Big)^2 - 4|x'|^2 \geq 4c^2\Big(t - \frac{T}{2}\Big)^2 - 4\sup_{x' \in \Omega}|x'|^2$$

(30) imply that the vector $\nabla\phi$ is time–like and $(H2)(ii)$, (iii) are fulfilled. It remains to verify that the function ϕ is pseudoconvex with respect to the wave operator $\partial_t^2 - \Delta$ in $]0,T[\times\Omega$. We have that

$$\{p, \phi\} = 4\xi' \cdot x' + 4c\xi_0\Big(t - \frac{T}{2}\Big) \, .$$

Moreover, the fact that $c < 1$ implies that

$$\{p, \{p, \phi\}\} = 8(|\xi'|^2 - c\xi_0^2) > 0$$

for $p(\xi) = 0$. Hence, condition (4) is satisfied. On the other hand,

$$p(\nabla\phi) = |2x'|^2 - |2c(t - T/2)|^2$$

and $c < 1$ imply that

$$-\{p(\nabla\phi), \{p\,,\, \phi\}\} = 8(|2x'|^2 - c|2c(t - T/2)|^2) > 0$$

on $p(\nabla\phi) = 0$. Thus (5) is fulfilled and the proof is complete. ♢

[3] This is strictly speaking not necessary. To avoid it one needs to work with a piecewise smooth ϕ.

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