Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 57, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ftp ejde.math.unt.edu (login: ftp)

Variational and penalization methods for studying connecting orbits of Hamiltonian systems *

Chao-Nien Chen & Shyuh-yaur Tzeng

Abstract

In this article, we consider a class of second order Hamiltonian systems that possess infinite or finite number of equilibria. Variational arguments will be used to study the existence of connecting orbits joining pairs of equilibria. Applying penalization methods, we obtain various patterns for multibump homoclinics and heteroclinics of Hamiltonian systems.

0 Introduction

In recent years, some new tools have been developed in the calculus of variations for studying the existence of connecting orbits of nonlinear differential equations; see for example [1, 4, 5, 6, 8, 14, 18, 19, 22, 23, 27, 28, 29, 30, 33, 34, 35]. A great deal of attention has been focused on the second order Hamiltonian system

$$\ddot{q} - V'(t,q) = 0\,, \tag{HS}$$

where $q : \mathbb{R} \to \mathbb{R}^n$, $V \in C^2(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$, and $V'(t,y) = D_yV(t,y)$. If $V'(t,\eta) = 0$ for all $t \in \mathbb{R}$, then η is an equilibrium of (HS). A solution q of (HS), that satisfies

$$\lim_{t\to-\infty} q(t) = \eta_i \text{ and } \lim_{t\to\infty} q(t) = \eta_j \tag{0.1}$$

for a pair of equilibria η_i and η_j, is called a heteroclinic solution or heteroclinic orbit of (HS). In case that $\eta_i = \eta_j$ and $q \not\equiv \eta_i$, the solution is called a homoclinic orbit.

Rabinowitz [27] considered a class of (HS) where V is periodic in t and has a local minimum at 0. Under certain growth conditions for V at $y = 0$ and

* *Mathematics Subject Classifications:* 34C37, 49M30, 58E99, 58F09.
Key words: Hamiltonian system, homoclinic, heteroclinic, calculus of variations.

Submitted November 11, 1999. Published August 22, 2000.
Partially supported by the National Science Council of Republic of China

infinity, he applied Mountain Pass Lemma to obtain a homoclinic orbit q of (HS) which satisfies

$$\lim_{|t|\to\infty} q(t) = 0\,. \tag{0.2}$$

Subsequently, Coti Zelati and Rabinowitz [8] studied the multibump homoclinic solutions for this system. Qualitatively, if the one bump homoclinic solution satisfies certain nondegeneracy condition, a chain of one bump homoclinics can be concatenated to form a multibump homoclinic. Such techniques have been extended to study semilinear elliptic partial differential equations [9, 34]. Results in this spirit in dynamical systems are the Shadowing Lemma and the Smale-Birkhoff Theorem (see e.g. [15]).

The existence of heteroclinic orbits of (HS) has been studied in [5, 35], where the function V satisfies the following conditions:

(V1) There is a set $\mathcal{K}_1 \subset \mathbb{R}^n$ such that if $\eta \in \mathcal{K}_1$ then $V(t,\eta) = \inf_{y\in\mathbb{R}^n} V(t,y) = V_0$ for all $t \in \mathbb{R}$.

(V2) There are positive numbers μ_1, μ_2 and ρ_0 such that if $|y-\eta| \le \rho_0$ for some $\eta \in \mathcal{K}_1$ then $\mu_2|y-\eta|^2 \ge V(t,y) - V_0 \ge \mu_1|y-\eta|^2$ for all $t \in \mathbb{R}$. Moreover, if $\eta_i, \eta_j \in \mathcal{K}_1$ and $i \ne j$, then $|\eta_i - \eta_j| > 8\rho_0$.

(V3) There is a $\mu_0 > 0$ such that if $V(t,y) \le V_0 + \mu_0$ for some $t \in \mathbb{R}$ then $|y-\eta| \le \rho_0$ for some $\eta \in \mathcal{K}_1$.

(V4) For any $r_0 > 0$ there is an $M > 0$ such that $\sup_{t\in\mathbb{R}} \parallel D_y^2 V(t,y) \parallel_\infty \le M$ if $|y| \le r_0$.

If V is periodic in t and in each component of y, Strobel [35] showed that, for any $\eta_i \in \mathcal{K}_1$, there is a heteroclinic solution q of (HS) which satisfies

$$q(t) \to \eta_i \text{ as } t \to -\infty$$

and

$$q(t) \to \mathcal{K}_1 \backslash \{\eta_i\} \text{ as } t \to \infty.$$

Moreover, for any pair of $\eta_i, \eta_j \in \mathcal{K}_1$, they can be joined by a chain of heteroclinics. If additional nondegeneracy conditions are satisfied, there exist multibump heteroclinic orbits connecting η_i and η_j. Such kinds of results have also been proved by Rabinowitz [28] and Maxwell [22, 23] for orbits connecting periodic solutions instead of equilibria.

For the first order Hamiltonian system, the existence of multibump homoclinic solutions was proved by Sèrè [33]. The interested readers may consult [6, 18, 19, 24, 30] for more references and various extensions for using variational methods to study connecting orbits of Hamiltonian systems.

In this paper, we treat the case where V is not necessarily periodic in its variables. By (HS), the potential V is only determined up to an additive constant, so we may assume that $V_0 = 0$. Let $E = W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ with the norm

$$\| z \| = (\int_{-\infty}^{\infty} (|\dot{z}|^2 + |z|^2) dt)^{1/2}.$$

By changing variables of V, we may take $\eta_1 = 0$. For $z \in E$, define

$$I(z) = \int_{-\infty}^{\infty} [\frac{1}{2}|\dot{z}|^2 + V(t,z)]dt. \tag{0.3}$$

Since $I \in C^1(E, \mathbb{R})$ and $E \subset C_0(\mathbb{R}, \mathbb{R}^n)$, the space of continuous functions z on $\mathbb{R}$ such that $z(t) \to 0$ as $|t| \to \infty$, if $I'(q) = 0$ and $I(q) > 0$ then q is a homoclinic orbit of (HS).

A sequence $\{z_m\} \subset E$ is called a $(\mathrm{PS})_c$ sequence if $I(z_m) \to c$ and $I'(z_m) \to 0$ as $m \to \infty$. Our approach is to search critical points of I by investigating the convergence of Palais-Smale sequences. The investigation will be based on a comparison argument described as follows. Define

$$\theta(\rho) = \min(\mu_1 \rho^2, \mu_0)\,,$$
$$\Lambda = \sup\{\|V'(t,y)\| + 1/2 | t \in \mathbb{R} \text{ and } y \in \bigcup_{\eta \in \mathcal{K}_1} \overline{B_{\rho_0}(\eta)}\}\,.$$

For $j_1 < j_2$, let

$$\hat{E}(j_1, j_2) = \{z \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) | z(t) = \eta_1 \text{ if } t \le j_1 \text{ and } z(t) = \eta_2 \text{ if } t \ge j_2\},$$
$$\tilde{E}(j_1, j_2) = \{z \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) | z(t) = \eta_2 \text{ if } t \le j_1 \text{ and } z(t) = \eta_1 \text{ if } t \ge j_2\}\,,$$
$$\hat{\alpha}(j_1, j_2) = \inf_{z \in \hat{E}(j_1, j_2)} I(z)\,,$$
$$\tilde{\alpha}(j_1, j_2) = \inf_{z \in \tilde{E}(j_1, j_2)} I(z)\,.$$

Set $\hat{\alpha}(-\infty, j_2) = \lim_{j_1 \to -\infty} \hat{\alpha}(j_1, j_2)$ and $\hat{\alpha}(j_1, \infty) = \lim_{j_2 \to \infty} \hat{\alpha}(j_1, j_2)$.

Theorem 1 *Assume (V1)-(V4) are satisfied. Suppose there are $k_1 < k_2 < k_3 < k_4$ such that*

$$\hat{\alpha}(k_1, k_2) < \min(\hat{\alpha}(-\infty, k_1), \hat{\alpha}(k_2, k_3))\,, \tag{0.4}$$
$$\tilde{\alpha}(k_3, k_4) < \min(\tilde{\alpha}(k_2, k_3), \tilde{\alpha}(k_4, \infty)), \tag{0.5}$$
$$k_3 - k_2 > 6\rho_0 + 2(\hat{\alpha}(k_1, k_2) + \tilde{\alpha}(k_3, k_4) + \rho_0 \sqrt{2\theta(\rho_0)})/\theta(r), \tag{0.6}$$

where

$$r = \min\left(1, \frac{\rho_0}{2}, \sqrt[4]{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0\sqrt{2\theta(\rho_0)}}{\Lambda}, \frac{\hat{\theta}}{3\Lambda}\right) \tag{0.7}$$

and $\hat{\theta} = \min(\hat{\alpha}(-\infty, k_1) - \hat{\alpha}(k_1, k_2),\ \hat{\alpha}(k_2, k_3) - \hat{\alpha}(k_1, k_2),\ \tilde{\alpha}(k_2, k_3) - \tilde{\alpha}(k_3, k_4),\ \tilde{\alpha}(k_4, \infty) - \tilde{\alpha}(k_3, k_4))$. Then there is a homoclinic orbit of (HS) which satisfies (0.2).

