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Uniqueness of solutions to a system of differential inclusions *

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Abstract

In this paper we study the uniqueness of solutions to the initial and Dirichlet boundary-value problem of differential inclusions

$$\Delta u_i + \nabla\cdot \vec{B_i}\,(u_1, u_2, \dots, u_N) \in \frac{\partial F_i(u_i)}{\partial t}, \quad i = 1, 2, \dots, N,$$

where $\vec{B_i}\,(s_1, s_2, \dots, s_N)$ is an n-dimensional vector continuously differentiable on $\mathbb{R}^N$, and $F_i(u_i) = \{w_i : u_i = A_i(w_i)\}$, $i = 1, 2, \dots, N$ with $A_i(s)$ continuously differentiable functions on $\mathbb{R}$ and $A_i'(s) \geq 0$.

1 Introduction

This paper concerns with the system of differential inclusions

$$\Delta u_i + \nabla\cdot \vec{B_i}\,(u_1, u_2, \dots, u_N) \in \frac{\partial F_i(u_i)}{\partial t}, \quad (x,t) \in Q_T, \quad i = 1, 2, \dots, N, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0, T)$, with $T > 0$, n and N are positive integers, $\vec{B_i}\,(s_1, s_2, \dots, s_N)$ is an n-dimensional vector continuously differentiable on $\mathbb{R}^N$, and

$$F_i(u_i) = \{w_i : u_i = A_i(w_i)\}, \quad i = 1, 2, \dots, N$$

with $A_i(s)$ continuously differentiable functions on $\mathbb{R}$ and $A_i'(s) \geq 0$. Note that if $A_i(s)$ is strictly increasing, then $F_i(u_i)$ is single-valued, and (1.1) becomes equality. However, we are interested in the case when some or all $A_i(s)$'s are only nondecreasing, and so the $F_i(u_i)$'s are interval-valued functions.

System (1.1) arises from mathematical models describing the nonlinear diffusion phenomena which exist in nature extensively. An important classical case of (1.1) is that with $\vec{B_i} = \vec{0}$ and $N = 1$. In this case (1.1) can be changed to

$$\frac{\partial w}{\partial t} = \Delta A(w).$$

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Brézis and Crandall [2] proved the uniqueness of bounded measurable solutions for the Cauchy problem of the equation, where the nonlinear function $A(s)$ is assumed to be only non-decreasing. In other words, if $A(s)$ is differentiable, then

$$A'(s) \geq 0\,;$$

namely, the equation is permitted to be strongly degenerate. Thereafter some authors tried to extend the uniqueness results to the equation with convection, i.e.,

$$\frac{\partial w}{\partial t} = \Delta A(w) + \nabla \cdot \vec{B}\,(w).$$

However, in most of those works, the nonlinear function $A(s)$ is assumed to be strictly increasing. In other words, the equation is weakly degenerate, see for example [4, 3, 8, 9].

In this paper we study the uniqueness of solutions of the initial and Dirichlet boundary-value problem of (1.1). The initial-boundary conditions are

$$u_i = 0, \quad (x,t) \in \partial\Omega \times [0,T], \quad i = 1,2,\dots,N, \tag{1.2}$$

$$F_i(u_i)(x,0) = \{f_i(x)\}, \quad x \in \Omega, \quad i = 1,2,\dots,N. \tag{1.3}$$

Definition For $i = 1,2,\dots,N$, let f_i's be bounded and measurable functions. $(u_1,u_2,\dots,u_N)$ is called a solution of the initial and Dirichlet boundary-value problem (1.1)–(1.3), if the u_i's are bounded and measurable functions and there exist bounded measurable functions $w_i \in F_i(u_i)$ such that for arbitrary test function φ in $C^\infty(\overline{Q_T})$ with value zero for $x \in \partial\Omega$ and for $t = T$, the following integral equalities hold

$$\begin{aligned}&\iint_{Q_T} \left(u_i \Delta\varphi - \vec{B_i}\,(u_1,u_2,\dots,u_N) \cdot \nabla\varphi + w_i \frac{\partial \varphi}{\partial t} \right) dx\,dt \\ &+ \int_\Omega f_i(x)\varphi(x,0)dx = 0, \quad i = 1,2,\dots,N.\end{aligned}$$

The main result of this paper is the following theorem.

