Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 31, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

The limiting equation for Neumann Laplacians on shrinking domains *

Yoshimi Saito

Abstract

Let $\{\Omega_\epsilon\}_{0<\epsilon\leq 1}$ be an indexed family of connected open sets in $\mathbb{R}^2$, that shrinks to a tree Γ as ϵ approaches zero. Let H_{Ω_ϵ} be the Neumann Laplacian and f_ϵ be the restriction of an $L^2(\Omega_1)$ function to Ω_ϵ. For $z \in \mathbb{C}\backslash[0,\infty)$, set $u_\epsilon = (H_{\Omega_\epsilon} - z)^{-1} f_\epsilon$. Under the assumption that all the edges of Γ are line segments, and some additional conditions on Ω_ϵ, we show that the limit function $u_0 = \lim_{\epsilon\to 0} u_\epsilon$ satisfies a second-order ordinary differential equation on Γ with Kirchhoff boundary conditions on each vertex of Γ.

1 Introduction

Let Ω be a connected open set in $\mathbb{R}^2$. Consider a family of the Neumann Laplacians H_{Ω_ϵ}, $0 < \epsilon \leq 1$, on the sub-domain Ω_ϵ such that $\{\Omega_\epsilon\}$ shrinks to Γ in the sense that

$$\begin{gathered} \Omega = \Omega_1 \supset \Omega_{\epsilon_2} \supset \Omega_{\epsilon_1} \quad (1 > \epsilon_2 > \epsilon_1 > 0), \\ \lim_{\epsilon\to 0} \overline{\Omega_\epsilon} = \Gamma, \end{gathered} \tag{1.1}$$

where the bar over a set means the closure of the set. We continue here the study started in [11] regarding the following question: In what sense does the operator H_{Ω_ϵ} converges to an operator on Γ as $\epsilon \to 0$? That is, we try to find conditions under which, given a thin domain and an operator on the domain, an operator on an imbedded tree or network gives a good approximation of the operator on the domain. This investigation is part of the general question on replacing the study of a thin domain by the study of an imbedded tree or network, which has been proposed in many branches of science such as physics and chemistry. For references on this problem, see for example Ruedenberg-Scherr [9], Exner-Seba [5], Kuchment [7], Schatzman [12], Rubinstein-Schatzman [10], and Kuchment-Zeng [8]. In [12], a family of "fattened" domains Ω_ϵ of a C^2 manifold M are considered. It is shown that the k-th eigenvalue $\lambda_k(\epsilon)$ of the Neumann Laplacian

* *Mathematics Subject Classifications:* 35J05, 35Q99.
Key words and phrases: Neumann Laplacian, tree, shrinking domains.

Submitted March 9, 2000. Published April 26, 2000.
Partially supported by Deutche Forschungs Gemeinschaft grant SFB 359.

H_{Ω_ϵ} on Ω_ϵ converges asymptotically to the k-th eigenvalue λ_k of the Laplace-Beltrami operator of M. In [10] the above results are extended to the case where the manifold M is replaced by a graph G; see for example [8], and for a simplified proof [10].

In [11], we discussed the convergence of the resolvent $(H_{\Omega_\epsilon} - z)^{-1}$ as $\epsilon \to 0$. After introducing the Hilbert spaces $L_2(\Gamma)$ and $H^1(\Gamma)$ on the tree Γ and defining the selfadjoint "Neumann Laplacian" operator H_Γ in $L^2(\Gamma)$ as in [4], we presented a set of general conditions under which the the resolvent $(H_{\Omega_\epsilon} - z)^{-1} f_\epsilon$ converges as as $\epsilon \to 0$, where f_ϵ is the restriction of a function $f \in L^2(\Omega)$ to Ω_ϵ ([11], Theorem 4.5). Also in [11], we studied the case where Ω is a bounded convex set and the tree Γ is a straight line segment (ridge).

Let Ω be a bounded convex set in $\mathbb{R}^2$ and suppose that Ω and Ω_ϵ are given by

$$\begin{gathered} \Omega = \{x = (x_1, x_2) : -\ell_-(x_1) < x_2 < \ell_+(x_1),\ a < x_1 < b\}, \\ \Omega_\epsilon = \{x = (x_1, x_2) : -\epsilon\ell_-(x_1) < x_2 < \epsilon\ell_+(x_1),\ a < x_1 < b\}, \\ \Gamma = \{x = (x_1, 0) : a \le x_1 \le b\}, \end{gathered} \tag{1.2}$$

where $-\infty < a < b < \infty$, $0 < \epsilon \le 1$ and $\ell_\pm(t)$ are positive C^1 functions on $[a, b]$. Then we have, for $f \in H^1(\Omega)$ and $z \in \mathbb{C}\backslash[0, \infty)$,

$$\lim_{\epsilon \to 0} \gamma\big[(H_{\Omega_\epsilon} - z)^{-1} f_\epsilon\big] = (H_\Gamma - z)^{-1}(\gamma f), \tag{1.3}$$

in a weighted Hilbert space $L^2_{a_0}(\Gamma)$, where H_Γ is the "Neumann Laplacian" on Γ defined in [4] (see §2), γ is the trace operator on Γ, and

$$\begin{gathered} L^2_{a_0}(\Gamma) = L^2(\Gamma; a_0(\sigma)d\sigma), \\ a_0(\sigma) = \ell_-(\sigma) + \ell_+(\sigma) \end{gathered} \tag{1.4}$$

([11], Theorem 5.5).

In this work, we consider the case when Γ is a tree such that all the edges are line segments (Assumption 4.2, (i)). Suppose that the family $\{\Omega_\epsilon\}$ is given by

$$\Omega_\epsilon = \{(\sigma, s) : -\epsilon\ell_-(\sigma) < s < \epsilon\ell_+(\sigma),\ \sigma \in \Gamma\}, \tag{1.5}$$

where σ is the arc length along the edges of Γ and s is the arc length along the curve $C_\sigma = \tau^{-1}(\sigma)$, τ being a map from Ω into $\Omega \cap \Gamma$ which is Lipschitz continuous almost everywhere in Ω (see §2). Then we assume that, for σ belonging to the edge e_j of Γ, the curve C_σ is perpendicular to the edge near e_j except its vertices (Assumption 4.2, (ii)). Set

$$u_\epsilon(x) = u_\epsilon(\sigma, s) = (H_{\Omega_\epsilon} - z)^{-1} f_\epsilon, \tag{1.6}$$

where $f \in H^1(\Omega) \cap C^1(\Omega)$. Then there exists a subsequence $\{u_{\epsilon_k}\}_{k=1}^\infty$ such that $\{u_{\epsilon_k}(\sigma, 0)\}$, the restriction of u_{ϵ_k} on the tree Γ, converges to u_0 weakly in $L^2_{a_0}(\Gamma)$ as $k \to \infty$, and u_0 satisfies the equation

$$-a_0^{-1}(\sigma)\frac{d}{d\sigma}(a_0(\sigma)u') - zu = f(\sigma, 0) \tag{1.7}$$

on each edge with the Kirchhoff boundary condition at each vertex (Theorems 4.5 and 4.8), where a_0 is given by (1.4) and u' means the derivative of u_0 with respect to the arc length σ along the edge.

In §2, after introducing the tree Γ imbedded in the open connected set Ω, we discuss the change of variables $x = (x_1, x_2) \to (\sigma, s)$. In §3 some estimates of $u_\epsilon(\sigma, 0)$ are given. These estimates will be used to guarantee the weak convergence of $\{u_{\epsilon_k}\}_{k=1}^{\infty}$. §4 is devoted to showing the above convergence of $\{u_{\epsilon_k}\}_{k=1}^{\infty}$ to a solution u_0 of the equation (1.6) (Theorems 4.5 and 4.8). The main tools are Lemmas 4.1 and 4.4 whose proof will be given in §6. We shall discuss the continuity of the limiting function u_0 at each vertex in §5.

2 Preliminaries

In this section we are going to introduce a domain Ω in $\mathbb{R}^2$, a tree Γ contained in Ω and a family $\{\Omega_\epsilon\}_{0<\epsilon\leq 1}$ of sub-domains of Ω.

Let Ω be a domain (i.e., a connected open set) in $\mathbb{R}^2$. Let $\Gamma \subset \overline{\Omega}$ be a tree, that is, a connected graph without loops or cycles, where $\overline{\Gamma}$ is the closure of Γ. Its edges e_j, $j \in J$, are non-degenerate open curve such that the closure $\overline{e_j}$ is a smooth curve, where J is an index set. The endpoints $\overline{e_j}\backslash e_j$ are the vertices. Here we should note that we allow these edges to be smooth curves, not just line segments. We shall assume that Γ has, at most, a countably infinite number of edges, and hence the index set J is a subset of the natural numbers $\mathbf{N}$. We also assume that each vertex of Γ is of *finite degree*, that is, only a finite number of edges emanate from each vertex, and that only one edge emanates from a vertex c if c belongs to the boundary $\partial\Omega$ of Ω. For every $x, y \in \Gamma$ there is a unique path in Γ joining x and y. Thus, by introducing the distance between x and y by the length of a unique path connecting x and y, Γ becomes a metric space. Also, if Γ is endowed with the natural one-dimensional Lebesgue measure, it is a σ-finite measure space. The tree Γ is rooted at an arbitrary fixed point $a \in \Gamma$. We define $t \succeq_a x$ (or equivalently $x \preceq_a t$) to mean that x lies on the path from a to t.

Throughout this work we assume the following: **(I)** Assumptions on Ω and Γ:

(1-i) Ω be a domain (i.e., a connected open set) in $\mathbb{R}^2$ and $\Gamma \subset \overline{\Omega}$ be a connected tree which has at most countable number of edges e_j, $j \in J$, where $\overline{\Omega}$ is the closure of Ω. Each edge e_j is an open curve with finite length such that the closure $\overline{e_j}$ is a C^2 curve. The endpoints $\overline{e_j}\backslash e_j$ are called the vertices. When any two edges are connected, they are connected only at their vertices. Also they are not tangential at the vertex from which the two edges emanate.

(1-ii) We have $E(\Gamma) \subset \Omega$, where $E(\Gamma)$ is the set of all edges of Γ.

(1-iii) For $v \in V(\Gamma) \cap \partial\Omega$, only one edge emanates from v, where $V(\Gamma)$ is the set of all vertices of Γ, and $\partial\Omega$ is the boundary of Ω.

(II) Assumptions on τ. There exists a map τ from Ω into $\Omega \cap \Gamma$ which satisfies the following:

(2-i) The subset $\tau^{-1}(V(\Gamma))$ is a (2-dimensional) null set. For each $e_j \in E(\Gamma)$, set $\Omega_j = \tau^{-1}(e_j)$. Then Ω_j is an open set and τ is locally Lipschitz continuous on Ω_j, that is, for each $x \in \Omega_j$ there exists a neighborhood $V(x) \subset \Omega_j$ of x and a positive constant $\gamma(x)$ such that for all $y \in V(x)$

$$d_\Gamma(\tau(x), \tau(y)) \le \gamma(x)|x - y|, \tag{2.1}$$

where d_Γ denotes the metric on Γ and $|\cdot|$ the Euclidean metric (for definiteness) on $\mathbb{R}^2$.