The homoclinic orbit described in Theorem 1 will be obtained by finding a local minimizer of I. Our strategy is to add penalization to I so that a local minimizer of I becomes a global minimzer to a penalized functional. An example for which Theorem 1 holds is $n = 1$ and $V(t, y) = G(t)(1 - (y - 1)^2)^2$, where $G > 0$ and is relatively large on $(-\infty, k_1)$, (k_2, k_3) and (k_4, ∞). Such oscillation conditions may repeat infinitely many times when V is periodic in t. In this case, I possesses a large number of local minimizers. To find multiple homoclinics of (HS), we add different penalty functions to single out such local minimzers. Roughly speaking, the homoclinics obtained here comprise two heteroclinics nicely concatenated. It will also be discussed under what conditions a chain of more than two heteroclinics can be concatenated to form a multibump homoclinic. Strobel [35] has obtained such multibump solutions by using delicate deformation arguments. There additional nondegeneracy conditions are required and in general such conditions are hard to verify. Also, as in most results for multibump solutions, Strobel's arguments indicate that the distance between two bumps needs to be long; however, there is no estimates for how long it has to be. We do not assume any nondegenercy condition on single bump solutions and a lower bound of distance between two bumps will be obtained. Our methods can be applied to aperiodic Hamiltonian systems as well. The detailed analysis will be given in Section 2 and Section 4.

In Sections 3, a minimax method, in the framework of Mountain Pass Lemma, will be used to obtain homoclinic solutions of (HS). Since the local minimizers obtained by penalization methods are not necessary to be isolated, additional works are required to verify the existence of minimax critical values of I. Also, the justification of convergence of minimaxing sequences is more complicated than that of minimizing sequences.

A simple example for which the potential of (HS) possesses more than two minima is a forced pendulum problem, where $V(t, y) = G(t)(1 + \cos(y - \pi))$, $y \in \mathbb{R}$ and $G > 0$. Such examples can easily be found in case of $n > 1$. The distribution of the wells of V in general need not to be equi-distanced. Naturally, one expects to find a connecting orbit whose trajectory is close to some other equilibria in some time intervals. Results for (HS) involving connecting orbits of this type will be presented in Section 5.

1 Preliminaries

This section contains several technical results such as qualitative properties of I and various estimates of Palais-Smale sequences.

Proposition 1 *For any $t_1, t_2 \in \mathbb{R}$, $q \in W^{1,2}([t_1, t_2], \mathbb{R}^n)$ and $\rho \in (0, \rho_0]$, if $\inf_{\substack{t\in[t_1,t_2]\\ \eta\in\mathcal{K}_1}} |q(t) - \eta| \geq \rho$ then $\int_{t_1}^{t_2} V(t, q)dt \geq (t_2 - t_1)\theta(\rho)$.*

The proof of this proposition follows from (V2) and (V3).

Proposition 2 *Suppose* $q(t_1) \in \partial B_\rho(\eta_i)$, $q(t_2) \in \partial B_\rho(\eta_j)$, *and* $q(t) \in \mathbb{R}^n \backslash (\cup_{\eta \in \mathcal{K}_1} B_\rho(\eta))$ *for* $t \in (t_1, t_2)$. *If* $i \neq j$ *and* $\rho \in (0, \rho_0]$, *then*

$$\int_{t_1}^{t_2} [\frac{1}{2}|\dot{q}(t)|^2 + V(t,q)]dt \geq \frac{1}{2(t_2-t_1)}(|\eta_i - \eta_j| - 2\rho)^2 + \theta(\rho)(t_2 - t_1). \quad (1.1)$$

Proof. Since

$$|\eta_i - \eta_j| - 2\rho \leq |q(t_2) - q(t_1)| = |\int_{t_1}^{t_2} \dot{q}(t)dt| \leq \sqrt{t_2 - t_1}(\int_{t_1}^{t_2} |\dot{q}(t)|^2 dt)^{1/2},$$

Proposition 1 yields (1.1).

Lemma 1 *Let* $\{z_m\} \subset E$ *be a (PS)*$_c$ *sequence. Then there is a constant* $C_0 > 0$ *such that*

$$\sup_{m \in \mathbb{N}} \parallel \dot{z}_m \parallel_{L^2(\mathbb{R})} \leq C_0.$$

The proof of this lemma is trivial.

Lemma 2 *If* $\{z_m\}$ *is a (PS)*$_c$ *sequence then* $\{z_m\}$ *is bounded in* $L^\infty(\mathbb{R}, \mathbb{R}^n)$.

Proof. Without loss of generality, we may assume that $I(z_m) \leq c + 1$ for all $m \in \mathbb{N}$. Let $d_m(\tau) = \inf_{\eta \in \mathcal{K}_1} |z_m(\tau) - \eta|$ and $\tilde{S}_m = \{\tau \in \mathbb{R} | d_m(\tau) < \rho_0\}$. By Proposition 2, for any $t \in \mathbb{R}$,

$$\tilde{S}_m \cap [-\hat{n} + t, \hat{n} + t] \neq \phi,$$

where $\hat{n} = [\frac{c+1}{2\theta(\rho_0)}] + 2$. Pick a $t_m \in \tilde{S}_m \cap [-\hat{n} + t, \hat{n} + t]$. It follows from Lemma 1 that

$$|z_m(t)| \leq |z_m(t_m)| + |\int_{t_m}^{t} \dot{z}_m(s)ds| \leq \sup_{\eta \in \mathcal{K}_1} |\eta| + \rho_0 + \sqrt{\hat{n}}C_0.$$

This completes the proof of the case of Card $\mathcal{K}_1 < \infty$. An argument used in [5][Lemma 2] takes care of the other case.

2 A Penalization Method

We now prove the existence of homoclinic orbits of (HS). As mentioned earlier, the homoclinic orbit will be obtained as a local minimizer of I. To find such a local minimizer, we use a penalization method descirbed as follows.

Proof of Theorem 1. To illustrate the basic idea of penalization, here we give a proof for the special case where $\mathcal{K} = \{\eta_1, \eta_2\}$. The extension to the general case can easily be carried out once we have seen the proof of Theorem 4. Let $\bar{\rho} = 5\rho_0/2$, $\hat{t}_1 = k_1 - 3\rho_0 - (\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4))/\theta(r)$, $\hat{t}_2 = k_2 + 3\rho_0 + (\hat{\alpha}(k_1,k_2) + \rho_0\sqrt{2\theta(\rho_0)})/\theta(r)$, $\hat{t}_3 = k_3 - 3\rho_0 - (\tilde{\alpha}(k_3,k_4) + \rho_0\sqrt{2\theta(\rho_0)})/\theta(r)$, $\hat{t}_4 = k_4 + 3\rho_0 + (\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4))/\theta(r)$ and

$$M_0 = \Big[\frac{\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4) + 1}{\rho_0}\Big]^2 + \frac{\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4)}{\hat{t}_3 - \hat{t}_2} + \theta(r).$$

Let $\psi \in C^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ such that $0 \le \psi \le M_0$ and

$$\psi(t,y) = \begin{cases} 0 & \text{if } y \in B_{\bar{\rho}}(\eta_2) \text{ or } t \in [\hat{t}_1 + \rho_0, \hat{t}_2 - \rho_0] \cup [\hat{t}_3 + \rho_0, \hat{t}_4 - \rho_0] \\ M_0 & \text{if } y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [\hat{t}_2, \hat{t}_3] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_1) \text{ and } t \notin (\hat{t}_2 - \rho_0, \hat{t}_3 + \rho_0) \\ M_0 & \text{if } y \in \mathbb{R}^n \backslash (\bigcup_{i=1,2} B_{3\rho_0}(\eta_i)) \text{ and } t \in (-\infty, \hat{t}_1] \cup [\hat{t}_4, \infty). \end{cases}$$

Set

$$I_0(z) = I(z) + \int_{-\infty}^{\infty} \psi(t,z)\,dt\,, \tag{2.1}$$

$$\alpha = \inf_{z\in E} I_0(z). \tag{2.2}$$

It is clear that

$$\alpha < \hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4). \tag{2.3}$$