Theorem 1 *The initial and Dirichlet boundary-value problem (1.1)–(1.3) has at most one solution.*

The method of the proof is inspired by Brézis and Crandall [2]. Here what we consider is not the Cauchy problem but the initial and Dirichlet boundary-value problem, so we adopt the self-adjoint operators with homogeneous Dirichlet boundary condition to prove the uniqueness instead of the self-adjoint operators on the whole space. Moreover, the problem which we consider is a system of differential inclusions with convection, so we must overcome some other technical difficulties.

2 Proof of the main theorem

We first introduce a family of operators. The L^2 theory for elliptic equations (see, e.g., [5]) implies that for each $\lambda > 0$ and $f \in H^{-1}(\Omega)$, the Dirichlet problem

$$-\Delta u + \lambda u = f, \quad x \in \Omega, \tag{2.1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{2.2}$$

has a unique solution $u \in H_0^1(\Omega)$. For $0 < \lambda < 1$, we define the operator

$$T_\lambda : H^{-1}(\Omega) \to H_0^1(\Omega), \quad f \mapsto u,$$

where u is the unique solution to (2.1)–(2.2). It is easy to see that T_λ is self-adjoint, namely, for arbitrary $f, g \in H^{-1}(\Omega)$,

$$\langle f, T_\lambda g\rangle = \langle g, T_\lambda f\rangle$$

holds, where $\langle\cdot,\cdot\rangle$ represents the dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Specially, for $f \in L^2(\Omega)$ and $g \in H^{-1}(\Omega)$, we have

$$\langle f, T_\lambda g\rangle = \int_\Omega f T_\lambda g dx.$$

In addition, for arbitrary $f \in L^2(\Omega)$, the L^2 theory for elliptic equations also implies $T_\lambda f \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\|T_\lambda f\|_{H^2(\Omega)} \le C_0 \|f\|_{L^2(\Omega)}, \tag{2.3}$$

here C_0 is a constant depending only on n and Ω, but independent of λ.

Proof of Theorem 1. Let $(u_1, u_2, \dots, u_N)$ and $(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$ be two solutions to (1.1)–(1.3). For $i = 1, 2, \dots, N$, the bounded measurable functions in $F_i(u_i)$ and $F_i(\hat{u}_i)$ satisfying the definition are denoted by w_i and $\hat{w}_i$ correspondingly. For $i = 1, 2, \dots, N$, we set

$$v_i = u_i - \hat{u}_i, \quad z_i = w_i - \hat{w}_i,$$

$$\vec{H}_i = \vec{B}_i\,(u_1, u_2, \dots, u_N) - \vec{B}_i\,(\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N).$$

The definition of solutions implies that z_i, v_i and $\vec{H}_i$ $(i = 1, 2, \dots, N)$ are all bounded measurable functions, and for arbitrary test function φ, namely, $\varphi \in C^\infty(\overline{Q_T})$ with $\varphi = 0$ for $x \in \partial\Omega$ and for $t = T$, the integral equalities

$$\iint_{Q_T} \left(v_i \Delta\varphi - \vec{H}_i \cdot \nabla\varphi + z_i \frac{\partial\varphi}{\partial t} \right) dx\, dt = 0, \quad i = 1, 2, \dots, N \tag{2.4}$$

hold. Let $\psi \in C^\infty([0, T])$ with $\psi(T) = 0$ and $k \in C_0^\infty(\Omega)$. Then we see that $T_\lambda k \in H^2(\Omega) \cap H_0^1(\Omega)$. By an approximate process, we may choose $\psi T_\lambda k$ as a test function. Letting $\varphi = \psi T_\lambda k$ in (2.4), we get

$$\iint_{Q_T} \left(\lambda\psi v_i T_\lambda k - \psi v_i k - \psi\, \vec{H}_i \cdot \nabla T_\lambda k + \frac{\partial\psi}{\partial t} z_i T_\lambda k \right) dx\, dt = 0.$$

Using integration by parts and the self-adjointness of T_λ, we get

$$\iint_{Q_T}\Big(\lambda\psi kT_\lambda v_i-\psi kv_i+\psi kT_\lambda(\nabla\cdot\vec{H}_i)-\psi k\frac{\partial T_\lambda z_i}{\partial t}\Big)\,dx\,dt=0.$$