(2-ii) Let $C(t) = \tau^{-1}(t)$ for $t \in E(\Gamma)$. Then $C(t)$ is a rectifiable curve. Further, $C(t) \cap \Gamma = \{t\}$ and $C(t) \backslash \{t\}$ has two components, $C_\pm(t)$ say. Also we assume that $C(t)$ is not tangential to Γ at t. Let $C_+(t)$ and $C_-(t)$ be parameterized by arc length s which is measured from t with $0 \le s \le \ell_+(t)$ on $C_+(t)$ and $-\ell_-(t) \le s \le 0$ on $C_-(t)$. Let $\tau(x) = (\tau_1(x), \tau_2(x)) \in \Gamma$ for $x \in \Omega$. Then, for $t \in E(\Gamma)$, there exists a null set $e(t) \subset C(t)$ with respect to ds, the measure induced by the arc length parameter s on $C(t)$, such that τ_1 and τ_2 are differentiable at $x \in C(t) \backslash e(t)$.

(2-iii) Let $|\nabla\tau(x)| = [|\nabla\tau_1(x)|^2 + |\nabla\tau_2(x)|^2]^{1/2}$. For $t \in E(\Gamma)$ fixed, define $|\nabla\tau(s)|$ on $C(t)$ by $|\nabla\tau(s)| = |\nabla\tau(x)|$ with $x \in C(t)$ and $d_{C(t)}(t, x) = s$, where $d_{C(t)}(t, x)$ is the distance between t and x along $C(t)$. Then $|\nabla\tau(s)|, |\nabla\tau(s)|^{-1} \in L^1(C(t), ds)$.

(2-iv) For any vertex $v \in \Omega$ the functions $\ell_\pm$ are bounded below from 0 around v, i.e., for a vertex $v \in \Omega$, there exists a neighborhood $U(v) \subset \Gamma$ of v such that

$$\begin{aligned} &\inf_{t \in U(v)\backslash\{v\}} \ell_-(t) > 0, \\ &\inf_{t \in U(v)\backslash\{v\}} \ell_+(t) > 0\,. \end{aligned} \tag{2.2}$$

Some examples of the triples (Ω, Γ, τ) are given in [2, 3, 4, 11] including horn-shaped domains, room and passages domains and fractal domains.

For $j \in J$ let the edge e_j have the vertices a_j and b_j such that $b_j \succeq_a a_j$, where the tree Γ is rooted at a. Then we parameterize e_j by $\sigma_j(t) = dist(a_j, t)$, where $dist(a_j, t)$ is the arc length from a_j to $t \in e_j$ along e_j. From now on we may drop the subscript j in σ_j if there is no danger of misunderstanding. If $x = (x_1, x_2) \in \Omega_j = \tau^{-1}(e_j) \subset \Omega$ is such that $\tau(x) = t(\sigma)$ and $dist(t(\sigma), x) = s$, where $dist(t(\sigma), x)$ is the distance between x and $t(\sigma)$ along the curve $C_{t(\sigma)}$, then a co-ordinate system on Ω_j is defined by

$$x = x(\sigma, s), \quad \tau(x) = t(\sigma), \quad s \in (-\ell_-(\sigma),\ \ell_+(\sigma)), \tag{2.3}$$

where $\ell_\pm(\sigma) = \ell_\pm(t(\sigma))$. A family $\{\Omega_\epsilon\}_{0<\epsilon\le 1}$ of sub-domains of Ω is defined as follows:

Definition For $j \in J$ and $0 < \epsilon \le 1$ let

$$\Omega_j^{(\epsilon)} = \{x = x(\sigma, s) \ / \ \sigma \in e_j, \ -\epsilon\ell_-(\sigma) < s < \epsilon\ell_+(\sigma)\} \tag{2.4}$$

and

$$\Omega_\epsilon = \big[\cup_{j\in J} \overline{\Omega_j^{(\epsilon)}} \big]^\circ, \tag{2.5}$$

where A° is the interior of A. By definition we have $\Omega = \Omega_1$.

Definition For each $0 < \epsilon \le 1$ let H_{Ω_ϵ} be the Neumann Laplacian on Ω_ϵ.

It is known ((2.4) in [4]) that

$$\frac{\partial(x_1, x_2)}{\partial(\sigma, s)} = \frac{1}{|\nabla\tau(\sigma, s)|}. \tag{2.6}$$

Let $I \in e_j$. Then we have, for $f \in L^1(\tau^{-1}(I))$,

$$\int_{\tau^{-1}(I)} f(x)dx = \int_I d\sigma \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} f(\sigma, s)|\nabla\tau(\sigma, s)|^{-1}\, ds. \tag{2.7}$$

Note that we have again simplified the notation by writing (σ, s) for $x(\sigma, s)$. Of particular importance is the case when $f = F \circ \tau$ in (2.7) with $F \in L^1(I)$:

$$\begin{aligned} \int_{\tau^{-1}(I)} F\circ\tau(x)\, dx &= \int_I F(\sigma)\, d\sigma \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |\nabla\tau(\sigma, s)|^{-1}\, ds \\ &=: \int_I F(\sigma)\alpha_\epsilon(\sigma)\, d\sigma, \end{aligned} \tag{2.8}$$

where

$$\alpha_\epsilon(\sigma) := \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \frac{1}{|\nabla\tau(\sigma, s)|}\, ds. \tag{2.9}$$

If $I = e_j$ in (2.7) and (2.8), then $\tau^{-1}(I)$ should be replaced by $\Omega_j^{(\epsilon)}$.

3 Evaluation of $u_\epsilon = (H_{\Omega_\epsilon} - z)^{-1}f$ on Γ

We shall start with an additional assumption on the tree Γ and the family $\{\Omega_\epsilon\}_{0<\epsilon\le 1}$. Then we shall show some evaluation for the restriction of

$$u_\epsilon = u_\epsilon(f, z)) = (H_{\Omega_\epsilon} - z)^{-1} f_\epsilon \tag{3.1}$$

on Γ, where $z \in \mathbf{C}\backslash[0, \infty)$ and f_ϵ is the restriction of $f \in L^2(\Omega)$ on Ω_ϵ.

Assumption 3.1. (i) For each $j \in J$ $\ell_\pm(\sigma)$ are positive C^1 function on e_j and are continuously extended on $\overline{e_j}$. Also $\ell_\pm(\sigma)$ satisfy

$$\begin{aligned} \sup\nolimits_{e_j}(\ell_-(\sigma)+\ell_+(\sigma)) \equiv L_j < \infty, \\ \sup\nolimits_{e_j}(|\ell'_-(\sigma)|+|\ell'_+(\sigma)|) \equiv R_j < \infty, \end{aligned} \tag{3.2}$$

where $\ell'_\pm(\sigma) = \frac{d}{d\sigma}\ell_\pm(\sigma)$.

(ii) For $j \in J$ there exists $\epsilon_j \in (0,1]$ such that $|\nabla\tau(\sigma,s)|$ is continuous on $\Omega_j^{(\epsilon_j)}$,

$$\begin{aligned} 0 < m_j \equiv \inf\nolimits_{x(\sigma,s)\in\Omega_j^{(\epsilon_j)}} |\nabla\tau(\sigma,s)| \le \sup\nolimits_{x(\sigma,s)\in\Omega_j^{(\epsilon_j)}} |\nabla\tau(\sigma,s)| \equiv M_j < \infty, \\ \sup\nolimits_{x(\sigma,s)\in\Omega_j^{(\epsilon_j)}} \left|\tfrac{\partial x(\sigma,s)}{\partial s}\right| \equiv K_j < \infty. \end{aligned} \tag{3.3}$$

Now we introduce a positive function on each e_j which will play an important role.

Definition 3.2. For each $j \in J$, set

$$a_0^{(j)}(\sigma) = \frac{\ell_-(\sigma)+\ell_+(\sigma)}{|\nabla\tau(\sigma,0)|} \quad (\sigma \in e_j). \tag{3.4}$$

Note that $a_0^{(j)}$ is a bounded positive function on e_j. From now on we may drop the superscript j in $a_0^{(j)}$ if there is no risk of misunderstanding, i.e. $a_0(\sigma) = a_0^{(j)}(\sigma)$. Also note that

$$a_0(\sigma) = \lim_{\epsilon\to 0} \epsilon^{-1}\alpha_\epsilon(\sigma) \quad (\sigma \in e_j), \tag{3.5}$$

where α_ϵ is given by (2.9).

For a subset Ω' of Ω, $\|f\|_{\Omega'}$ denotes the L^2 norm of f on Ω'. Also we set

$$\begin{aligned} \|\psi\|^2_{e_j,a_o} = \textstyle\int_{e_j} |\psi(\sigma)|^2 a_0(\sigma)\,d\sigma, \\ \|\psi\|^2_{e_j} = \textstyle\int_{e_j} |\psi(\sigma)|^2\,d\sigma, \end{aligned} \tag{3.6}$$

Lemma 3.3. *We have*

$$\|u(\cdot,0)\|^2_{e_j,a_0} \le 2\Big(\frac{M_j}{m_j}\Big)\big\{\epsilon L_j^2 K_j^2 \|\nabla u\|^2_{\Omega_j^{(\epsilon)}} + \epsilon^{-1}\|u\|^2_{\Omega_j^{(\epsilon)}}\big\}, \tag{3.7}$$

and

$$\|u\|^2_{\Omega_j^{(\epsilon)}} \le 2\epsilon\Big(\frac{M_j}{m_j}\Big)\big\{\epsilon L_j^2 K_j^2 \|\nabla u\|^2_{\Omega_j^{(\epsilon)}} + \|u(\cdot,0)\|^2_{e_j,a_0}\big\} \tag{3.8}$$

for $u \in H^1(\Omega_j^{(\epsilon)}) \cap C^1(\Omega_j^{(\epsilon)})$, *where* $\|\ \|_A$, $A \subset \mathbb{R}^2$, *is the norm of* $L^2(A)$.