Let $\{u_m\}$ be a sequence which satisfies $\lim_{m\to\infty} I_0(u_m) = \alpha$ and

$$\sup_{m\in\mathbb{N}} I_0(u_m) \le \alpha + 1. \tag{2.4}$$

By Lemma 2, $\{u_m\}$ is bounded in $W^{1,2}_{loc}(\mathbb{R},\mathbb{R}^n)$. Hence there is a $q \in W^{1,2}_{loc}(\mathbb{R},\mathbb{R}^n)$ such that along a subsequence

$$u_m \to q \text{ weakly in } W^{1,2}_{loc}(\mathbb{R},\mathbb{R}^n) \text{ and strongly in } L^\infty_{loc}(\mathbb{R},\mathbb{R}^n) \tag{2.5}$$

and

$$I_0(q) \le \alpha. \tag{2.6}$$

To show that q is a homoclinic orbit of (HS), we first note that $q \not\equiv 0$. Indeed,

$$\text{there is a } t_0 \in (\hat{t}_2, \hat{t}_3) \text{ such that } q(t_0) \in B_r(\eta_2); \tag{2.7}$$

for otherwise, we would obtain $I_0(q) > \alpha$ which violates (2.6). Hence there is a $m_0 \in \mathbb{N}$ such that if $m \ge m_0$ then $u_m(t_0) \in B_r(\eta_2)$.

We are now going to show that $q \in E$. Let

$$\begin{aligned} \tau_1(m) &= \sup\left\{s | u_m(t) \in \overline{B_r(0)} \quad \text{if} \quad t \le s\right\}, \\ \tau_2(m) &= \sup\{t | t < t_0 \text{ and } u_m(t) \in \overline{B_r(0)}\}. \end{aligned} \tag{2.8}$$

We claim

$$\tau_2(m) \ge \hat{t}_1 - \frac{1}{\theta(r)}(\alpha + 1). \tag{2.9}$$

Suppose (2.9) is false. Letting

$$\tau_3(m) = \inf\{t | t > \tau_2(m) \text{ and } u_m(t) \in \overline{B_r(\eta_2)}\}, \tag{2.10}$$

we get

$$\tau_3(m) \le \tau_2(m) + \frac{1}{\theta(r)}(\alpha + 1) < \hat{t}_1.$$

Then there are $t_1, t_2 \in (\tau_2(m), \tau_3(m))$ such that $u_m(t_1) \in \partial B_{3\rho_0}(0)$, $u_m(t_2) \in \partial B_{3\rho_0}(\eta_2)$ and $u_m(t) \in \mathbb{R}^n \backslash (\bigcup_{i=1,2} B_{3\rho_0}(\eta_i))$ if $t \in (t_1, t_2)$. It follows from Proposition 2 and (2.3) that

$$\begin{aligned} I_0(u_m) &\ge \frac{2\rho_0^2}{(t_2 - t_1)} + (t_2 - t_1)\left[\frac{1 + \hat{\alpha}(k_1, k_2) + \tilde{\alpha}(k_3, k_4)}{\rho_0}\right]^2 \\ &> 2(1 + \hat{\alpha}(k_1, k_2) + \tilde{\alpha}(k_3, k_4)) > \alpha + 1. \end{aligned} \tag{2.11}$$

This violates (2.4) and thus (2.9) must be true. Similarly $\tau_5(m) \le \hat{t}_4 + (\alpha + 1)/\theta(r)$, where

$$\tau_5(m) = \inf\left\{t | t > t_0 \text{ and } u_m(t) \in \overline{B_r(0)}\right\}.$$

By (2.5) there are $\hat{\tau} \in (\hat{t}_1 - (\alpha+1)/\theta(r), t_0)$ and $\bar{\tau} \in (t_0, \hat{t}_4 + (\alpha+1)/\theta(r))$ such that $\max(|q(\hat{\tau})|, |q(\bar{\tau})|) < r$. Let

$$\tau_3 = \tau_3(q) = \inf\{t | t \in (\hat{\tau}, t_0] \text{ and } q(t) \in \overline{B_r(\eta_2)}\}, \tag{2.12}$$

$$\tau_2 = \tau_2(q) = \sup\{t | t < \tau_3 \text{ and } q(t) \in \overline{B_r(0)}\}, \tag{2.13}$$

$$\tau_4 = \tau_4(q) = \sup\{t | t \in [t_0, \bar{\tau}) \text{ and } q(t) \in \overline{B_r(\eta_2)}\}, \tag{2.14}$$

$$\tau_5 = \tau_5(q) = \inf\left\{t | t > \tau_4 \text{ and } q(t) \in \overline{B_r(0)}\right\}. \tag{2.15}$$

We claim that

$$\text{there is no } t < \tau_2 \text{ such that } q(t) \in \overline{B_r(\eta_2)}. \tag{2.16}$$

Suppose (2.16) is false. Then there are $\tau_0, \tau_1 \in (-\infty, \tau_2)$ such that $q(\tau_0) \in \partial B_{\rho_0}(\eta_2)$, $q(\tau_1) \in \partial B_{\rho_0}(0)$ and $q(t) \in \mathbb{R}^n \backslash (\bigcup_{i=1,2} B_{\rho_0}(\eta_i))$ if $t \in (\tau_0, \tau_1)$. It follows from Proposition 2 that

$$\int_{\tau_0}^{\tau_1} [\frac{1}{2}|\dot{q}|^2 + V(t, q)]dt \ge 2\rho_0\sqrt{18\theta(\rho_0)}.$$

Set

$$Z(t)=\begin{cases} 0 & \text{if } t\in(-\infty,\tau_2-r]\cup[\tau_5+r,\infty)\\ \frac{t-\tau_2+r}{r}q(\tau_2) & \text{if } t\in(\tau_2-r,\tau_2)\\ q(t) & \text{if } t\in[\tau_2,\tau_5]\\ \frac{\tau_5+r-t}{r}q(\tau_5) & \text{if } t\in(\tau_5,\tau_5+r). \end{cases}$$

Applying the mean value theorem yields

$$\int_{\tau_2-r}^{\tau_2}[\frac{1}{2}|\dot Z|^2+V(t,Z)]dt+\int_{\tau_5}^{\tau_5+r}[\frac{1}{2}|\dot Z|^2+V(t,Z)]dt\le 2\Lambda r. \tag{2.17}$$

Hence

$$\begin{aligned} I_0(Z) &= I_0(q)-\int_{-\infty}^{\tau_2}[\frac{1}{2}|\dot q|^2+V(t,q)+\psi(t,q)]dt\\ &\quad -\int_{\tau_5}^{\infty}[\frac{1}{2}|\dot q|^2+V(t,q)+\psi(t,q)]dt+\int_{\tau_2-r}^{\tau_2}[\frac{1}{2}|\dot Z|^2+V(t,Z)]dt\\ &\quad +\int_{\tau_5}^{\tau_5+r}[\frac{1}{2}|\dot Z|^2+V(t,Z)]dt\\ &\le \alpha-2\rho_0\sqrt{18\theta(\rho_0)}+2\Lambda r<\alpha. \end{aligned}$$

This violates (2.2) and thus (2.16) must be true. Likewise, there is no $t>\tau_5$ such that $q(t)\in\overline{B_r(\eta_2)}$. Let

$$S_0=\{t|q(t)\notin B_r(0)\}\cap((-\infty,\tau_2)\cup(\tau_5,\infty)).$$

It is easy to see that $|S_0|\le[\frac{\alpha}{\theta(r)}]+1$, where $|S_0|$ is the Lebesgue measure of S_0. Let $S_1=S_0\cup[\tau_2,\tau_5]$ and $S_2=\mathbb{R}\backslash S_1$. It follows from (V2) that

$$I_0(q)\ge\int_{-\infty}^{\infty}\frac{1}{2}|\dot q|^2dt+\int_{S_1}[V(t,q)+\psi(t,q)]dt+\int_{S_2}\mu_1|q|^2dt.$$

Since $|S_1|\le[\alpha/\theta(r)]+1+\tau_5-\tau_2$, we conclude that $q\in E$ and thus $q(t)\to 0$ as $|t|\to\infty$.