Owing to the arbitrariness of ψ and k, we see that

$$\frac{\partial T_\lambda z_i}{\partial t}=\lambda T_\lambda v_i-v_i+T_\lambda(\nabla\cdot\vec{H}_i) \tag{2.5}$$

in the sense of distribution. It follows that $\frac{\partial T_\lambda z_i}{\partial t}\in L^2(Q_T)$ and $T_\lambda z_i\in H^2(\Omega)\cap H_0^1(\Omega)$. Let $\psi\in C^\infty([0,T])$ with $\psi(T)=0$. By an approximate process, we may choose $\psi T_\lambda z_i$ as a test function. Letting $\varphi=\psi T_\lambda z_i$ in (2.4), we get

$$\iint_{Q_T}\Big(\lambda\psi v_iT_\lambda z_i-\psi v_iz_i-\psi\,\vec{H}_i\cdot\nabla T_\lambda z_i+\frac{\partial\psi}{\partial t}z_iT_\lambda z_i+\psi z_i\frac{\partial T_\lambda z_i}{\partial t}\Big)\,dx\,dt=0. \tag{2.6}$$

Combining (2.5) with (2.6), we see that

$$\begin{aligned}\iint_{Q_T}\Big(&\lambda\psi v_iT_\lambda z_i+\lambda\psi z_iT_\lambda v_i-2\psi v_iz_i+\psi z_iT_\lambda(\nabla\cdot\vec{H}_i)\\&-\psi\,\vec{H}_i\cdot\nabla T_\lambda z_i+\frac{\partial\psi}{\partial t}z_iT_\lambda z_i\Big)\,dx\,dt\quad=\quad0\,.\end{aligned}$$

Using integration by parts and the self-adjointness of T_λ, for $i=1,2,\dots,N$, we get

$$\iint_{Q_T}\Big(2\lambda\psi v_iT_\lambda z_i-2\psi v_iz_i-2\psi\,\vec{H}_i\cdot\nabla T_\lambda z_i+\frac{\partial\psi}{\partial t}z_iT_\lambda z_i\Big)\,dx\,dt=0. \tag{2.7}$$

Let

$$g_{i\lambda}(t)=\int_\Omega z_iT_\lambda z_idx,\quad t\in[0,T],\quad i=1,2,\dots,N.$$

Now we prove that $g_{i\lambda}(t)$ converges to zero on $[0,T]$ uniformly as $\lambda\to0$ for $i=1,2,\dots,N$.

First, we show that $g_{i\lambda}(t)$ is absolutely continuous. From (2.7), we get

$$\iint_{Q_T}\Big(2\lambda\psi v_iT_\lambda z_i-2\psi v_iz_i-2\psi\,\vec{H}_i\cdot\nabla T_\lambda z_i-\psi\frac{\partial(z_iT_\lambda z_i)}{\partial t}\Big)\,dx\,dt=0.$$

From the arbitrariness of ψ, we see that

$$\begin{aligned}g'_{i\lambda}(t)&=\frac{d}{dt}\int_\Omega z_iT_\lambda z_idx=\int_\Omega\frac{\partial(z_iT_\lambda z_i)}{\partial t}\,dx\\&=2\lambda\int_\Omega v_iT_\lambda z_idx-2\int_\Omega v_iz_idx-2\int_\Omega\vec{H}_i\cdot\nabla T_\lambda z_idx,\quad\text{a.e. }t\in[0,T].\end{aligned}$$

Since z_i, v_i and $\vec{H}_i$ are all bounded measurable functions and (2.3) holds, we get that $g'_{i\lambda}(t)\in L^1(0,T)$. Thus $g_{i\lambda}(t)$ is absolutely continuous.

Next, we show that $g_{i\lambda}(0+0) \equiv \lim_{t\to 0^+} g_{i\lambda}(t) = 0$. Let

$$\psi_\varepsilon(t) = \int_t^{+\infty} \alpha_\varepsilon(s-\varepsilon)ds, \quad \alpha_\varepsilon(s) = \frac{1}{\varepsilon}\alpha\left(\frac{s}{\varepsilon}\right),$$

where $\alpha(s)$ denotes the kernel of one-dimensional mollifier, namely, α is in the space $C_0^\infty(-\infty,+\infty)$, $\alpha \ge 0$, $\operatorname{supp}\alpha = [-1,1]$ and $\int_{-1}^1 \alpha(s)\,ds = 1$. Thus $\psi_\varepsilon \in C^\infty([0,T])$ and $\psi_\varepsilon(T) = 0$ for sufficiently small $\varepsilon > 0$. Letting $\psi = \psi_\varepsilon$ in (2.7), we get

$$\iint_{Q_T} (2\lambda\psi_\varepsilon v_i T_\lambda z_i - 2\psi_\varepsilon v_i z_i - 2\psi_\varepsilon \vec{H_i}\cdot\nabla T_\lambda z_i - \alpha_\varepsilon(t-\varepsilon) z_i T_\lambda z_i)\,dx\,dt = 0.$$