Proof. (I) Let I be a closed subset of e_j. Since $I \subset e_j \subset \Omega_j^{(\epsilon)}$, $u_\epsilon(\sigma, 0)$ and $u'_\epsilon(\sigma, 0)$ are bounded on I. Then we have

$$\begin{aligned}
\|u(\cdot,0)\|^2_{I,a_0} &= \int_I \epsilon^{-1}|\nabla\tau(\sigma,0)|^{-1}\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |u(\sigma,0)|^2\,ds\,d\sigma \\
&\le 2\epsilon^{-1}m_j^{-1}\int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |u(\sigma,0)-u(\sigma,s)|^2\,ds\,d\sigma \\
&\quad +2\epsilon^{-1}m_j^{-1}\int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |u(\sigma,s)|^2\,ds\,d\sigma \\
&\equiv 2\epsilon^{-1}m_j^{-1}(I_1+I_2).
\end{aligned} \tag{3.9}$$

Using the second inequality of (3.3), wee see that

$$\big|\frac{\partial u}{\partial s}\big| \le K_j|\nabla u|, \tag{3.10}$$

which is combined with (2.7) to give

$$\begin{aligned}
I_1 &\le \int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big|\int_0^s \big|\frac{\partial u}{\partial s}(\sigma,\eta)\big|\,d\eta\big|^2\,ds\,d\sigma \\
&\le \int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big(\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big|\frac{\partial u}{\partial s}(\sigma,\eta)\big|\,d\eta\big)^2\,ds\,d\sigma \\
&\le L_j\epsilon\int_I \big(\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big|\frac{\partial u}{\partial s}(\sigma,s)\big|\,ds\big)^2\,d\sigma \\
&\le (L_j\epsilon)^2\int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big|\frac{\partial u}{\partial s}(\sigma,s)\big|^2\,ds\,d\sigma \\
&\le (L_j\epsilon)^2K_j^2M_j\int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |\nabla u(\sigma,s)|^2\frac{ds\,d\sigma}{|\nabla\tau(\sigma,s)|} \\
&\le (L_j\epsilon)^2K_j^2M_j\|\nabla u\|^2_{\Omega_j^{(\epsilon)}},
\end{aligned} \tag{3.11}$$

where we have used the fact that $\tau^{-1}(I) \subset \Omega_j^{(\epsilon)}$. As for I_2 we have

$$I_2 \le M_j\int_I\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |u(\sigma,s)|^2\frac{ds\,d\sigma}{|\nabla\tau(\sigma,s)|} \le M_j\|u\|^2_{\Omega_j^{(\epsilon)}}. \tag{3.12}$$

Thus we have from (3.11) and (3.12)

$$\|u(\cdot,0)\|^2_{I,a_0} \le 2\big(\frac{M_j}{m_j}\big)\big\{L_j^2K_j^2\epsilon\|\nabla u\|^2_{\Omega_j^{(\epsilon)}} + \epsilon^{-1}\|u\|^2_{\Omega_j^{(\epsilon)}}\big\}. \tag{3.13}$$

Since $I \subset e_j$ is arbitrary, (3.7) follows from (3.13).

(II) As in (I), we have

$$
\begin{aligned}
\|u\|^2_{\Omega_j^{(\epsilon)}} &\leq 2\int_{e_j}\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |u(\sigma,s)-u(\sigma,0)|^2|\nabla\tau(\sigma,s)|^{-1}\,ds\,d\sigma \\
&\quad +2\int_{e_j}\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |u(\sigma,0)|^2|\nabla\tau(\sigma,s)|^{-1}\,ds\,d\sigma \\
&\equiv 2(J_1+J_2).
\end{aligned} \tag{3.14}
$$

Then we can proceed as in evaluation I_j to obtain

$$
J_1 \leq \Big(\frac{M_j}{m_j}\Big)L_j^2K_j^2\epsilon^2\|\nabla u\|^2_{\Omega_j^{(\epsilon)}}\,, \tag{3.15}
$$

$$
J_2 \leq \Big(\frac{M_j}{m_j}\Big)\epsilon\|u(\cdot,0)\|^2_{e_j,a_0}, \tag{3.16}
$$

which completes the proof. ♢

Proposition 3.4. *Suppose that Assumptions* 2.1 *and* 3.1 *hold. Let* $f\in H^1(\Omega)$. *Set*

$$
u_\epsilon = u_\epsilon(f_\epsilon,z) = (H_{\Omega_\epsilon}-z)^{-1}f_\epsilon, \tag{3.17}
$$

where $z\in\mathbb{C}\backslash[0,\infty)$ *and* f_ϵ *is the restriction of* f *on* $\Omega_j^{(\epsilon)}$. *Suppose that*

$$
\limsup_{\epsilon\to 0}\sum_{k\in J}\Big(\frac{M_k}{m_k}\Big)\big\{\epsilon L_k^2K_k^2\|\nabla f\|^2_{\Omega_k^\epsilon}+\|f(\cdot,0)\|^2_{e_k,a_0}\big\}<\infty, \tag{3.18}
$$

where $f(\sigma,0)$ *on each edge* e_j *is given by the trace of* f *on* e_j, *Then, for sufficiently small* $\epsilon\in(0,1]$ *and* $j\in J$,

$$
\begin{aligned}
\|u_\epsilon(\cdot,0)\|^2_{e_j,a_0} &\leq 4\Big(\frac{M_j}{m_j}\Big)|z|^{-1}\Big[L_j^2K_j^2\|f\|^2_{\Omega_\epsilon} \\
&\quad +|z|^{-1}\sum_{k\in J}\Big(\frac{M_k}{m_k}\Big)\big\{\epsilon L_k^2K_k^2\|\nabla f\|^2_{\Omega_k^\epsilon}+\|f(\cdot,0)\|^2_{e_k,a_0}\big\}\Big].
\end{aligned} \tag{3.19}
$$

Remark 3.5 It has been known (see,e.g., Gilbarg-Trudinger [6], Theorem 8.10) that the condition $f\in H^1(\Omega)$ implies $u_\epsilon\in H^3(\Omega)_{\rm loc}$, since $u_\epsilon\in H^1(\Omega)$. Then, by the Sobolev imbedding theorem (see, e.g. Adams [1], Theorem 5.4), we have $u_\epsilon\in C^1(\Omega)$.

Proof of Proposition 3.4. (I) It is easy to see that

$$
\|u_\epsilon\|^2_{\Omega_j^{(\epsilon)}} \leq \|u_\epsilon\|^2_{\Omega_\epsilon} \leq |z|^{-2}\|f\|^2_{\Omega_\epsilon}, \tag{3.20}
$$

$$
\|\nabla u_\epsilon\|^2_{\Omega_j^{(\epsilon)}} \leq \|\nabla u_\epsilon\|^2_{\Omega_\epsilon} \leq |z|\|u\|^2_{\Omega_\epsilon}+\|f\|_{\Omega_\epsilon}\|u_\epsilon\|_{\Omega_\epsilon} \leq 2|z|^{-1}\|f\|^2_{\Omega_\epsilon}.
$$

(II) It follows from (3.20) and (3.7) with u replaced by u_ϵ that

$$\|u_\epsilon(\cdot,0)\|^2_{e_j,a_0} \le 2\big(\frac{M_j}{m_j}\big)\big\{2\epsilon L_j^2 K_j^2 |z|^{-1}\|f\|^2_{\Omega_j^{(\epsilon)}} + \epsilon^{-1}|z|^{-2}\|f\|^2_{\Omega_j^{(\epsilon)}}\big\}. \qquad (3.21)$$

The inequality (3.19) is obtained from (3.21) and (3.8) with u replaced by f. $\diamondsuit$

4 The limiting equation

Let $u_\epsilon = (H_{\Omega_\epsilon} - z)^{-1} f$ be as in (3.17). Since $\|u_\epsilon(\cdot,0)\|_{e_j,a_0}$ is uniformly bounded for $\epsilon \in (0,1]$ by Proposition 3.4 if $f \in H^1(\Omega)$ satisfies (3.18), u_ϵ converges weakly along some subsequence $\{\epsilon_m\}_{m=1}^\infty$ with the limiting function u_0. In this section we shall prove that u_0 is a solution of a second-order ordinary differential equation on Γ with the Kirchhoff boundary condition on each vertex (Theorems 4.5 and 4.8). The equation is independent from choice of the subsequence $\{\epsilon_m\}_{m=1}^\infty$. First we shall state two lemmas (Lemmas 4.1 and 4.4) which will play crucial roles in this section. These lemmas will be shown in §6. In order to prove Lemma 4.4, we need another important assumption (Assumption 4.2). Let Γ_0 be a measurable subset of the tree Γ and let a be a positive measurable function defined on $\Gamma_0 \cap \cup_{j\in J} e_j$. Then the Hilbert space $L^2(\Gamma_0, a)$ is a weighted L^2 space with inner product

$$(F,\ G)_{\Gamma_0,a} = \sum_{j\in J}\int_{\Gamma_0\cap e_j} F(\sigma)\overline{G(\sigma)}\, a(\sigma) d\sigma \qquad (4.1)$$

and norm $\|F\|_{\Gamma_0} = [(F,\ F)_{\Gamma_0,a}]^{1/2}$. We denote $L^2(\Gamma_0, 1)$ by $L^2(\Gamma_0)$.

Lemma 4.1. *Suppose that* Assumptions 2.1 *and* 3.1 *are satisfied. Let* $j \in J$ *and let* $\{u_\epsilon\}_{0<\epsilon\le 1}$ *be a family of functions such that*

$$u_\epsilon \in H^1(\Omega_j^{(\epsilon)}) \bigcap C^1(\Omega_j^{(\epsilon)}) \quad (0 < \epsilon \le 1). \qquad (4.2)$$

Let $F \in L^2(e_j, a_0)$*, i.e.,* $\|F\|_{e_j,a_0} < \infty$*. Set* $v(x) = F(\tau(x))$*. Then there exists* $C_j = C_j(M_j, m_j, L_j, K_j)$*, a positive constant depending only on* M_j, m_j, L_j, K_j*, such that*

$$\begin{aligned} &\big|\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} u_\epsilon(x)\overline{v(x)}\, dx - \int_{e_j} u_\epsilon(\sigma,0)\overline{F(\sigma)} a_0(\sigma)\, d\sigma\big| \qquad (4.3)\\ &\le \quad C_j\big\{\sqrt{\epsilon}\|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}}\|F\|_{e_j,a_0} + \|u_\epsilon(\cdot,s)\|_{e_j,a_0}\big[\int_{e_j}|F(\sigma)|^2\psi_\epsilon(\sigma)^2 a_0(\sigma)\, d\sigma\big]^{1/2}\big\}, \end{aligned}$$

where

$$\psi_\epsilon(\sigma) = \frac{1}{\epsilon a_0(\sigma)}\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} (|\nabla\tau(\sigma,s)|^{-1} - |\nabla\tau(\sigma,0)|^{-1})\, ds. \qquad (4.4)$$

Here we need another assumption.

Assumption 4.2. (i) All the edges e_j of the tree Γ are finite (non-degenerate) *line segments.* (ii) Let $j \in J$. Let E be a closed subset of the (open) edge e_j. Then there exists a positive number $\epsilon_j(E) \in (0,1]$, depending only on j and E, such that, for any $t \in E$, the portion of the curve $C_t \cap \Omega_j^{(\epsilon_j(E))}$ is a line segment which is *perpendicular* to e_j.