To show q satisfies (HS), we first prove that

$$q(t)\in B_{2\rho_0}(0) \text{ if } t\in(-\infty,\tau_2)\cup(\tau_5,\infty). \tag{2.18}$$

We only treat the case of $t\in(-\infty,\tau_2)$, the other is analogue. Suppose there is a $\tau<\tau_2$ such that $q(\tau)\notin B_{2\rho_0}(0)$. Then we can find $t_3<t_4<t_5\le t_6<t_7<t_8\le\tau_2$ such that $q(t_i)\in\partial B_r(0)$ if $i=3,8$, $q(t_i)\in\partial B_{\rho_0}(0)$ if $i=4,7$, $q(t_i)\in\partial B_{2\rho_0}(0)$ if $i=5,6$ and $\rho_0<|q(t)|<2\rho_0$ if $t\in(t_4,t_5)\cup(t_6,t_7)$. If $t_8-t_3<2r$, setting

$$Z_1(t)=\begin{cases} q(t) & \text{if } t\notin(t_3,t_8)\\ \frac{t_8-t}{t_8-t_3}q(t_3)+\frac{t-t_3}{t_8-t_3}q(t_8) & \text{if } t\in(t_3,t_8), \end{cases}$$

we get

$$\begin{aligned} I_0(Z_1) - I_0(q) &= \int_{t_3}^{t_8} [\frac{1}{2(t_8-t_3)^2}|q(t_8)-q(t_3)|^2 + V(t,Z_1)]dt \\ &\quad - \int_{t_3}^{t_8} [\frac{1}{2}|\dot q|^2 + V(t,q) + \psi(t,q)]dt\,. \end{aligned}$$

Now

$$\rho_0 \le |q(t_5)-q(t_4)| = |\int_{t_4}^{t_5} \dot q(t)dt| \le \sqrt{t_5-t_4}(\int_{t_4}^{t_5} |\dot q(t)|^2 dt)^{1/2}$$

which implies that

$$\int_{t_4}^{t_5} [\frac{1}{2}|\dot q|^2 + V(t,q)]dt \ge \frac{\rho_0^2}{2(t_5-t_4)} > \frac{\rho_0^2}{2(t_8-t_3)}.$$

Likewise,

$$\int_{t_6}^{t_7} [\frac{1}{2}|\dot q|^2 + V(t,q)]dt > \frac{\rho_0^2}{2(t_8-t_3)}.$$

Moreover, by (V2)

$$\int_{t_3}^{t_8} [\frac{1}{2(t_8-t_3)^2}|q(t_8)-q(t_3)|^2 + V(t,Z_1)]dt < \frac{2r^2}{t_8-t_3} + 2\mu_2 r^3.$$

Invoking (0.7) yields

$$I_0(Z_1) - I_0(q) < \frac{2r^2}{t_8-t_3} + 2\mu_2 r^3 - \frac{\rho_0^2}{t_8-t_3} \le 0,$$

which is absurd since $I_0(q) = \min_{z\in E} I(z)$.

We next consider the case that $t_8 - t_3 \ge 2r$. Let

$$Z_2(t) = \begin{cases} q(t) & \text{if } t \notin (t_3,t_8) \\ 0 & \text{if } t \in [t_3+r, t_8-r] \\ \frac{t_3+r-t}{r} q(t_3) & \text{if } t \in (t_3, t_3+r) \\ \frac{t-t_8+r}{r} q(t_8) & \text{if } t \in (t_8-r, t_8)\,. \end{cases}$$

Arguing as in (2.17), we obtain

$$\int_{t_3}^{t_8} [\frac{1}{2}|\dot Z_2|^2 + V(t,Z_2) + \psi(t,Z_2)]dt \le 2\Lambda r.$$

On the other hand, by (V2)

$$\int_{t_4}^{t_5} [\frac{1}{2}|\dot q|^2 + V(t,q)]dt \ge \frac{\rho_0^2}{2(t_5-t_4)} + \theta(\rho_0)(t_5-t_4) \ge \rho_0\sqrt{2\theta(\rho_0)}.$$

Likewise,

$$\int_{t_6}^{t_7}[\frac{1}{2}|\dot{q}|^2+V(t,q)]dt \ge \rho_0\sqrt{2\theta(\rho_0)}.$$

Hence using (0.7) yields

$$\begin{aligned} & I_0(Z_2)-I_0(q) \\ = & \int_{t_3}^{t_8}[\frac{1}{2}|\dot{Z}_2|^2+V(t,Z_2)+\psi(t,Z_2)-\frac{1}{2}|\dot{q}|^2-V(t,q)-\psi(t,q)]\,dt \\ < & \ 2\Lambda r-2\rho_0\sqrt{2\theta(\rho_0)}\le 0\,, \end{aligned}$$

which leads to the same contradiction as above. This completes the proof of (2.18). Also, a slight modification in the above argument shows that

$$q(t)\in B_{2\rho_0}(\eta_2) \text{ if } t\in(\tau_3,\tau_4). \tag{2.19}$$

Next, we claim

$$\tau_3<\hat{t}_2-2\rho_0. \tag{2.20}$$

If (2.20) is false, then

$$\begin{aligned} \int_{-\infty}^{\tau_3}[\frac{1}{2}|\dot{q}|^2+V(t,q)+\psi(t,q)]dt &= \inf_{z\in A_1}\int_{-\infty}^{\tau_3}[\frac{1}{2}|\dot{z}|^2+V(t,z)+\psi(t,z)]dt \\ &< \hat{\alpha}(k_1,k_2)+\Lambda r, \end{aligned} \tag{2.21}$$

where $A_1=\{z|z\in E \text{ and } z(\tau_3)=q(\tau_3)\}$. Combining (2.21) with (0.7) gives

$$\int_{-\infty}^{\tau_3}[\frac{1}{2}|\dot{q}|^2+V(t,q)+\psi(t,q)]dt<\hat{\alpha}(k_1,k_2)+\rho_0\sqrt{2\theta(\rho_0)}.$$

It follows from Proposition 2 that

$$\tau_2\ge\tau_3-\frac{1}{\theta(r)}\left(\hat{\alpha}(k_1,k_2)+\rho_0\sqrt{2\theta(\rho_0)}\right)\ge k_2+\rho_0.$$

Setting

$$Z_3(t)=\begin{cases} \eta_2 & \text{if } t\ge\tau_3+r \\ \frac{\tau_3+r-t}{r}q(\tau_3)+\frac{t-\tau_3}{r}\eta_2 & \text{if } t\in(\tau_3,\tau_3+r) \\ q(t) & \text{if } t\in[\tau_2,\tau_3] \\ \frac{t-\tau_2+r}{r}q(\tau_2) & \text{if } t\in(\tau_2-r,\tau_2) \\ 0 & \text{if } t\le\tau_2-r, \end{cases}$$

we see that $Z_3\in\hat{E}(k_2,k_3)$ and

$$\begin{aligned} \int_{\tau_2}^{\tau_3}[\frac{1}{2}|\dot{q}|^2+V(t,q)+\psi(t,q)]dt &\ge I_0(Z_3)-2\Lambda r \\ &> \hat{\alpha}(k_2,k_3)-2\Lambda r \\ &\ge \hat{\alpha}(k_1,k_2)+\Lambda r. \end{aligned}$$

This is incompatible with (2.21), so (2.20) must hold. The above argument also shows that

$$\tau_4 > \hat{t}_3 + 2\rho_0. \tag{2.22}$$

It remains to prove that

$$\tau_2 > \hat{t}_1 + 2\rho_0 \tag{2.23}$$

and

$$\tau_5 < \hat{t}_4 - 2\rho_0. \tag{2.24}$$

Again, we argue indirectly. If (2.23) is false, then $\tau_3 \le k_1 - \rho_0$. This implies that

$$\int_{\tau_2}^{\tau_3} [\frac{1}{2}|\dot{q}|^2 + V(t,q) + \psi(t,q)]dt \ge \hat{\alpha}(\tau_2 - r, \tau_3 + r) - 2\Lambda r > \hat{\alpha}(-\infty, k_1) - 2\Lambda r\,.$$

Define

$$Z_4(t) = \begin{cases} q(t) & \text{if } t \ge \tau_4 \\ \frac{\tau_4 - t}{r}\eta_2 + \frac{t-\tau_4+r}{r}q(\tau_4) & \text{if } t \in (\tau_4 - r, \tau_4) \\ \eta_2 & \text{if } t \in (k_2, \tau_4 - r) \\ Z_5(t) & \text{if } t \in (-\infty, k_2), \end{cases}$$

where $Z_5 \in \hat{E}(k_1,k_2)$ and satisfies $I_0(Z_5) = \hat{\alpha}(k_1,k_2)$. Since $Z_4 \in E$ and

$$\begin{aligned} I_0(Z_4) &\le \hat{\alpha}(k_1,k_2) + \Lambda r + \int_{\tau_4}^{\infty} [\frac{1}{2}|\dot{q}|^2 + V(t,q) + \psi(t,q)]dt \\ &= \hat{\alpha}(k_1,k_2) + \Lambda r + I_0(q) - \int_{-\infty}^{\tau_4} [\frac{1}{2}|\dot{q}|^2 + V(t,q) + \psi(t,q)]dt \\ &< \hat{\alpha}(k_1,k_2) + \alpha - \hat{\alpha}(-\infty,k_1) + 3\Lambda r \le \alpha, \end{aligned} \tag{2.25}$$

we see that (2.25) contradicts (2.2) and thus (2.23) must hold. The proof of (2.24) is similar.