The dominated convergence theorem implies

$$\begin{aligned} g_{i\lambda}(0+0) &= \lim_{\varepsilon\to 0^+}\int_0^{2\varepsilon} \alpha_\varepsilon(t-\varepsilon) g_{i\lambda}(t)dt \\ &= \lim_{\varepsilon\to 0^+}\iint_{Q_T} \alpha_\varepsilon(t-\varepsilon) z_i T_\lambda z_i\,dx\,dt \\ &= 2\lambda \lim_{\varepsilon\to 0^+}\iint_{Q_T} \psi_\varepsilon v_i T_\lambda z_i\,dx\,dt - 2\lim_{\varepsilon\to 0^+}\iint_{Q_T} \psi_\varepsilon v_i z_i\,dx\,dt \\ &\quad -2\lim_{\varepsilon\to 0^+}\iint_{Q_T} \psi_\varepsilon \vec{H_i}\cdot\nabla T_\lambda z_i\,dx\,dt\,. \end{aligned}$$

Since z_i, v_i and $\vec{H_i}$ are all bounded measurable functions and (2.3) holds, we get that $g_{i\lambda}(0+0) = 0$.

Finally, we prove that $g_{i\lambda}(t)$ converges to zero on $[0,T]$ uniformly as $\lambda \to 0$. It follows easily from the above arguments that

$$\begin{aligned} g_{i\lambda}(t) &= g_{i\lambda}(0+0) + \int_0^t g'_{i\lambda}(s)ds \\ &= 2\lambda\int_0^t\int_\Omega v_i T_\lambda z_i\,dx\,ds - 2\int_0^t\int_\Omega v_i z_i\,dx\,ds \\ &\quad -2\int_0^t\int_\Omega \vec{H_i}\cdot\nabla T_\lambda z_i\,dx\,ds\,. \end{aligned}$$

Since w_i and $\hat{w}_i$ are bounded measurable and A_i and $\vec{B_i}$ are continuously differentiable, there exist three positive constants M_0, M_1 and M_2 such that for $i = 1, 2, \ldots, N$, the following estimates hold

$$|z_i| \le M_0, \quad |v_i| \le M_1|z_i|, \quad |\vec{H_i}| \le M_2\left(\sum_{j=1}^N v_j^2\right)^{1/2}.$$

Noticing that z_i and v_i have the same sign for $A_i'(s) \ge 0$, we get

$$v_i z_i = |v_i||z_i| \ge \frac{1}{M_1} v_i^2.$$

By Schwarz's inequality and Young's inequality, we get

$$\begin{aligned}
&|\int_\Omega \vec{H}_i \cdot \nabla T_\lambda z_i\,dx| \\
&\leq \Big(\int_\Omega |\,\vec{H}_i\,|^2 dx\Big)^{1/2}\Big(\int_\Omega |\nabla T_\lambda z_i|^2 dx\Big)^{1/2} \\
&\leq M_2 \sum_{j=1}^N \Big(\int_\Omega v_j^2 dx\Big)^{1/2}\Big(\int_\Omega |\nabla T_\lambda z_i|^2 dx\Big)^{1/2} \\
&\leq \frac{1}{NM_1}\sum_{j=1}^N \int_\Omega v_j^2 dx + \frac{N^2 M_1 M_2^2}{4}\int_\Omega (\nabla T_\lambda z_i \nabla T_\lambda z_i) dx \\
&= \frac{1}{NM_1}\sum_{j=1}^N \int_\Omega v_j^2 dx + \frac{N^2 M_1 M_2^2}{4}\int_\Omega (-T_\lambda z_i \Delta T_\lambda z_i) dx \\
&= \frac{1}{NM_1}\sum_{j=1}^N \int_\Omega v_j^2 dx + \frac{N^2 M_1 M_2^2}{4}\int_\Omega \big(-\lambda (T_\lambda z_i)^2 + z_i T_\lambda z_i\big)\, dx \\
&\leq \frac{1}{NM_1}\sum_{j=1}^N \int_\Omega v_j^2 dx + \frac{N^2 M_1 M_2^2}{4} g_{i\lambda}(t).
\end{aligned}$$