Remark 4.3. (i) Roughly speaking, (ii) of the above assumption claims that the curve C_t is perpendicular to e_j near e_j except the vertices. (ii) Assumption 4.2, (ii) also implies

$$\begin{aligned} |\nabla\tau(\sigma,s)| = 1 \quad ((\sigma,s) \in \tau^{-1}(E) \cap \Omega_j^{(\epsilon_j(E))}), \\ |\nabla\tau(\sigma,0)| = 1 \quad (\sigma \in e_j), \\ a_0(\sigma) = \ell_-(\sigma) + \ell_+(\sigma) \quad (\sigma \in e_j). \end{aligned} \tag{4.5}$$

Lemma 4.4. *Suppose that* Assumptions 2,1, 3.1 *and* 4.2 *hold. Let $j \in J$ and let $\{u_\epsilon\}_{0<\epsilon\le 1}$ be a family of functions such that*

$$u_\epsilon \in H^1(\Omega_j^{(\epsilon)}) \bigcap C^1(\Omega_j^{(\epsilon)}) \quad (0 < \epsilon \le 1). \tag{4.6}$$

Let $F \in C^2(e_j)$ with $F' \in C_0^1(e_j)$, where F' is the derivative of F with respect to σ. Let $\epsilon_0 = \epsilon_j(supp\, F')$. Then, by setting $v(x) = F(\tau(x))$, the inequality

$$\begin{aligned} &\Big|\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} \nabla u_\epsilon \cdot \overline{\nabla v}\, dx - \int_{e_j} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma,0)\overline{F'(\sigma)} a_0(\sigma)\, d\sigma\Big| \\ &\le \sqrt{\epsilon} C(L_j,R_j)(\|F'\|_{e_j,a_0} + \|F''\|_{e_j,a_0})\|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}} \end{aligned} \tag{4.7}$$

holds for $\epsilon \in (0,\epsilon_0)$, where $C(L_j,R_j)$ is a positive constant depending only on L_j and R_j.

Theorem 4.5. *Suppose that Assumptions* 2,1,3.1 *and* 4.2 *hold. Let $f \in H^1(\Omega)$ which satisfies* (3.17). *Let f_ϵ be the restriction of f on $\Omega_j^{(\epsilon)}$ for $\epsilon \in (0,1]$. Let $u_\epsilon = (H_{\Omega_\epsilon} - z)^{-1} f_\epsilon$ be as in* (3.16). *Let $j \in J$. Let $\{\epsilon_m\}_{m=1}^\infty \subset (0,1]$ be a decreasing sequence such that $\{\epsilon_m\}_{m=1}^\infty$ converges to 0 and the sequence $\{u_{\epsilon_m}(\cdot,0)\}_{m=1}^\infty$ converges weakly in $L^2(e_j,a_0)$. Then the limit function u_0 satisfies*

$$\int_{e_j} u_0(\sigma)\big\{-(a_0(\sigma)\overline{F'(\sigma)})' - z\overline{F(\sigma)}a_0(\sigma) - f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\big\}\, d\sigma = 0 \tag{4.8}$$

for any $F \in C_0^2(e_j)$, i.e., u_0 is a weak solution of the equation

$$-\frac{1}{a_0}(a_0 u')' - zu = f(\cdot,0) \tag{4.9}$$

on e_j.

Remark 4.6. (i) The sequence $\{\epsilon_m\}_{m=1}^{\infty}$ which satisfies the conditions in the above theorem does exist since $\|u_\epsilon(\cdot,0)\|_{e_j,a_0}$ is uniformly bounded for $\epsilon \in (0,1]$ by Proposition 3.4. (ii) Thus, the limit function u_0 is, not only a weak solution of (4.9), but also a strong solution with $u_0 \in C^2(e_j)$.

Proof of Theorem 4.5. (I) Let $v(x) = F(\tau(x))$. We extend F on Γ by setting $F = 0$ outside e_j. Then we have $v \in H^1(\Omega_\epsilon)$ and $v = 0$ outside $\Omega_j^{(\epsilon)}$. Let ϵ_0 be as in Lemma 4.1. Note that $\psi_\epsilon(\sigma) = 0$ on e_j for $\epsilon \in (0,\epsilon_0]$, where $\psi_\epsilon(\sigma)$ is given by (4.4). Therefore, replacing u_ϵ by zu_ϵ in Lemma 4.1 and using the second inequality in (3.20), we obtain

$$
\begin{aligned}
&\Big|\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} zu_\epsilon(x)\overline{v(x)}\,dx - \int_{e_j} zu_\epsilon(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\,d\sigma\Big| \\
&\le C_j|z|\sqrt{\epsilon}\|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}}\|F\|_{e_j,a_0} \qquad (4.10)\\
&\le 2C_j\sqrt{\epsilon}\|f\|_{\Omega_\epsilon}\|F\|_{e_j,a_0}
\end{aligned}
$$

for $\epsilon \in (0,\epsilon_0]$, which implies that

$$
\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} zu_\epsilon(x)\overline{v(x)}\,dx = \int_{e_j} zu_\epsilon(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\,d\sigma + O(\sqrt{\epsilon}) \quad (\epsilon \to 0), \tag{4.11}
$$

when $F \in C_0^2(e_j)$ is fixed. Next we set $u_\epsilon = f_\epsilon$ in Lemma 4.1 to obtain

$$
\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} f_\epsilon(x)\overline{v(x)}\,dx = \int_{e_j} f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\,d\sigma + O(\sqrt{\epsilon}) \quad (\epsilon \to 0), \tag{4.12}
$$

(II) Similarly we have from Lemma 4.4

$$
\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} \nabla u_\epsilon(x)\cdot\overline{\nabla v(x)}\,dx = \int_{e_j} \frac{\partial u_\epsilon}{\partial\sigma}(\sigma,0)\overline{F'(\sigma)}a_0(\sigma)\,d\sigma + O(\sqrt{\epsilon}) \quad (\epsilon \to 0), \tag{4.13}
$$

(III) It follows from (4.11), (4.12) and (4.13) that

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} \big\{\nabla u_\epsilon\cdot\overline{\nabla v} - zu_\epsilon\overline{v} - f_\epsilon\overline{v}\big\}\,dx \\
&= \int_{e_j}\Big\{\frac{\partial u_\epsilon}{\partial\sigma}(\sigma,0)\overline{F'(\sigma)}a_0(\sigma) - zu_\epsilon(\sigma,0)\overline{F(\sigma)}a_0(\sigma) \qquad (4.14)\\
&\quad - f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\Big\}\,d\sigma + O(\sqrt{\epsilon}).
\end{aligned}
$$

Noting that the domain of integration in the left-hand side of (4.13) can be extended to Ω_ϵ and that $v \in H^1(\Omega_\epsilon)$, by the definition of the Neumann Laplacian H_{Ω_ϵ}

$$
\begin{aligned}
0 &= \frac{1}{\epsilon}\int_{\Omega_\epsilon} \big\{\nabla u_\epsilon\cdot\overline{\nabla v} - zu_\epsilon\overline{v} - f_\epsilon\overline{v}\big\}\,dx \\
&= \int_{e_j}\Big\{\frac{\partial u_\epsilon}{\partial\sigma}(\sigma,0)\overline{F'(\sigma)}a_0(\sigma) - zu_\epsilon(\sigma,0)\overline{F(\sigma)}a_0(\sigma) \qquad (4.15)\\
&\quad - f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\Big\}\,d\sigma + O(\sqrt{\epsilon}).
\end{aligned}
$$

Thus, by using partial integration, we have

$$0 = \int_{e_j} u_\epsilon(\sigma)\big\{ - (a_0(\sigma)\overline{F'(\sigma)})' - z\overline{F(\sigma)}a_0(\sigma) - f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\big\}\, d\sigma + O(\sqrt{\epsilon}). \tag{4.16}$$

Set $\epsilon = \epsilon_m$ in (4.15) and let $m \to \infty$. Then we have (4.7), where we should note that, since a_0 is positive and bounded below from zero on any closed subset of, $u_{\epsilon_m}(\cdot, 0)$ converges to u_0 weakly in $L^2(I)$ as well as in $L^2(I, a_0)$ with any closed subset I of e_j. This completes the proof. ♢

Corollary 4.7. *Let u_0 be as in* Theorem 4.5. *Then following limits exist*

$$\lim_{\sigma\to a_j+0} a_0(\sigma)u_0'(\sigma), \quad \lim_{\sigma\to b_j-0} a_0(\sigma)u_0'(\sigma)\,.$$

Proof. The proof is obvious. With $\sigma, \sigma_0 \in e_j$ we have

$$a_0(\sigma)u_0'(\sigma) = -\int_{\sigma_0}^{\sigma} \big(zu_\epsilon(\eta,0) - f(\eta,0)\big)\, d\eta + a_0(\sigma_0)u_0'(\sigma_0) \tag{4.17}$$

♢ Noting that Γ consists of at most countably infinite edges, we may assume that there exists a sequence $\{\epsilon_m\}_{m=1}^{\infty}$ such that there exists $u_0 \in L^2(\Gamma, a_0)_{\rm loc}$ such that

$$\begin{gathered} \epsilon_m \to 0 \quad (m\to\infty), \\ u_{\epsilon_m}(\cdot,0) \to u_0 \quad \text{in } L^2(e_j) \quad (j\in J), \end{gathered} \tag{4.18}$$

Theorem 4.8. *Suppose that* Assumptions 2,1, 3.1 *and* 4.2 *hold. Let $u_{\epsilon_m} = (H_{\Omega_\epsilon} - z)^{-1}f_{\epsilon_m}$ be as in* (4.18), *where f is as in* Theorem 4.5. *Let c be a vertex of Γ and set*

$$J(c) = \{j \in J : a_j = c \text{ or } b_j = c\}, \tag{4.19}$$

where a_j and b_j are the endpoints of e_j with $b_j \succeq_a a_j$. Then it follows that

$$\sum_{j\in J(c)} \eta(j)a_0(c)u_0'(c) = 0, \tag{4.20}$$

i.e., The Kirchhoff boundary condition is satisfied at each vertex of Γ, where

$$\eta(j) = \begin{cases} 1 & \text{if } c = b_j, \\ -1 & \text{if } c = a_j, \end{cases} \tag{4.21}$$

and

$$a_0(c)u_0'(c) = \begin{cases} \lim_{\sigma\to b_j, \sigma\in e_j} a_0(\sigma)u_0'(\sigma) & \text{if } \eta(j) = 1, \\ \lim_{\sigma\to a_j, \sigma\in e_j} a_0(\sigma)u_0'(\sigma) & \text{if } \eta(j) = -1. \end{cases} \tag{4.22}$$

Proof. (I) Let F be a function defined on Γ such that

$$\begin{aligned} &\operatorname{supp} F \subset (\cup_{j\in J(c)} e_j) \cup \{c\},\\ &F = 1 \ \text{ in a neighborhood of } c,\\ &F \text{ is } C^2 \text{ on each } e_j \text{ with } j \in J(c). \end{aligned} \tag{4.23}$$