3 A Minimax Approach

We now use a minimax approach, which is in the same spirit of Mountain Pass Lemma [2], to obtain the existence of homoclinic solutions of (HS). Let q be the homoclinic solution obtained in Theorem 1,

$$\Gamma = \{v \in C([0,1],E) | v(0) = q \text{ and } v(1) = 0\} \tag{3.1}$$

$$\beta = \inf_{v\in\Gamma} \max_{a\in[0,1]} I(v(a))\,. \tag{3.2}$$

Proposition 3 *If the hypotheses of Theorem 1 are satisfied, then $\beta > \alpha$.*

Proof. We argue indirectly. If $\beta = \alpha$, then there exists a sequence $\{v_m\} \subset \Gamma$ such that

$$\max_{a\in[0,1]} I(v_m(a)) \to \alpha \text{ as } m \to \infty.$$

For $z \in E$, define

$$G(z) = \{(t, z(t))|t \in \mathbb{R}\}.$$

Let $A = \{(t, y)|\psi(t, y) = 0\}$, where ψ was the penalty function defined in the proof of Theorem 1. For fixed m, we set

$$a_m = \sup\{\bar{a}|G(v_m(a)) \subset \mathring{A} \text{if } a < \bar{a}\} \tag{3.3}$$

and $u_m = v_m(a_m)$. Then $G(u_m) \cap \partial A \neq \phi$. Since

$$\lim_{m\to\infty} I_0(u_m) = \lim_{m\to\infty} I(u_m) = \alpha,$$

the arguments used to prove (2.9), (2.11) and (2.17) show that if $u_m(t) \in \partial A$ and m is sufficiently large, then $t \in [\hat{t_1} - (\alpha+1)/\theta(r), \hat{t}_4 + (\alpha+1)/\theta(r)]$. Furthermore, we may proceed as the proof of Theorem 1 to obtain a $\bar{q} \in E$ and a subsequence, still denoted by $\{u_m\}$, such that $u_m \to \bar{q}$ weakly in $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n)$ and strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$. Consequently there is an $s \in [\hat{t_1} - (\alpha + 1)/\theta(r), \hat{t}_4 + (\alpha + 1)/\theta(r)]$ such that $\bar{q}(s) \in \partial A$. On the other hand, repeating the proof of Theorem 1 yields $I(\bar{q}) = \alpha$ and $G(\bar{q}) \subset \mathring{A}$. Thus we get a contradiction which completes the proof.

To show β is a critical value of I, we use comparison arguments to justify the convergence of $(\mathrm{PS})_\beta$ sequences. Let $\mathcal{E}_k = \{z \in E|z(t) = 0 \text{ if } t \in [-k, k]\}$. For $w \in \mathcal{E}_k$, we define

$$J_k(w) = \int_{-\infty}^{-k} [\frac{1}{2}|\dot{w}|^2 + V(t, w)]dt + \int_k^\infty [\frac{1}{2}|\dot{w}|^2 + V(t, w)]dt \tag{3.4}$$

and

$$P_k = \Big\{c : \text{ there exists a sequence } \{w_m\} \subset \mathcal{E}_k \text{ with } J_k(w_m) \to c \text{ and } J_k'(w_m) \to 0 \text{ as } m \to \infty\Big\}.$$

To simplify the presentation, we focus on the case where $\mathcal{K}_1 = \{\eta_1, \eta_2\}$ in the remaining of this section.

Theorem 2 *Assume the hypotheses of Theorem 1 are satisfied. If*

$$\beta < \min(\hat{\alpha}(-\infty, k_1), \tilde{\alpha}(k_4, \infty)), \tag{3.5}$$

$$\beta \notin P_k \text{ and } \beta - \alpha \notin P_k \text{ for some } k, \tag{3.6}$$

then there are at least two homoclinic orbits of (HS) which satisfy (0.2).

We now state a technical lemma to be used in the proof of Theorem 2.

Lemma 3 *Let $\{z_m\}$ be a $(PS)_c$ sequence and q_0 be a function which satisfies $I'(q_0) = 0$. Suppose there is an increasing sequence $\{t_m\} \subset (0, \infty)$ such that $t_m \to \infty$ and*

$$\int_{-t_m}^{t_m} |z_m - q_0|^2 dt \to 0 \text{ as } m \to \infty. \tag{3.7}$$

Then, for $k \geq 1$, there is a sequence $\{w_m\} \subset \mathcal{E}_k$ such that $\lim_{m\to\infty} J_k(w_m) = c - I(q_0)$ and $\lim_{m\to\infty} J_k'(w_m) = 0$.

A detailed proof of the lemma can be found in [8].

Proof of Theorem 2. By the standard deformation theory (e.g., [26]), there exists a sequence $\{z_m\} \subset E$ such that $\lim_{m\to\infty} I(z_m) = \beta$ and $\lim_{m\to\infty} I'(z_m) = 0$. It follows from Lemma 2 that $\{z_m\}$ is bounded in $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n)$. Hence there is a $\tilde{q} \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n) \cap L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$ such that along a subsequence $z_m \to \tilde{q}$ weakly in $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n)$ and strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$. Moreover,

$$I(\tilde{q}) \leq \beta.$$

Applying (3.5) and the arguments used to prove Theorem 1, we get $\tilde{q} \in E$. It follows from Lemma 2 and the Dominated Convergence Theorem that

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} V'(t, z_m)\phi dt = \int_{-\infty}^{\infty} V'(t, \tilde{q})\phi dt \tag{3.8}$$

for all $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ is dense in E, it follows that $I'(\tilde{q})\phi = 0$ for all $\phi \in E$.

It remains to show that $\tilde{q} \not\equiv 0$ and $\tilde{q} \neq q$. if $\tilde{q} \equiv 0$, invoking Lemma 3 would yield $\beta \in P_k$. Similarly, using $\beta - \alpha \notin P_k$, we obtain $\tilde{q} \neq q$.

In the next existence result, we add an additional assumption on V so that (3.6) holds.

(V5) There exist positive numbers $e, \theta_1, \theta_2, R_1, R_2, R_3, R_4$ such that

$$V(t, y) \geq \begin{cases} \theta_1 & \text{if } t \leq -k \text{ and } R_3 \leq |y| \leq R_1 \\ \theta_2 & \text{if } t \geq k \text{ and } R_4 \leq |y| \leq R_2 \end{cases}$$

and $y \cdot V'(t, y) \geq e|y|^2$ if $(t, y) \in ((-\infty, -k] \times \overline{B_{R_1}(0)}) \cup ([k, \infty) \times \overline{B_{R_2}(0)})$.

Theorem 3 *Assume (V5), (3.5) and the hypotheses of Theorem 1 are satisfied. If*

$$\beta < \min(\sqrt{2\theta_1}(R_1 - R_3), \sqrt{2\theta_2}(R_2 - R_4)), \tag{3.9}$$

then there are at least two homoclinic orbits of (HS) which satisfy (0.1).

Proof. It suffices to verify (3.6). Let $\{w_m\} \subset \mathcal{E}_k$ be a sequence such that $J_k'(w_m) \to 0$ as $m \to \infty$. If $\|w_m\|_{L^\infty(-\infty,-k]} \le R_1$ and $\|w_m\|_{L^\infty[k,\infty)} \le R_2$ for all large m, then it follows from (V5) that

$$J_k'(w_m)\frac{w_m}{\|w_m\|_k} \ge \|w_m\|_k^{-1}\int_{|t|\ge k}(|\dot{w}_m|^2 + e|w_m|^2)dt \ge \bar{e}\|w_m\|_k,$$

where $\bar{e} = \min(1,e)$ and $\|\cdot\|_k$ denotes the restriction of $\|\cdot\|$ on $\mathcal{E}_k$. Then we have $\|w_m\|_k \to 0$ and consequently

$$J_k(w_m) \to 0 \text{ as } m \to \infty. \tag{3.10}$$

Suppose that there is a subsequence, still denoted by $\{w_m\}$, such that $\|w_m\|_{L^\infty(-\infty,-k]} > R_1$. Since $w_m(-k) = 0$ and $\lim_{t\to-\infty} w_m(t) = 0$, there is a $s_m \in (-\infty,-k)$ such that $|w_m(s_m)| > R_1$. Using (V5) and arguing like (2.11), we get

$$J_k(w_m) \ge (R_1 - R_3)\sqrt{2\theta_1}. \tag{3.11}$$

Likewise,

$$J_k(w_m) \ge (R_2 - R_4)\sqrt{2\theta_2} \tag{3.12}$$

if $\|w_m\|_{L^\infty[k,\infty)} > R_2$. Now (3.10),(3.11) and (3.12) imply (3.6). The proof is complete.