Let

$$g_\lambda(t) = \sum_{i=1}^N g_{i\lambda}(t).$$

Therefore,

$$\begin{aligned}
&g_\lambda(t) \\
&= 2\sum_{i=1}^N \Big(\lambda \int_0^t \int_\Omega v_i T_\lambda z_i\,dx\,ds - \int_0^t \int_\Omega v_i z_i\,dx\,ds - \int_0^t \int_\Omega \vec{H}_i \cdot \nabla T_\lambda z_i\,dx\,ds\Big) \\
&\leq 2\sum_{i=1}^N \Big(\lambda \int_0^t \int_\Omega v_i T_\lambda z_i\,dx\,ds - \frac{1}{M_1}\int_0^t \int_\Omega v_i^2\,dx\,ds \\
&\quad + \frac{1}{NM_1}\sum_{j=1}^N \int_0^t \int_\Omega v_j^2\,dx\,ds + \frac{N^2 M_1 M_2^2}{4}\int_0^t g_{i\lambda}(s) ds\Big) \\
&\leq 2\lambda \sum_{i=1}^N \int_0^T \int_\Omega |v_i T_\lambda z_i|\,dx\,ds + \frac{N^2 M_1 M_2^2}{2}\int_0^t g_\lambda(s) ds.
\end{aligned}$$

Moreover, it follows that

$$g_\lambda(t) \geq 0$$

by

$$g_{i\lambda}(t) \;=\; \int_\Omega z_i T_\lambda z_i dx$$

$$\begin{aligned}
&= \int_\Omega (-\Delta T_\lambda z_i T_\lambda z_i + \lambda T_\lambda z_i T_\lambda z_i)dx \\
&= \int_\Omega (\nabla T_\lambda z_i \nabla T_\lambda z_i + \lambda T_\lambda z_i T_\lambda z_i)dx \\
&\geq 0\,.
\end{aligned}$$

Hence by Gronwall's inequality, we get

$$g_\lambda(t) \leq C_1\lambda,$$

where C_1 is a constant depending only on N, M_0, M_1, M_2, C_0, T and the measure of Ω, but independent of λ and t. So $g_\lambda(t)$ converges to zero on $[0,T]$ uniformly as $\lambda \to 0$. Noticing that $g_{i\lambda}(t) \geq 0$, we get that $g_{i\lambda}(t)$ converges to zero on $[0,T]$ uniformly as $\lambda \to 0$.

Now we prove

$$z_i(x,t) = 0, \quad \text{a.e. } (x,t) \in Q_T, \quad i = 1, 2, \dots, N.$$

For any $\varphi \in C_0^\infty(Q_T)$, we have

$$\begin{aligned}
&|\iint_{Q_T} z_i\varphi\,dx\,dt\Big|^2 \\
&= \Big|\iint_{Q_T} (-\Delta T_\lambda z_i + \lambda T_\lambda z_i)\varphi\,dx\,dt\Big|^2 \\
&= \Big|\iint_{Q_T} (\nabla T_\lambda z_i \nabla\varphi + \lambda\varphi T_\lambda z_i)\,dx\,dt\Big|^2 \\
&\leq 2\|\nabla\varphi\|^2_{L^2(Q_T)}\|\nabla T_\lambda z_i\|^2_{L^2(Q_T)} + 2\lambda^2\|\varphi\|^2_{L^2(Q_T)}\|T_\lambda z_i\|^2_{L^2(Q_T)} \\
&\leq C_2\Big(\iint_{Q_T} |\nabla T_\lambda z_i|^2\,dx\,dt + \lambda\iint_{Q_T} (T_\lambda z_i)^2\,dx\,dt\Big) \\
&\leq C_2\iint_{Q_T} (-\Delta T_\lambda z_i + \lambda T_\lambda z_i)T_\lambda z_i\,dx\,dt \\
&= C_2\iint_{Q_T} z_i T_\lambda z_i\,dx\,dt \\
&\leq C_2 T \sup_{t\in[0,T]} g_{i\lambda}(t) \to 0, \quad (\lambda \to 0),
\end{aligned}$$

where $C_2 = 2\|\nabla\varphi\|^2_{L^2(Q_T)} + 2\|\varphi\|^2_{L^2(Q_T)}$ independent of λ. Therefore,

$$\iint_{Q_T} z_i\varphi\,dx\,dt = 0, \quad \forall\varphi \in C_0^\infty(Q_T).$$

It follows that

$$z_i(x,t) = 0, \quad \text{a.e. } (x,t) \in Q_T, \quad i = 1, 2, \dots, N.$$

Thus

$$w_i(x,t) = \hat{w}_i(x,t), \quad \text{a.e. } (x,t) \in Q_T, \quad i = 1,2,\dots,N,$$

which implies

$$u_i(x,t) = \hat{u}_i(x,t), \quad \text{a.e. } (x,t) \in Q_T, \quad i = 1,2,\dots,N.$$

The proof is complete.

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