Let F_j be the restriction of F on $\overline{e_j}$. Then $F_j \in C^2(\Gamma_j)$ and $F_j' \in C_0^1(e_j)$. Set $v_j(x) = F_j(\tau(x))$. Then, using Lemmas 4.1 and 4.4, and proceeding as in the proof of Theorem 4.5, we have

$$\begin{aligned} &\frac{1}{\epsilon}\int_{\Omega_j^{(\epsilon)}} \{\nabla u_\epsilon \cdot \overline{\nabla v_j} - z u_\epsilon \overline{v_j} - f_\epsilon \overline{v_j}\}\,dx\\ &= \int_{e_j} \big\{\frac{\partial u_\epsilon}{\partial \sigma}(\sigma,0)\overline{F_j(\sigma)}a_0(\sigma) - z u_\epsilon(\sigma,0)\overline{F_j(\sigma)}a_0(\sigma)\\ &\quad -f(\sigma,0)\overline{F_j(\sigma)}a_0(\sigma)\big\}\,d\sigma + O(\sqrt{\epsilon}). \end{aligned} \tag{4.24}$$

By summing up both sides of (4.24) with respect to $j \in J(c)$, which is a finite set since Γ is of finite degree, we obtain, as in (4.14),

$$\begin{aligned} 0 &= \sum_{j\in J(c)} \int_{e_j} \big\{\frac{\partial u_\epsilon}{\partial \sigma}(\sigma,0)\overline{F'(\sigma)}a_0(\sigma) - z u_\epsilon(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\\ &= -f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\big\}\,d\sigma + O(\sqrt{\epsilon}), \end{aligned} \tag{4.25}$$

where, and in the sequel, we shall use $F(\sigma)$ in place of F_j. Here, by partial integration,

$$\int_{e_j} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma,0)\overline{F'(\sigma)}a_0(\sigma)\,d\sigma = -\int_{e_j} u_\epsilon(\sigma,0)(a_0(\sigma)\overline{F'(\sigma)})'\,d\sigma, \tag{4.26}$$

where we should note that F' has a compact support in e_j. Combine (4.25) and (4.26) and let $\epsilon \to 0$ along ϵ_m to give

$$\begin{aligned} 0 &= \sum_{j\in J(c)} \int_{e_j} \big\{ -u_0(\sigma,0)(\overline{F'(\sigma)}a_0(\sigma))' - z u_0(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\\ &\quad -f(\sigma,0)\overline{F(\sigma)}a_0(\sigma)\big\}\,d\sigma + O(\sqrt{\epsilon}). \end{aligned} \tag{4.27}$$

(II) Suppose that $c = a_j$. Then, repeating partial integration, and noting that $F = 1$ near c, we obtain

$$\begin{aligned} &-\int_{e_j} u_0(\sigma,0)(\overline{F'(\sigma)}a_0(\sigma))'\,d\sigma\\ &= \int_{e_j} u_0'(\sigma,0)\overline{F'(\sigma)}a_0(\sigma)\,d\sigma\\ &= \lim_{\sigma\to a_j}\int_\sigma^{b_j} u_0'(\sigma)\overline{F'(\sigma)}a_0(\sigma)\,d\sigma \end{aligned} \tag{4.28}$$

$$
\begin{aligned}
&= -\int_{e_j} (a_0(\sigma)u_0'(\sigma,0))'\overline{F'(\sigma)}\,d\sigma - a_0(c)u_0'(c) \\
&= -\int_{e_j} (a_0(\sigma)u_0'(\sigma,0))'\overline{F'(\sigma)}\,d\sigma + \eta(j)a_0(c)u_0'(c)
\end{aligned}
$$

Similarly, we have, for $c = b_j$,

$$
\begin{aligned}
&-\int_{e_j} u_0(\sigma,0)(\overline{F'(\sigma)}a_0(\sigma))'\,d\sigma \\
&\quad = -\int_{e_j} (a_0(\sigma)u_0'(\sigma,0))'\overline{F'(\sigma)}\,d\sigma + \eta(j)a_0(c)u_0'(c),
\end{aligned}
\tag{4.29}
$$

where we should note that $u_0 \in C^2(e_j)$ ((ii) of Remark 4.6).

(III) It follows from (4.26), (4.27) and (2.28) that

$$
\begin{aligned}
0 &= \sum_{j\in J(c)} \int_{e_j} \big\{ -(a_0(\sigma)u_0'(\sigma,0))' - zu_0(\sigma,0)a_0(\sigma,0) \\
&\quad -f(\sigma,0)a_0(\sigma,0)\big\}\overline{F(\sigma)}\,d\sigma + \sum_{j\in J(c)} \eta(j)a_0(c)u_0'(c)
\end{aligned}
\tag{4.30}
$$

Since u_0 is now a strong solution of the equation (4.9), the first term of the left-hand side of (4.30) is zero, and hence the Kirchhoff boundary condition (4.20) follows from (4.30). $\diamondsuit$

5 Continuity of the limit function

Let u_0 be a limit function on Γ given by (4.17). Since u_0 is a solution of the differential equation on each e_j, $j \in J$, u_0 is smooth on each e_j (Remark 4.6, (ii)). In this section, we shall show, under some additional conditions, that $\{u_\epsilon\}$ converges to u_0 in stronger senses, and that u_0 is continuous at the vertices of Γ.

The proof of the following proposition will be given in §6.

Proposition 5.1. *Suppose that* Assumptions 2,1, 3.1 *and* 4.2 *hold. Let $u_\epsilon = (H_{\Omega_\epsilon} - z)^{-1}f_\epsilon$, where $z \in \mathbf{C}\backslash[0,\infty)$, $f \in H^1(\Omega)$, and f_ϵ is the restriction of f on Ω_ϵ. Let f satisfy* (3.18) *Then there exists a positive constants $C_j = C_j(K_j, K_j', L_j, M_j, m_j, z)$, depending only on K_j, L_j, M_j, m_j and z, such that*

$$
\begin{aligned}
&\|u_\epsilon'(\cdot,0)\|^2_{e_j,a_0} \\
&\quad \le C_j\big[\epsilon\|u_\epsilon\|^2_{2,\Omega_\epsilon} + \sum_{k\in J}\big(\frac{M_k}{m_k}\big)\big(\epsilon L_k^2K_k^2\|\nabla f\|^2_{\Omega_\epsilon^{(k)}} + \|f(\cdot,0)\|^2_{e_k,a_0}\big)\big],
\end{aligned}
\tag{5.1}
$$

where $\|\cdot\|_{2,\Omega_\epsilon}$ is the norm of the second-order Sobolev space $H^2(\Omega_\epsilon)$ on Ω_ϵ.

Theorem 5.2. *Suppose that* Assumptions 2,1, 3.1 *and* 4.2 *hold. Suppose that*

$$\limsup_{\epsilon\to 0} \sqrt{\epsilon}\|u_\epsilon\|_{2,\Omega_\epsilon} < \infty. \tag{5.2}$$

Let $f \in H^1(\Omega)$ which satisfies (3.18), *and let $u_\epsilon = (H_{\Omega_\epsilon} - z)^{-1} f_\epsilon$, where $z \in \mathbf{C}\backslash[\mathbf{0},\infty]$, and f_ϵ is the restriction of f on $\Omega_j^{(\epsilon)}$. Let $\{\epsilon_m\}_{m=1}^\infty$ such that $\epsilon_m \to 0$ as $m \to \infty$ and $u_{\epsilon_m}(\cdot,0)$ converges weakly in each $L^2(e_j, a_0)$. Then the limit function u_0 is continuous at any vertex c of Γ such that $c \in \Omega$.*

Example 5.3. Let

$$\begin{gathered}\Omega = \{x = (x_1, x_2) : -\ell_-(x_1) < x_2 < \ell_+(x_1),\ a < x_1 < b\},\\ \Omega_\epsilon = \{x = (x_1, x_2) : -\epsilon\ell_-(x_1) < x_2 < \epsilon\ell_+(x_1),\ a < x_1 < b\},\\ \Gamma = \{x = (x_1, 0) : a \le x_1 \le b\},\end{gathered} \tag{5.3}$$

where $-\infty < a < b < \infty$, $0 < \epsilon \le 1$ and $\ell_\pm(t)$ are positive C^1 functions on $[a,b]$. and the map τ is given by

$$\tau(x_1,x_2) = (x_1, 0) \quad ((x_1,x_2) \in \Omega). \tag{5.4}$$

Suppose that Ω is a bounded convex set. Note that each Ω_ϵ is a convex set, too. Then it has been known ([11]) that

$$\|u_\epsilon\|_{2,\Omega_\epsilon} \le C(z)\|f_\epsilon\|_{\Omega_\epsilon} \le C(z)\|f\|_\Omega, \tag{5.5}$$

where $C(z)$ is a positive constant depending only on z. Thus the condition (5.2) is satisfied in this case.

Proof of Theorem 5.2. Let

$$J(c) = \{j \in J : a_j = c \text{ or } b_j = c\}, \tag{5.6}$$

For each $j \in J(c)$ let $c_j \in e_j$ and let $\overline{e_j}'$ be all points on $\overline{e_j}$ between c and $c_j \in e_j$ (including c and c_j). Set $\Gamma(c)' = \cup_{j\in J(c)}\overline{e_j}'$. Since a_0 is bounded below from 0, it follows from (3.19) and (5.2) that there exists $\epsilon_0 \in (0,1]$ such that the sequences $\{\|u_\epsilon(\cdot,0)\|_{e_j}\}_{0<\epsilon<\epsilon_0}$ and $\{\|u'_\epsilon(\cdot,0)\|_{e_j}\}_{0<\epsilon<\epsilon_0}$ are uniformly bounded, where the norm $\|\ \ \|_{e_j}$ is given in (3.6). Therefore there exist a sequence $\{\epsilon_m\}$, $\epsilon_m \to 0$ $(m \to \infty)$, and $c'_j \in \overline{e_j}'$, $(j \in J(c))$, such that $\lim_{m\to\infty} u_{\epsilon_m}(c'_j, 0)$ exists for each $j \in J(c)$. Then we see from

$$\begin{gathered}u_{\epsilon_m}(\sigma,0) = \textstyle\int_{c'_j}^{\sigma} u'_\epsilon(\eta,0)\,d\eta + u_{\epsilon_m}(c'_j,0) \quad (\sigma \in \Gamma'_j),\\ u_{\epsilon_m}(\sigma,0) - u_{\epsilon_m}(\sigma',0) = \textstyle\int_{\sigma'}^{\sigma} u'_\epsilon(\eta,0)\,d\eta \quad (\sigma,\sigma' \in \Gamma'_j)\,,\end{gathered} \tag{5.7}$$