4 Multibump Homoclinic Solutions

As we have seen, the homoclinic solution obtained in Theorem 1 comprises two heteroclinics nicely concatenated. Our aim in this section is to discuss under what conditions a chain of more than two heteroclinics can be concatenated to form a multibump homoclinic. To avoid complicated notation, we will mainly focus on the case where V is periodic in t.

(V6) $V(t+T,y) = V(t,y)$ for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

Let us first deal with the case that two heteroclinics are concatenated to form a homoclinic solution of (HS).

Theorem 4 *Assume that (V1)-(V3) and (V6) are satisfied. Suppose there are* $k_0 < k_1 < k_2 < k_3 < k_4 = k_0 + T$ *such that*

$$\hat{\alpha}(k_1,k_2) < \min(\hat{\alpha}(k_0,k_1), \hat{\alpha}(k_2,k_3))\,, \tag{4.1}$$

$$\tilde{\alpha}(k_3,k_4) < \min(\tilde{\alpha}(k_2,k_3), \tilde{\alpha}(k_0,k_1))\,, \tag{4.2}$$

$$\min(k_3-k_2, k_1-k_0) > 6\rho_0 + 2(\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4) + \rho_0\sqrt{2\theta(\rho_0)})/\theta(r)\,, \tag{4.3}$$

where

$$r = \min\left(1, \frac{\rho_0}{2}, \sqrt[4]{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0\sqrt{2\theta(\rho_0)}}{\Lambda}, \frac{\bar{\theta}}{3\Lambda}\right) \tag{4.4}$$

and $\bar{\theta} = \min(\hat{\alpha}(k_0,k_1) - \hat{\alpha}(k_1,k_2), \hat{\alpha}(k_2,k_3) - \hat{\alpha}(k_1,k_2), \tilde{\alpha}(k_2,k_3) - \tilde{\alpha}(k_3,k_4), \tilde{\alpha}(k_0,k_1) - \tilde{\alpha}(k_3,k_4))$. *Then there exist a sequence of homoclinics* $\{q_j\}$ *of (HS) which satisfy (0.2) and* $I(q_j) < \hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4)$. *Moreover, for any pair of* q_i *and* q_j*, there does not exist* $m \in \mathbb{N}$ *such that* $q_i(t+mT) = q_j(t)$.

Proof. The proof is similar to that of Theorem 1, so we only sketch the significant difference. Let $\psi_1 \in C^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ such that $0 \le \psi_1 \le M_1$ and

$$\psi_1(t,y) = \begin{cases} 0 & \text{if } t \in [\hat{t}_1+\rho_0, \hat{t}_2-\rho_0] \cup [\hat{t}_3+\rho_0, \hat{t}_4-\rho_0] \\ M_1 & \text{if } y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [\hat{t}_2, \hat{t}_3] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_2) \text{ and } t \in (\hat{t}_2-\rho_0, \hat{t}_3+\rho_0) \\ M_1 & \text{if } y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in [\hat{t}_0, \hat{t}_1] \cup [\hat{t}_4, \hat{t}_1+T] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_1) \text{ and } t \in (-\infty, \hat{t}_2-\rho_0] \cup [\hat{t}_3+\rho_0, \infty), \end{cases}$$

where $\bar{\rho} = 5\rho_0/2$, $\hat{t}_0 = k_0 + 3\rho_0 + \left(\tilde{\alpha}(k_3,k_4) + \rho_0\sqrt{2\theta(\rho_0)}\right)/\theta(r)$, $\hat{t}_1 = k_1 - 3\rho_0 - \left(\hat{\alpha}(k_1,k_2) + \rho_0\sqrt{2\theta(\rho_0)}\right)/\theta(r)$, $\hat{t}_2 = k_2 + 3\rho_0 + \left(\hat{\alpha}(k_1,k_2) + \rho_0\sqrt{2\theta(\rho_0)}\right)/\theta(r)$, $\hat{t}_3 = k_3 - 3\rho_0 - \left(\tilde{\alpha}(k_3,k_4) + \rho_0\sqrt{2\theta(\rho_0)}\right)/\theta(r)$, $\hat{t}_4 = \hat{t}_0 + T$, $t^\star = \min(\hat{t}_3 - \hat{t}_2, \hat{t}_1 - \hat{t}_0)$ and $M_1 = \theta(r) + (\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4))/t^\star$. Set

$$I_1(z) = \int_{-\infty}^{\infty} [\frac{1}{2}|\dot{z}|^2 + V(t,z) + \psi_1(t,z)]dt. \tag{4.5}$$

We may proceed as in the proof of Theorem 1 to get a $q \in E$ which satisfies

$$I_1(q) = \inf_{z\in E} I_1(z) < \hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4), \tag{4.6}$$

(2.18), (2.20), and (2.22).

Next, we prove that q satisfies (2.23), (2.24) and (2.19). If (2.23) is false, arguing like (2.21) yields

$$\begin{aligned} \int_{\tau_2}^{\tau_3} [\frac{1}{2}|\dot{q}| + V(t,q) + \psi_1(t,q)]dt &< \int_{-\infty}^{t_0} [\frac{1}{2}|\dot{q}|^2 + V(t,q) + \psi_1(t,q)]dt \\ &< \hat{\alpha}(k_1,k_2) + \Lambda r\,, \end{aligned} \tag{4.7}$$

where as (2.7) t_0 is a point in $(\hat{t}_2, \hat{t}_3)$ and $q(t_0) \in B_r(\eta_2)$. Combining (4.7) with (4.4) gives

$$\tau_3 \le \tau_2 + \frac{1}{\theta(r)}(\hat{\alpha}(k_1,k_2) + \rho_0\sqrt{2\theta(\rho_0)}) \le k_1 - \rho_0. \tag{4.8}$$

Arguing like (2.7), we get an $s \in (\hat{t}_0, \hat{t}_1)$ such that $q(s) \in B_r(0)$. Hence $\tau_2 > \hat{t}_0$ and

$$\int_{\tau_2}^{\tau_3} [\frac{1}{2}|\dot{q}|^2 + V(t,q) + \psi_1(t,q)]dt \geq \hat{\alpha}(k_0,k_1) - 2\Lambda r \geq \hat{\alpha}(k_1,k_2) + \Lambda r. \quad (4.9)$$

This completes the proof of (2.23), since (4.9) is incompatible with (4.7).

The proofs of (2.24) and (2.19) follow from the same lines of reasoning as above. Thus we have shown that q is a homoclinic solution of (HS).

To obtain more homoclinic solutions of (HS), we add different penalty functionals to I. It will be seen that the distance between two bumps can be as long as we please. For $j \in \mathbb{N}$, let $\psi_j(t,y)$ be defined as follows:

$$\begin{cases} 0 & \text{if } t \in [\hat{t}_1 + \rho_0, \hat{t}_2 - \rho_0] \cup [\hat{t}_3 + \rho_0 + (j-1)T, \hat{t}_4 - \rho_0 + (j-1)T] \\ M_1 & \text{if } y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [\hat{t}_2, \hat{t}_3 + (j-1)T] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_2) \text{ and } t \in (\hat{t}_2 - \rho_0, \hat{t}_3 + \rho_0 + (j-1)T) \\ M_1 & \text{if } y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in [\hat{t}_0, \hat{t}_1] \cup [\hat{t}_4 + (j-1)T, \hat{t}_1 + jT] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_1) \text{ and } t \in (-\infty, \hat{t}_2 - \rho_0] \cup [\hat{t}_3 + \rho_0 + (j-1)T, \infty). \end{cases}$$

If

$$I_j(z) = I(z) + \int_{-\infty}^{\infty} \psi_j(t,z)dt, \quad (4.10)$$

there is a $q_j \in E$ which satisfies

$$I_j(q_j) = \inf_{z \in E} I_j(z) < \tilde{\alpha}(k_1,k_2) + \tilde{\alpha}(k_3,k_4). \quad (4.11)$$

Moreover, a slight modification in the above argument shows that q_j is a solution of (HS). Observe that

$$\tau_4(q_j) > \hat{t}_3 + 2\rho_0 + (j-1)T\,,$$
$$\tau_5(q_j) < \hat{t}_4 - 2\rho_0 + (j-1)T,$$

where $\tau_i(q_j)$ defined as (2.14), (2.15). Thus the last assertion of the theorem is verified.