(3.19) and (5.1) that $\{u_{\epsilon_m}\}$ is uniformly bounded and equicontinuous on $\Gamma(c)'$. Therefore there exists a subsequence of $\{\epsilon_m\}$, which will be denoted again by $\{\epsilon_m\}$, such that $\{u_{\epsilon_m}\}$ converges to u_0 uniformly on $\Gamma(c)'$, and hence u_0 is continuous on $\Gamma(c)'$. This completes the proof. ♢

Example 5.4 (Rooms and passages domain). Let $\{h_k\}$, $\{\delta_{2k}\}$, $k = 1, 2, \cdots$, be infinite sequences of positive numbers such that

$$\sum_{k=1}^{\infty} h_k = b \le \infty, \quad 0 < \text{const.} \le \frac{h_{k+1}}{h_k} \le 1, \quad 0 < \delta_{2k} \le h_{2k+1}, \tag{5.8}$$

and let $H_k := \sum_{j=1}^{k} h_j, k = 1, 2, \cdots$. Then $\Omega \subset \mathbb{R}^2$ is defined as the union of the rooms R_k and passages P_{k+1} given by

$$\begin{aligned} R_k &= (H_k - h_k, H_k) \times (-\tfrac{h_k}{2}, \tfrac{h_k}{2}), \\ P_{k+1} &= [H_k, H_k + h_{k+1}] \times (-\tfrac{\delta_{k+1}}{2}, \tfrac{\delta_{k+1}}{2}), \end{aligned} \tag{5.9}$$

for $k = 1, 3, 5, \ldots$. In §6.1 of [2], this was analyzed as an example of a generalized ridged domain with generalized ridge $\Gamma = [0, b]$ $(b < \infty)$ or $\Gamma = [0, \infty)$ $(b = \infty)$. In order to make Γ a tree, each point on Γ which connects a room and the adjacent passage can be called a vertex ($V_0, V_1, V_2, \ldots$ in Fig. 1)

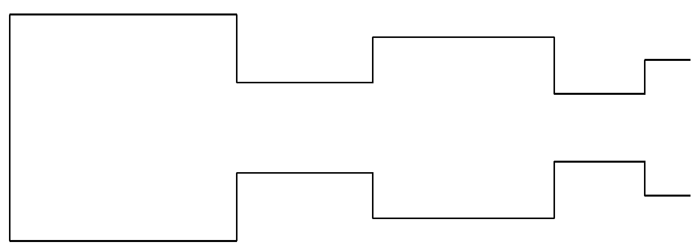

Figure 1: A domain of Rooms and passages

A mapping τ is defined as follows: (i) in a passage P: $\tau(x_1, x_2) = x_1$; (ii) in the first half of the room R succeeding the passage P:

$$\tau(x_1, x_2) = \max(x_1, |x_2| - \frac{\delta}{2}), \quad 0 \le x_1 \le \frac{h}{2}, \tag{5.10}$$

where P is of width δ and $0 \le x_1 \le h$ in R after translation. Hence τ is Lipschitz and $|\nabla\tau| = 1$ almost everywhere in Ω. It is easy to see that $a_0(x_1)$ in this case is a bounded continuous function on Γ, and hence the Kirchhoff boundary condition will be imposed only at $x_1 = 0, b$ $(b < \infty)$ or $x_1 = 0$ $(b = \infty)$. Since a_0 is positive on Γ, the limit function u_0 is continuous on Γ, and the differential equation for u_0 can be explicitly written using h_k and δ_{2k}.

Finally we are going to show, under some additional conditions. that $u_\epsilon(\cdot, 0)$ converges as $\epsilon \to 0$ without taking a subsequence. We are now in a position to introduce another weighted L^2 spaces on the tree Γ.

Definition 5.5. Suppose that a tree Γ satisfies (I) of Assumption 2.1. Let $a(\sigma)$ and $b(\sigma)$ be positive functions defined on $\cup_{j \in J} e_j$ such that $a(\sigma)$ and $b(\sigma)$ are bounded from 0 near each vertex $v \in \Omega$. Then let $H^1(\Gamma, a, b)$ be a subspace of $L^2(\Gamma, a)$ such that $F \in H^1(\Gamma, a, b)$ satisfies the following conditions.

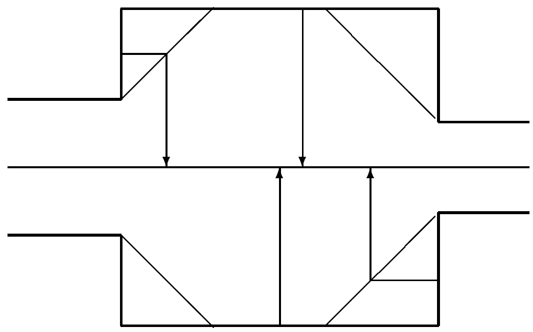

Figure 2: C_t for a domain of rooms and passages

(a) F is continuous on $\Gamma \cap \Omega$.

(b) F is absolutely continuous on each $e_j \cap \Omega$.

(c) F satisfies

$$\|F\|^2_{\Gamma,a,b,1} = \sum_{j \to J} \int_{e_j} |F'(\sigma)|^2 \, b(\sigma) d\sigma + \|F\|^2_{\Gamma,a} < \infty, \tag{5.11}$$

where F' denotes the derivative of F with respect to σ.

Note that it is assumed that $\Gamma \cap \partial\Omega$ consists only of the vertices of the tree Γ. The next lemma guarantees that $H^1(\Gamma, a, b)$ is a Hilbert space with inner product

$$(F,\ G)_{\Gamma,a,b,1} = \sum_{j \in J} \int_{e_j} F'(\sigma)\overline{G'(\sigma)}\, b(\sigma) d\sigma + (F, G)_{\Gamma,a}. \tag{5.12}$$

and norm

$$\|F\|_{\Gamma,a,b,1} = [(F,\ F)_{\Gamma,a,b,1}]^{1/2} \tag{5.13}$$

Lemma 5.6. *Suppose that* $H^1(\Gamma, a, b)$ *be as in* Definition 5.5. *Then* $H^1(\Gamma, a, b)$ *is a Hilbert space with its norm and inner product given by* (5.12) *and* (5.13).

The proof of this lemma will be given in §6. Set $H^1(\Gamma, a_0) = H^1(\Gamma, a_0, a_0)$. Then, under Assumption 3.1, it follows from Lemma 5.6 that $H^1(\Gamma, a_0)$ is a Hilbert space.

Let $H_{\Gamma,0}$ be the selfadjoint operator in $L^2(\Gamma, a_0)$ associated with the sesquilinear form

$$\ell_0[F, G] = \int_\Gamma F'(\sigma)\overline{G'(\sigma)}\, a_0(\sigma) d\sigma \quad (F, G \in H^1(\Gamma, a_0)) \tag{5.14}$$

Theorem 5.7. *Suppose that* Assumptions 2,1, 3.1, 4.2 *hold. Suppose that*

$$\limsup_{\epsilon\to 0}\sqrt{\epsilon}\|u_\epsilon\|_{2,\Omega_\epsilon}<\infty. \tag{5.15}$$

Suppose that the tree Γ *has a finite number of edges. Let* $f\in H^1(\Omega)$*, and let* $u_\epsilon=(H_{\Omega_\epsilon}-z)^{-1}f_\epsilon$*, where* $z\in\mathbf{C}\backslash[0,\infty]$*, and* f_ϵ *is the restriction of* f *on* $\Omega_j^{(\epsilon)}$*. Then we have* $u_\epsilon(\cdot,0)\in H^1(\Gamma,a_0)$ *and*

$$u_\epsilon(\cdot,0)\to(H_{\Gamma,0}-z)^{-1}f(\cdot,0)\quad(\epsilon\to 0) \tag{5.16}$$

weakly in $H^1(\Gamma,a_0)$*.*

Proof. (I) Let N be the number of the edges of Γ. Then it follows from Propositions 3.4 and 5.1 that

$$\begin{aligned}
&\sum_{j=1}^{N}\big(\|u_\epsilon(\cdot,0)\|^2_{e_j,a_0}+\|u'_\epsilon(\cdot,0)\|^2_{e_j,a_0}\big)\\
&\le\ 4|z|^{-1}\Big[\big(\sum_{j=1}^{N}\frac{M_jK_j^2L_j^2}{m_j}\big)\|f\|^2_{\Omega_\epsilon} \\
&\quad+|z|^{-1}\big\{\sum_{j=1}^{N}\big(\frac{M_j}{m_j}\big)\big(\sum_{k=1}^{N}\big(\frac{M_k}{m_k}\big)\big(\epsilon L_k^2K_k^2\|\nabla f\|^2_{\Omega_\epsilon^{(k)}}+\|f(\cdot,0)\|^2_{e_k,a_0}\big)\big\}\Big]\\
&\quad+\big(\sum_{j=1}^{N}C_j\big)\Big[\epsilon\|u_\epsilon\|^2_{2,\Omega_\epsilon}+\sum_{k=1}^{N}\big(\frac{M_k}{m_k}\big)\big(\epsilon L_k^2K_k^2\|\nabla f\|^2_{\Omega_\epsilon^{(k)}}+\|f(\cdot,0)\|^2_{e_k,a_0}\big)\Big],
\end{aligned} \tag{5.17}$$

which, together with the fact that $u_\epsilon\in C^1(\Omega)$, implies that $u_\epsilon(\cdot,0)\in H^1(\Gamma,a_0)$ for each $0<\epsilon\le 1$. Also, by replacing $\|\nabla f\|_{\Omega_\epsilon^{(k)}}$ by $\|\nabla f\|_\Omega$ in (5.15), we see that $\|u_\epsilon(\cdot,0)\|_{\Gamma,a_0,1}$ is uniformly bounded for $\epsilon\in(0,1]$.

(II) Let $\{\epsilon_n\}_{n=1}^\infty\subset(0,1]$ be a sequence such that $\epsilon_n\to 0$ as $n\to\infty$ and u_{ϵ_n} converges weakly in $H^1(\Gamma,a_0)$. Let $u_0\in H^1(\Gamma,a_0)$ be the limit function. Since u_{ϵ_n} is a bounded sequence in $L^2(\Gamma,a_0)$ and $H^1(\Gamma,a_0)$ is dense in $L^2(\Gamma,a_0)$, u_0 is also the weak limit of u_{ϵ_n} in each $L^2(e_j,a_0)$ $(j\in J)$. Therefore from Remark 4.6, (ii) and Theorem 4.7 we see that u_0 is a solution of the equation $-a_0(\sigma)^{-1}(a_0(\sigma)u')'-zu=f(\sigma,0)$ with the Kirchhoff boundary condition at each vertex of Γ. Let $F\in H^1(\Gamma,a_0)$. Then, by using partial integration and the fact that u_0 satisfies the above equation with Kirchhoff boundary condition, we have

$$\begin{aligned}
\ell_0[u_0,F]\ &=\ \sum_{n=1}^{N}\int_{e_j}u_0'(\sigma)\overline{F'(\sigma)}\,a_0(\sigma)d\sigma\\
&=\ -\int_\Gamma(a_0(\sigma)u_0(\sigma)')'\overline{F(\sigma)}\,d\sigma\\
&=\ (zu_0-f(\cdot,0),F)_{\Gamma,a_0},
\end{aligned} \tag{5.18}$$

where $\ell_0[u_0, F]$ is the sesquilinear form used to define the operator $H_{\Gamma,0}$ ((5.15)). Then, by the definition of $H_{\Gamma,0}$, u_0 belongs to the domain of $H_{\Gamma,0}$ and

$$u_0 = (H_{\Gamma,0} - z)^{-1} f(\cdot, 0). \tag{5.19}$$

Since the limiting function u_0 is now independent of the subsequence u_{ϵ_n}, we can conclude that the sequence $\{u_\epsilon(\cdot, 0)\}$ itself converges to $(H_{\Gamma,0} - z)^{-1} f(\cdot, 0)$ weakly in $H^1(\Gamma, a_0)$. This completes the proof. $\diamondsuit$

Remark 5.8. Let (Ω, Γ, τ) be as in Example 5.3. Then not only Theorem 5.7 can be applied to this case, but also it has been shown in [11], §5 that the sequence $\{u_\epsilon(\cdot, 0)\}$ converges to $(H_{\Gamma,0} - z)^{-1} f(\cdot, 0)$ strongly in $L^2(\Gamma, a_0)$.

6 Proofs

Proof of Lemma 4.1. (I) By (2.6) and (2.7) we have

$$\begin{aligned}
&\int_{\Omega_j^{(\epsilon)}} u_\epsilon(x) \overline{v(x)}\, dx \\
&= \int_{e_j} \overline{F(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} u_\epsilon(\sigma, s) |\nabla\tau(\sigma, s)|^{-1}\, ds\, d\sigma \\
&= \int_{e_j} \overline{F(\sigma)} \Big\{ \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} u_\epsilon(\sigma, 0) |\nabla\tau(\sigma, 0)|^{-1}\, ds \\
&\quad + \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big(u_\epsilon(\sigma, s)|\nabla\tau(\sigma, s)|^{-1} - u_\epsilon(\sigma, 0)|\nabla\tau(\sigma, 0)|^{-1} \big)\, ds \Big\}\, d\sigma \\
&\equiv \epsilon \int_{e_j} u_\epsilon(\sigma, 0) \overline{F(\sigma)} a_0(\sigma)\, d\sigma + G_1 + G_2,
\end{aligned} \tag{6.1}$$

where

$$G_1 = \textstyle\int_{e_j} \overline{F(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} (u_\epsilon(\sigma, s) - u_\epsilon(\sigma, 0)) |\nabla\tau(\sigma, s)|^{-1})\, ds\, d\sigma, \tag{6.2}$$

$$G_2 = \textstyle\int_{e_j} u_\epsilon(\sigma, 0) \overline{F(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} (|\nabla\tau(\sigma, s)|^{-1} - |\nabla\tau(\sigma, 0)|^{-1})\, ds\, d\sigma. \tag{6.3}$$

(II) Proceeding as in (3.11), we obtain

$$\begin{aligned}
\frac{1}{\epsilon}|G_1| &\le \frac{1}{\epsilon} \int_{e_j} |F(\sigma)| \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big(\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial u_\epsilon(\sigma, \eta)}{\partial s} \Big|\, d\eta \Big) |\nabla\tau(\sigma, s)|^{-1}\, ds\, d\sigma \\
&\le \frac{M_j}{m_j} \int_{e_j} |F(\sigma)| a_0(\sigma) \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial u_\epsilon(\sigma, s)}{\partial s} \Big|\, ds\, d\sigma \\
&\le \frac{M_j}{m_j} \int_{e_j} |F(\sigma)| a_0(\sigma) \Big(\Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial u_\epsilon(\sigma, s)}{\partial s} \Big|^2 |\nabla\tau(\sigma, s)|^{-1}\, ds \Big]^{1/2} \times
\end{aligned} \tag{6.4}$$

$$\Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |\nabla\tau(\sigma,s)|\,ds\Big]^{1/2}\Big)\,d\sigma$$

Using (3.3) and the first inequality of (3.2) we see that

$$\begin{aligned}
&\Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big|\frac{\partial u_\epsilon(\sigma,s)}{\partial s}\Big|^2 |\nabla\tau(\sigma,s)|^{-1}\,ds\Big]^{1/2} \\
&\leq\ K_j\Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |\nabla u_\epsilon(\sigma,s)|^2 |\nabla\tau(\sigma,s)|^{-1}\,ds\Big]^{1/2},
\end{aligned}$$

and

$$\begin{gathered}
\Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |\nabla\tau(\sigma,s)|\,ds\Big]^{1/2} \leq \sqrt{\epsilon L_j M_j}, \\
\sqrt{a_0(\sigma)} \leq \sqrt{\tfrac{L_j}{m_j}},
\end{gathered} \tag{6.5}$$

and hence we have

$$\begin{aligned}
\frac{1}{\epsilon}|G_1| \ &\leq\ \Big(\frac{M_j}{m_j}\Big)^{3/2} L_j K_j \sqrt{\epsilon}\int_{e_j} |F(\sigma)|\sqrt{a_0(\sigma)}\ \times \\
&\qquad \Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} |\nabla u_\epsilon(\sigma,s)|^2 |\nabla\tau(\sigma,s)|^{-1}\,ds\Big]^{1/2}\,d\sigma \\
&\leq\ \Big(\frac{M_j}{m_j}\Big)^{3/2} L_j K_j \sqrt{\epsilon}\,\|F\|_{e_j,a_0}\|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}}.
\end{aligned} \tag{6.6}$$

(III) As for G_2, we have

$$\frac{1}{\epsilon}|G_2| \leq \int_{e_j} |u_\epsilon(\sigma,0)||F(\sigma)|a_0(\sigma)\psi_\epsilon(\sigma)\,d\sigma, \tag{6.7}$$

where $\psi_\epsilon(\sigma)$ is given by (4.3). Thus, we obtain

$$\frac{1}{\epsilon}|G_2| \leq \|u_\epsilon(\cdot,0)\|_{e_j,a_0}\Big[\int_{e_j} |F(\sigma)|^2\psi_\epsilon(\sigma)^2 a_0(\sigma)\,d\sigma\Big]^{1/2}, \tag{6.8}$$

which, together with (6.5), completes the proof. ◇

Proof of Lemma 4.4. (I) Let a_j and b_j be the vertices of e_j such that $b_j \succeq_a a_j$. Since $\nabla u_\epsilon \cdot \overline{\nabla v}$ is invariant under the shift and rotation of the coordinate system, we may assume that our coordinate system has the origin at a_j and the x_1-axis in the direction of e_j. According to the change of coordinates system, the constant K_j in (3.3) in Assumption 3.1 may have to be replaced another (finite) positive constant which will be denoted again by K_j, while all other constants in Assumption 3.1 do not need to be changed. Then, since $v(x) = F(\tau(x)) = F(x_1)$

in $\Omega_j^{(\epsilon)}$, $\epsilon \in (0, \epsilon_0)$, we have

$$\begin{aligned}
\int_{\Omega_j^{(\epsilon)}} \nabla u_\epsilon \cdot \overline{\nabla v}\, dx &= \int_{\Omega_j^{(\epsilon)}} \frac{\partial u_\epsilon}{\partial x_1} \cdot \overline{F'(x_1)}\, dx \\
&= \int_{e_j} \overline{F'(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, s)\, ds\, d\sigma \\
&= \int_{e_j} \overline{F'(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0)\, ds\, d\sigma \qquad (6.9)\\
&\quad + \int_{e_j} \overline{F'(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big(\frac{\partial u_\epsilon}{\partial \sigma}(\sigma, s) - \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0)\big)\, ds\, d\sigma \\
&= \epsilon \int_{e_j} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0)\overline{F'(\sigma)} a_0(\sigma)\, d\sigma + H.
\end{aligned}$$

Here we should note that we may assume, by (4.5), that $|\nabla\tau(\sigma, s)| = 1$ in the change of variable formula (2.8).

(II) By noting that

$$\begin{aligned}
&\frac{d}{d\sigma} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} (u_\epsilon(\sigma, s) - u_\epsilon(\sigma, 0))\, ds \\
&= \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \big(\frac{\partial u_\epsilon}{\partial \sigma}(\sigma, s) - \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0)\big)\, ds \qquad (6.10)\\
&\quad + \epsilon\ell'_+(\sigma)(u_\epsilon(\sigma, \epsilon\ell_+(\sigma)) - u_\epsilon(\sigma, 0)) + \epsilon\ell'_-(\sigma)(u_\epsilon(\sigma, -\epsilon\ell_-(\sigma)) - u_\epsilon(\sigma, 0)),
\end{aligned}$$

we have

$$\begin{aligned}
H &= \int_{e_j} \overline{F'(\sigma)} \frac{d}{d\sigma} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} (u_\epsilon(\sigma, s) - u_\epsilon(\sigma, 0))\, ds\, d\sigma \\
&\quad -\epsilon \int_{e_j} \overline{F'(\sigma)} \ell'_+(\sigma)(u_\epsilon(\sigma, \epsilon\ell_+(\sigma)) - u_\epsilon(\sigma, 0))\, d\sigma \qquad (6.11)\\
&\quad -\epsilon \int_{e_j} \overline{F'(\sigma)} \ell'_-(\sigma)(u_\epsilon(\sigma, -\epsilon\ell_-(\sigma)) - u_\epsilon(\sigma, 0))\, d\sigma \\
&\equiv H_1 - H_2 - H_3.
\end{aligned}$$

Using partial integration and noting that $F'(\sigma) \in C_0^1(e_j)$, we obtain

$$H_1 = -\int_{e_j} \overline{F''(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} (u_\epsilon(\sigma, s) - u_\epsilon(\sigma, 0))\, ds\, d\sigma, \qquad (6.12)$$

and hence, by proceeding as in (3.11),

$$|H_1| \le \int_{e_j} |F''(\sigma)| \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \int_0^s \Big| \frac{\partial u_\epsilon}{\partial s}(\sigma, \eta)\Big|\, d\eta \Big|\, ds\, d\sigma$$

$$\begin{aligned} &\le \epsilon \int_{e_j} |F''(\sigma)| a_0(\sigma) \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial u_\epsilon}{\partial s}(\sigma, s)\Big|\, ds\, d\sigma \\ &\le \epsilon^{3/2} L_j \|F''\|_{e_j, a_0} \|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}}. \end{aligned} \tag{6.13}$$