Remark. From the proof of Theorem 4, we know that as long as suitable oscillation conditions like (4.1) and (4.2) hold, homoclinic orbits still exist even if V is not periodic in t.

A special case of Theorem 4 is the following:

Theorem 5 *Assume that (V1)-(V3) and (V6) are satisfied. Suppose there are* $k_1 < k_2 < k_3 = k_1 + T$ *such that* $\hat{\alpha}(k_1,k_2) < \hat{\alpha}(k_2,k_3)$, $\tilde{\alpha}(k_1,k_2) < \tilde{\alpha}(k_2,k_3)$ *and* $k_3 - k_2 > 6\rho_0 + 2\big(\hat{\alpha}(k_1,k_2) + \tilde{\alpha}(k_1,k_2) + \rho_0\sqrt{2\theta(\rho_0)}\big)/\theta(r)$, *where*

$$r = \min\left(1, \frac{\rho_0}{2}, \sqrt[4]{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0\sqrt{2\theta(\rho_0)}}{\Lambda}, \frac{\bar{\theta}}{3\Lambda}\right)$$

and $\bar{\theta} = \min(\hat{\alpha}(k_2,k_3) - \hat{\alpha}(k_1,k_2), \tilde{\alpha}(k_2,k_3) - \tilde{\alpha}(k_1,k_2))$. *Then the assertion of Theorem 4 holds.*

Next, we consider 2ℓ-bump solutions for $\ell > 1$.

Theorem 6 *Suppose the hypotheses of Theorem 4 are satisfied. For each $\ell \in \mathbb{N}$, there exist a sequence of 2ℓ-bump homoclinics $\{q_j\}$ of (HS) which satisfy (0.2).*

Proof. The homoclinic solutions will be obtained by adding different penalty functions. Set

$$M_\ell = \theta(r) + (\hat{\alpha}(k_1, k_2) + \tilde{\alpha}(k_3, k_4))\ell/t^\star.$$

Let $\hat{\psi}_1(t, y, \ell, m)$ be defined as follows:

$$\begin{cases} 0 & \text{if } t \in (-\infty, \hat{t}_0 - (\ell-1)T - \rho_0] \cup [\hat{t}_1 + (m-1)T + \rho_0, \infty) \\ M_\ell & \text{if } y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in [\hat{t}_0 - (\ell-1)T, \hat{t}_1 + (m-1)T] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_1) \text{ and} \\ & \quad t \in (-\infty, \hat{t}_0 - (\ell-1)T - \rho_0) \cup (\hat{t}_1 + (m-1)T + \rho_0, \infty) \end{cases}$$

and let $\hat{\psi}_2(t, y, \ell, m)$ be defined as follows:

$$\begin{cases} 0 & \text{if } t \in (-\infty, \hat{t}_2 - (\ell-1)T - \rho_0] \cup [\hat{t}_3 + (m-1)T + \rho_0, \infty) \\ M_\ell & \text{if } y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [\hat{t}_2 - (\ell-1)T, \hat{t}_3 + (m-1)T] \\ 0 & \text{if } y \in B_{\bar{\rho}}(\eta_2) \text{ and} \\ & \quad t \in (-\infty, \hat{t}_2 - (\ell-1)T - \rho_0) \cup (\hat{t}_3 + (m-1)T + \rho_0, \infty). \end{cases}$$

For every $\ell \in \mathbb{N}$, we may use $\hat{\psi}_1$ and $\hat{\psi}_2$ to construct penalty functions and obtain a sequence of 2ℓ-bump homoclinics. Since the notation is quite involved, we only carry out the case of $\ell = 2$ as follows: Let $m_1, m_2, m_3 \in \mathbb{N}$, $\ell = 2$ and

$$\begin{aligned} \tilde{\psi}(t, y, m_1, m_2, m_3) \;=\; & \hat{\psi}_1(t, y, \ell, 1) + \hat{\psi}_2(t - (\ell-1)T, y, \ell, m_1) \\ & + \hat{\psi}_1(t - (2\ell - 2 + m_1)T, y, \ell, m_2) \\ & + \hat{\psi}_2(t - (3\ell - 3 + m_1 + m_2)T, y, \ell, m_3) \\ & + \hat{\psi}_1(t - (4\ell - 4 + m_1 + m_2 + m_3)T, y, \ell, 1), \end{aligned}$$

where $\hat{t}_i$, $0 \le i \le 3$, are defined as in the proof of Theorem 4. Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_\ell = \mathbb{N} \backslash \{1, 2, \cdots, \ell - 1\}$. For every $(m_1, m_2, m_3) \in \mathbb{N}_2^3$, by adding $\int_{-\infty}^{\infty} \tilde{\psi}(t, z, m_1, m_2, m_3)dt$ to $I(z)$, the corresponding penalized functional possesses a minimizer which is a homoclinic orbit of (HS).

For $\ell \ge 3$, we can construct a penalty function $\tilde{\psi}_\ell(t, y, m_1, m_2, \cdots, m_{2\ell-1})$ for every $(m_1, m_2, \cdots, m_{2\ell-1}) \in \mathbb{N}_\ell^{2\ell-1}$ in the same vein. Then various patterns of multibump homoclinics of (HS) can be obtained as local minimizers of I.

5 Multibump Heteroclinic Solutions

When (HS) possesses more than two equilibria, there could be more different patterns of multibump connecting orbits. A connecting orbit is said of type

$(\eta_1, \eta_2, \cdots, \eta_\ell)$ if there exist $t_2 < t_3 < \cdots < t_{\ell-1}$ and $\{\eta_1, \eta_2, \cdots, \eta_\ell\} \subset \mathcal{K}_1$ such that $q(t_k) \in B_{\rho_0}(\eta_k)$ for $2 \le k \le \ell - 1$, $\lim_{t\to-\infty} q(t) = \eta_1$ and $\lim_{t\to\infty} q(t) = \eta_\ell$. Here it is understood that $\eta_m \neq \eta_{m+1}$ for all m. In this section, we study such types of heteroclinic solutions of (HS). Related results for homoclinics can be found in the same way. Since the notation is quite involved, we give a presentation for a heteroclinic of type (η_1, η_2, η_3) to illustrate the main idea for the construction of penalty functions. For $j_1 < j_2$, let

$$\begin{aligned} &F(\eta_i, \eta_m, j_1, j_2) \\ &\quad = \big\{ z \in W^{1,2}_{\rm loc}(\mathbb{R}, \mathbb{R}^n) : z(t) = \eta_i \text{ if } t \le j_1 \text{ and } z(t) = \eta_m \text{ if } t \ge j_2 \big\}. \end{aligned}$$

and

$$\gamma(\eta_i, \eta_m, j_1, j_2) = \inf_{z \in F(\eta_i, \eta_m, j_1, j_2)} I(z)\,.$$

Set

$$\begin{aligned} \gamma(\eta_i, \eta_m, -\infty, j_2) &= \lim_{j_1 \to -\infty} \gamma(\eta_i, \eta_m, j_1, j_2)\,, \\ \gamma(\eta_i, \eta_m, j_1, \infty) &= \lim_{j_2 \to \infty} \gamma(\eta_i, \eta_m, j_1, j_2). \end{aligned}$$

Theorem 7 *Assume (V1)-(V4) are satisfied. Suppose there are $k_1 < k_2 < k_3 < k_4$ such that*

$$\begin{gathered} \gamma(\eta_1, \eta_2, k_1, k_2) < \min(\gamma(\eta_1, \eta_2, -\infty, k_1), \gamma(\eta_1, \eta_2, k_2, k_3)) + 3\Lambda r\,, \\ \gamma(\eta_2, \eta_3, k_3, k_4) < \min(\gamma(\eta_2, \eta_3, k_2, k_3), \gamma(\eta_2, \eta_3, k_4, \infty)) + 3\Lambda r\,, \\ k_3 - k_2 > 6\rho_0 + 2(\gamma(\eta_1, \eta_2, k_1, k_2) + \gamma(\eta_2, \eta_3, k_3, k_4) + \rho\sqrt{2\theta(\rho_0)})/\theta(r)\,, \end{gathered}$$

where

$$r \le \min\left(1, \frac{\rho_0}{2}, \sqrt[4]{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0\sqrt{2\theta(\rho_0)}}{\Lambda}\right).$$

Then (HS) possesses a heteroclinic orbit of type (η_1, η_2, η_3).

Proof. Let $W \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ such that $0 \le W \le \tilde{M}$ and

$$W(t, y) = \begin{cases} 0 & \text{if } t \in [s_1 + \rho_0, s_2 - \rho_0] \cup [s_3 + \rho_0, s_4 - \rho_0] \\ \tilde{M} & \text{if } y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [s_2, s_3] \\ 0 & \text{if } y \in B_{\bar\rho}(\eta_2) \text{ and } t \in (s_2 - \rho_0, s_3 + \rho_0) \\ \tilde{M} & \text{if } y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in (-\infty, s_1] \\ 0 & \text{if } y \in B_{\bar\rho}(\eta_1) \text{ and } t \in (-\infty, s_2 - \rho_0] \\ \tilde{M} & \text{if } y \notin B_{3\rho_0}(\eta_3) \text{ and } t \in [s_4, \infty) \\ 0 & \text{if } y \in B_{\bar\rho}(\eta_3) \text{ and } t \in [s_3 + \rho_0, \infty), \end{cases}$$

where $\bar\rho = 5\rho_0/2$, $s_1 = k_1 - 3\rho_0 - (\rho_0\sqrt{2\theta(\rho_0)} + \gamma_{12})/\theta(r)$, $s_2 = k_2 + 3\rho_0 + (\rho_0\sqrt{2\theta(\rho_0)} + \gamma_{12})/\theta(r)$, $s_3 = k_3 - 3\rho_0 - (\rho_0\sqrt{2\theta(\rho_0)} + \gamma_{23})/\theta(r)$, $s_4 = k_4 + 3\rho_0 + (\rho_0\sqrt{2\theta(\rho_0)} + \gamma_{23})/\theta(r)$, $\gamma_{12} = \gamma(\eta_1, \eta_2, k_1, k_2)$, $\gamma_{23} = \gamma(\eta_2, \eta_3, k_3, k_4)$ and

$$\tilde{M} = \theta(r) + (\gamma_{12} + \gamma_{23})/(s_3 - s_2) + (\gamma_{12} + \gamma_{23} + 1)^2/\rho_0^2.$$

Let $Q \in C^2(\mathbb{R}, \mathbb{R}^n)$ be a fixed function which satisfies

$$Q(t) = \begin{cases} \eta_1 & \text{if } t \leq -\frac{1}{2} \\ \eta_3 & \text{if } t \geq \frac{1}{2}. \end{cases}$$

For $z \in E$, define

$$I_Q(z) = \int_{-\infty}^{\infty} [\frac{1}{2}|\dot{Q} + \dot{z}|^2 + V(t, Q + z) + W(t, Q + z)]dt.$$

We may procced as in the proof of Theorem 1 to get a $\bar{z} \in E$ which satisfies

$$I_Q(\bar{z}) = \inf_{z \in E} I_Q(z).$$

Then with a slight modification, the arguments used in Theorem 1 can be adapted to show that the function $Q + \bar{z}$ is a heteroclinic solution of (HS).

References

[1] A. Ambrosetti and M. Badiale, Homoclinics: Poincaré-Melnikov type results via a variational approach, *C. R. Acad. Sci. Sér. I Math.* **323** (1996), 753-758.

[2] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349-381.

[3] B. Buffoni, A. R. Champneys, and J. F. Toland, Bifurcation and coalescence of a plethora of homoclinic orbits, *J. Dynam. Differential Equations* **8** (1996), 221-279.

[4] B. Buffoni and E. Séré, A global condition for quasi-random behavior in a class of conservative systems, *Comm. Pure Appl. Math.* **49** (1996), 285–305.

[5] C. -N. Chen and S. -Y. Tzeng, Existence and multiplicity results for heteroclinic orbits of second order Hamiltonian systems, *J. Differential Equations*, **158** (1999), 211-250.

[6] V. Coti Zelati, I. Ekeland and E. Sèrè, A variational approach to homoclinic orbits in Hamiltonian systems, *Math. Ann.* **288** (1990), 133-160.

[7] P. Caldiroli and P. Montecchiari, Homoclinic orbits for second order Hamiltonian systems with potential changing sign. Comm. on Appl. Nonlinear Anal. **1** (1994), 97-129.

[8] V. Coti Zelati and P. H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.* **4** (1991), 693-727.

[9] V. Coti Zelati and P. H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$, Comm. Pure Appl. Math. **45** (1992), 1217-1269.

[10] R. L. Devaney, Homoclinic orbits in Hamiltonian systems, *J. Differential Equations* **21** (1976), 431–438.

[11] M. del Pino and P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var.* **4** (1996), 121-137.

[12] M. del Pino and P. Felmer, Multi-peak bound states for nonlinear Schrödinger equations, *Inst. H. Poincaré Anal. Non Linéaire* **15** (1998), 127-149.

[13] W.-Y. Ding and W.-M. Ni, On the existence of a positive entire solution of a semilinear elliptic equation, *Arch. Rational Mech. Anal.* **91** (1986), 283-308.

[14] P. Felmer, Heteroclinic orbits for spatially periodic Hamiltonian systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8** (1991), 477-497.

[15] J. Guckenheimer and P. J. Holmes. "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields." Springer-Verlag, Berlin, 1983.

[16] V. V. Kozlov. Calculus of variations in the large and classical mechanics. *Russian Math. Surveys* **40** (1985), 37-71.

[17] U. Kirchgraber and D. Stoffer, Chaotic behavior in simple dynamical systems, *SIAM Rev.* **32** (1990), 424-452.

[18] W. D. Kalies and R. C. A. M. VanderVorst, Multitransition homoclinic and heteroclinic solutions of the extended Fisher-Kolmogorov equation, *J. Differential Equations*, **131** (1996), 209–228.

[19] W. D. Kalies, J. Kwapisz, and R. C. A. M. VanderVorst, Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria, *Comm. Math. Phys.* **193** (1998), 337-371.

[20] P. L. Lions, The concentration-compactness principle in the calculus of variations, *Rev. Mat. Iberoam.* **1** (1985), 145–201.

[21] J. Mawhin and M. Willem, "Critical Point Theory and Hamiltonian Systems", Appl. Math. Sci., Vol. 74, Springer-Verlag, New York/Berlin, 1989.

[22] T. O. Maxwell, Heteroclinic chains for a reversible Hamiltonian system, *Nonlinear Anal.* **28** (1997), 871-887.

[23] T. O. Maxwell, Multibump heteroclinics for a reversible Hamiltonian system, preprint.

[24] P. Montecchiari, Multiplicity of homoclinic solutions for a class of asymptotically periodic second order Hamiltonian systems, *Rend. Mat. Accad. Lincei (9)* **4** (1994), 265–271.

[25] L. A. Peletier and W. C. Troy, Chaotic spatial patterns described by the extended Fisher-Kolmogorov equation, *J. Differential Equations* **129** (1996), 458–508.

[26] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in "CBMS Regional Conf. Ser. in Math.," Vol. 65, Amer. Math. Soc., Providence, 1986.

[27] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh*, **114A** (1990), 33-38.

[28] P. H. Rabinowitz, Heteroclinics for a reversible Hamiltonian system, *Ergodic Theory Dynam. Systems* **14** (1994), 817-829.

[29] P. H. Rabinowitz, Periodic and heteroclinic orbits for a periodic Hamiltonian system, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **6** (1989), 331-346.

[30] P. H. Rabinowitz, Homoclinic and heteroclinic orbits for a class of Hamiltonian systems, *Calc. Var.* **1** (1993), 1-36.

[31] P. H. Rabinowitz, Multibump solutions of differential equations: An overview, *Chinese J. Math.* **24** (1996), 1-36.

[32] P. H. Rabinowitz and K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, *Math. Z.* **206** (1991), 473-479.

[33] E. Sèrè, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.* **209** (1992), 27-42.

[34] G. Spradlin, Multibump solutions to a Hamiltonian systems with an almost periodic term, Part I: Finding a saddle point, preprint.

[35] K. Strobel, Multibump solutions for a class of periodic Hamiltonian systems, *University of Wisconsin*, Thesis, 1994.

Chao-Nien Chen
Department of Mathematics
National Changhua University of Education
Changhua, Taiwan, ROC
e-mail: chenc@math.ncue.edu.tw

Shyuh-yaur Tzeng
Department of Mathematics
National Changhua University of Education
Changhua, Taiwan, ROC
e-mail: tzeng@math.ncue.edu.tw