As for the term H_2, we have

$$\begin{aligned} |H_2| &\le \epsilon \int_{e_j} |F'(\sigma)||\ell'_+(\sigma)| \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial u_\epsilon}{\partial s}(\sigma, s)\Big|\, ds\, d\sigma \\ &\le \epsilon^{3/2} R_j \int_{e_j} |F'(\sigma)| \sqrt{a_0(\sigma)} \Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial u_\epsilon}{\partial s}(\sigma, s)\Big|^2 ds\Big]^{1/2} d\sigma \\ &\le \epsilon^{3/2} R_j \|F'\|_{e_j, a_0} \|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}}. \end{aligned} \tag{6.14}$$

Similarly,

$$|H_3| \le \epsilon^{3/2} R_j \|F'\|_{e_j, a_0} \|\nabla u_\epsilon\|_{\Omega_j^{(\epsilon)}}. \tag{6.15}$$

The inequality (4.6) is obtained from (6.12), (6.13) and (6.14). ◇

Proof of Proposition 5.1. (I) As in the proof of Lemma 4.4, we may assume that our coordinate system has the origin at a_j and the x_1-axis in the direction of e_j. Let $I \subset e_j$ be a closed interval and set

$$T \equiv \int_I \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0) \overline{F'(\sigma)} a_0(\sigma)\, d\sigma \tag{6.16}$$

for $F \in C^(I)$, where we should note that $u_\epsilon(\sigma, 0) \in C^(I)$, too, and hence T is well-defined. Then, we have

$$\begin{aligned} T &= \frac{1}{\epsilon} \int_I \overline{F'(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0)\, ds\, d\sigma \\ &= \frac{1}{\epsilon} \int_I \overline{F'(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big\{ \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, 0) - \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, s)\Big\}\, ds\, d\sigma \qquad (6.17) \\ &\quad + \frac{1}{\epsilon} \int_I \overline{F'(\sigma)} \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \frac{\partial u_\epsilon}{\partial \sigma}(\sigma, s)\, ds\, d\sigma \\ &\equiv T_1 + T_2 \end{aligned} \tag{6.18}$$

(II) We have

$$\begin{aligned} |T_1| &\le \frac{1}{\epsilon} \int_I |F'(\sigma)| \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \int_0^s \Big| \frac{\partial^2 u_\epsilon}{\partial\sigma\partial s}(\sigma, \eta)\Big|\, d\eta\Big|\, ds\, d\sigma \\ &\le \int_I |F'(\sigma)| a_0(\sigma) \int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial^2 u_\epsilon}{\partial\sigma\partial s}(\sigma, s)\Big|\, ds\, d\sigma \qquad (6.19) \\ &\le \sqrt{\epsilon} L_j \int_I |F'(\sigma)| \sqrt{a_0(\sigma)} \Big[\int_{-\epsilon\ell_-(\sigma)}^{\epsilon\ell_+(\sigma)} \Big| \frac{\partial^2 u_\epsilon}{\partial\sigma\partial s}(\sigma, s)\Big|^2 ds\Big]^{1/2} d\sigma. \end{aligned}$$

Since $\sigma = x_1$, we have from (3.3)

$$\Big|\frac{\partial^2 u_\epsilon}{\partial\sigma\partial s}\Big|^2 = \Big|\frac{\partial}{\partial s}\Big(\frac{\partial u_\epsilon}{\partial x_1}\Big)\Big|^2 \le K_j^2\Big(\Big|\frac{\partial^2 u_\epsilon}{\partial x_1^2}\Big|^2 + \Big|\frac{\partial^2 u_\epsilon}{\partial x_1\partial x_2}\Big|^2\Big), \tag{6.20}$$

which is combined with (6.17) to give

$$|T_1| \le \sqrt{\epsilon}L_j K_j\sqrt{M_j}\|F'\|_{I,a_0}\|u_\epsilon\|_{2,\Omega_\epsilon}, \tag{6.21}$$

where we have used the change of variable formula (2.6) and Assumption 3.1, (ii).

(III) As for T_2, we can proceed as in (II) to obtain

$$|T_2| \le \frac{1}{\sqrt{\epsilon}}K_j\sqrt{M_j}\|F'\|_{I,a_0}\|\nabla u_\epsilon\|_{\Omega_\epsilon}, \tag{6.22}$$

which, together with the second inequality of (3.19), yield

$$|T_2| \le \frac{1}{\sqrt{\epsilon}}|z|^{-1/2}\sqrt{2}K_j\sqrt{M_j}\|F'\|_{I,a_0}\|f\|_{\Omega_\epsilon}. \tag{6.23}$$

(IV) It follows from (6.16), (6.19) and (6.21) that

$$\Big|\int_I \frac{\partial u_\epsilon}{\partial\sigma}(\sigma,0)\overline{F'(\sigma)}a_0(\sigma)\,d\sigma\Big| \le C_j\big(\sqrt{\epsilon}\|u_\epsilon\|_{2,\Omega_\epsilon} + \frac{1}{\sqrt{\epsilon}}\|f\|_{\Omega_\epsilon}\big)\|F'\|_{I,a_0} \tag{6.24}$$

Setting $F(\sigma) = u_\epsilon(\sigma,0)$ in (6.22) and noting that $I \subset e_j$ is arbitrary, we obtain

$$\|u_\epsilon'(\cdot,0)\|_{e_j,a_0} \le \text{const.}\ \big(\sqrt{\epsilon}\|u_\epsilon\|_{2,\Omega_\epsilon} + \frac{1}{\sqrt{\epsilon}}\|f\|_{\Omega_\epsilon}\big). \tag{6.25}$$

As in the proof of Proposition 3.4, we can estimate $\|f\|_{\Omega_\epsilon}$ by using (3.8) with u replaced by f. Thus we have (5.1). $\diamondsuit$

Proof of Lemma 5.6. Since it is easy to see that $H^1(\Gamma,a,b)$ is a pre-Hilbert space, we have only to prove the completeness of $H^1(\Gamma,a,b)$. Let $\{F_n\}_{n=1}^\infty$ be a Cauchy sequence of $H^1(\Gamma,a,b)$. Let Γ_0 be a connected compact set of Γ such that $\Gamma_0 \subset \Gamma\cap\Omega$. Since Γ_0 is closed and meets only a finite number of edges ([3], Lemma 2.1), it follows that

$$\inf_{\sigma\in\Gamma_0\setminus V(\Gamma)} a(\sigma) > 0, \qquad \inf_{\sigma\in\Gamma_0\setminus V(\Gamma)} b(\sigma) > 0 \tag{6.26}$$

and hence $\{F_n\}_{n=1}^\infty$ is a Cauchy sequence with respect to the norm

$$|||F|||_{\Gamma_0,1} = \int_{\Gamma_0}\Big\{\Big|\frac{dF}{d\sigma}\Big|^2 + |F(\sigma)|^2\Big\}\,d\sigma. \tag{6.27}$$

Since a connected compact set $\Gamma_0 \subset \Gamma\cap\Omega$ can be chosen arbitrarily, we see that there is a function F on Γ such that

$$|||F - F_n|||_{\Gamma_0,1} \to 0 \quad (n\to\infty) \tag{6.28}$$

for any connected compact $\Gamma_0 \subset \Gamma \cap \Omega$. We may assume by taking a subsequence of $\{F_n\}_{n=1}^\infty$ if necessary that $F_n(\sigma)$ converges to $F(\sigma)$ for almost all σ in $\Gamma \cap \Omega$. Then, by using the inequality

$$\begin{aligned} |F_n(\sigma) - F_m(\sigma)| &\leq |\int_{\sigma_0}^{\sigma} |\frac{dF_n(\eta)}{d\eta} - \frac{dF_m(\eta)}{d\eta}| \, d\eta| + |F_n(\sigma_0) - F_m(\sigma_0)| \\ &\leq \int_{\Gamma_0} |\frac{dF_n(\eta)}{d\eta} - \frac{dF_m(\eta)}{d\eta}| \, d\eta + |F_n(\sigma_0) - F_m(\sigma_0)| \to 0 \end{aligned}$$

as $n, m \to \infty$ for $\sigma \in \Gamma_0$, where $\{F_n\}$ converges at $\sigma = \sigma_0$, we see that $\{F_n\}$ converges uniformly on Γ_0, and hence $\{F_n\}$ converges to a continuous function F on $\Gamma \cap \Omega$. This proves that F is continuous on $\Gamma \cap \Omega$. Let $\sigma, \sigma' \in e_j$. Then, by letting $n \to \infty$ in

$$F_n(\sigma') - F_n(\sigma) = \int_{P(\sigma,\sigma')} \frac{dF_n}{d\eta} \, d\eta,$$

we have

$$F(t') - F(t) = \int_{P(\sigma,\sigma')} \frac{dF_n}{d\eta} \, d\eta,$$

F is locally absolute continuous on each e_j. By noting that $\Gamma \backslash \Gamma \cap \Omega = \Gamma \cap \partial\Omega$ is a countable set, it is easy to see that $\{F_n\}$ converges to F in the norm $\| \quad \|_{\Gamma, a_0, 1}$, which completes the proof. ♢

Acknowledgments This work was started during the author's visit to the University of Heidelberg in the summer of 1999 under the support of SFB (Sonderforschungsbereich) 359. I would like to express my gratitude to Professor Willi Jäger for his hospitality and useful suggestions.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[2] W. D. Evans and D. J. Harris, *Sobolev embedding for generalized ridged domains*, Proc. Lond. Math. Soc. **54**, (1987), 141–175.

[3] W. D. Evans and D. J. Harris *Fractals, trees and the Neumann Laplacian*, Math. Ann. **296**, (1993), 493–527.

[4] W. D. Evans and Y. Saitō, *Neumann Laplacians on domains and operators on associated trees*, to appear in Quart. J. Math. Oxford.

[5] P. Exner and P. Seba, *Electrons in semiconductor microstructures: a challenge to operator theorists*, Proceedings on Schrödinger Operators, Standard and Nonstandard (Dublin 1988), World Scientific, Singapore, 79–100, 1988.

[6] D. Gilbarg and N. S. Trudinger, *Eliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1977.

[7] P. Kuchment, *The mathematics of photonic crystals*, to appear in Math. Modeling in Optical Science, SIAM.

[8] P. Kuchment and H. Zeng, *Convergence of spectra of mesoscopic system collapsing onto a graph*, preprint, 1999.

[9] K. Ruedenberg and C. W. Scherr, *Free-electron network model for conjugated systems. I, Theory*, J. Chem. Physics, **21** (1953), 1565-1581.

[10] J. Rubinstein and M. Schatzman *Variational problems on multiply connected thin strips I: Basic estimates and convergence of the Laplacian spectrum*, preprint, 1999.

[11] Y. Saitō, *Convergence of the Neumann Laplacians on Shrinking domains*, preprint, 1999.

[12] M. Schatzman, *On the eigenvalues of the Laplace operator on a thin set with Neumann boundary conditions*, Applicable Analysis, **61**, (1996), 293-306.

Yoshimi Saito
Department of mathematics
University of Alabama at Birmingham
Birmingham, AL 35294, USA.
e-mail: saito@math.uab.edu