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Exactness results for generalized Ambrosetti-Brezis-Cerami problem and related one-dimensional elliptic equations *

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Abstract

We consider the boundary-value problem

$$-(\varphi_p(u'))' = \varphi_\alpha(u) + \lambda\varphi_\beta(u) \quad \text{in } (0,1)$$
$$u(0) = u(1) = 0,$$

where $\varphi_p(x) = |x|^{p-2}\,x$, $p, \alpha, \beta > 1$ and $\lambda \in \mathbb{R}^*$. We give the exact number of solutions for all λ and most values of $\alpha, \beta, p > 1$. In the particular case where $1 < \beta < p = 2 < \alpha$, we resolve completely a problem suggested by A. Ambrosetti, H. Brezis and G. Cerami and which was partially solved by S. Villegas.

1 Introduction

The combined effects of concave and convex nonlinearities were considered by Ambrosetti, Brezis and Cerami in [7]. They consider the problem

$$\begin{aligned} -\Delta u &= u^{\alpha-1} + \lambda u^{\beta-1}, \quad x \in \Omega \\ u &> 0, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega \end{aligned} \tag{1.1}$$

with $1 < \beta < 2 < \alpha$ and $\lambda > 0$. They prove the existence of a constant $\Lambda > 0$ such that a solution u_λ of (1.1) exists if and only if $0 < \lambda \leq \Lambda$. Moreover, if the condition $\alpha \leq \alpha^*$ holds, then for all $\lambda \in (0, \Lambda)$ problem (1.1) has a second solution $v_\lambda > u_\lambda$, where $\alpha^* := (2N)/(N-2)$ if $N \geq 3$ and $\alpha^* = +\infty$ if $N = 1, 2$. Then several papers appeared where concave-convex nonlinearities were involved. We refer the reader to [7]-[10], [14], [17].

At the end of their paper [7], Ambrosetti, Brezis and Cerami suggested the study of the structure of the solution set of the one-dimensional problem

$$\begin{aligned} -u'' = |u|^{\alpha-2}\,u + \lambda\,|u|^{\beta-2}\,u,\ a < x < b \\ u(a) = u(b) = 0, \end{aligned} \tag{1.2}$$

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with $1 < \beta < 2 < \alpha$ and $\lambda > 0$. This study was done by S. Villegas [17] by means of a quadrature method. He shows that there exist two monotone divergent sequences $\{\varepsilon_n\}$ and $\{L_n\}$; $\varepsilon_n \leq L_n$ satisfying:

i) If $\lambda \in (0, \varepsilon_n)$ then (1.2) has exactly two pairs of opposite solutions with $n+1$ zeros.

ii) If $\lambda \in [\varepsilon_n, L_n)$ then (1.2) has at least two pairs of opposite solutions with $n+1$ zeros.

iii) If $\lambda = L_n$ then (1.2) has at least one pair of opposite solutions with $n+1$ zeros.

iv) If $\lambda > L_n$ then (1.2) has no solutions with $n+1$ zeros.

In the present paper, we consider the p-Laplacian version of problem (1.2), that is, we consider the boundary-value problem

$$\begin{gathered} -(\varphi_p(u'))' = f(\lambda, u) \quad \text{in } (0,1) \\ u(0) = u(1) = 0, \end{gathered} \tag{1.3}$$

with $\varphi_p(x) = |x|^{p-2}\, x$ and $f(\lambda, u) = \varphi_\alpha(u) + \lambda\varphi_\beta(u)$ and notice that when $p = 2$, problem (1.3) is reduced to problem (1.2). Our original interest was in answering the question: how does the solution set of (1.3) look like when $p \neq 2$ and $1 < \beta < p < \alpha$, $\lambda > 0$? The interest in this question comes from the fact that the structure of the solution set of problem (1.3) depends on $p > 1$ for some examples of second members $f(\lambda, u)$ (see for instance, Guedda and Veron [12], Addou [5]) and does not depend on p for some others (see for instance, Addou and Benmezaï [2], for positive solutions when $f(\lambda, u) = \lambda \exp(u)$). We shall prove that for $f(\lambda, u) = \varphi_\alpha(u) + \lambda\varphi_\beta(u)$, and $1 < \beta < p < \alpha$, Villegas' result holds for problem (1.3) for all $p > 1$ (part of Theorem 2.1, Assertion (C)).

Notice that Assertions (ii) and (iii), in Villegas' result, *do not provide the exact number* of solutions with $n+1$ zeros. So, when λ belongs to the range $[\varepsilon_n, L_n]$, the exact number of solutions with $n+1$ zeros *has yet to be studied.*

The exact number of solutions for problem (1.1) when λ ranges over the whole interval $(0, +\infty)$, was given by Ouyang and Shi [14], but under two restrictions: Ω was taken to be the unit ball in $\mathbb{R}^N$ and the space dimension $N \geq 4$. They proved the existence of some $\Lambda > 0$ such that problem (1.1) has exactly two solutions for $\lambda \in (0, \Lambda)$, exactly one solution for $\lambda = \Lambda$, and no solution for $\lambda > \Lambda$. Actually their result concerns more general nonlinearities.

Next, the purpose of our investigation was to complete our study by providing the exact number of solutions to (1.3) when $1 < \beta < p < \alpha$, for all $p > 1$ and all $\lambda > 0$. So, in the particular case where $p = 2$, Theorem 2.1, Assertion (C), completes Villegas' result and resolves completely the Ambrosetti-Brezis-Cerami problem [7, Section 6, (d)].

In pursuing our study, we were led quite naturally to study what may happens if λ is not necessarily positive or p is not necessarily between α and β. A precise description of the solution set of problem (1.3) for various values of p, α, and β was given

As it is well known, exactness results are difficult to obtain. The difficulties encountered in our study for $\lambda > 0$ and for $\lambda < 0$ are of different kinds. For $\lambda > 0$, we used an idea performed by the authors Addou and Benmezaï in [3], (see the proof of (iii) in [3, pp. 11-13]).

The paper is organized as follows. The main results are stated in Section 2. To prove our results we make use of a quadrature method which is described in Section 3. The main results for the case $\lambda > 0$ are proved in Section 4, while those for $\lambda < 0$ are proved in Section 5.

2 Notation and main results

Now we define some sets that will be used in the statements of the main results. For $k \geq 1$, let

$$S_k^+ = \left\{ \begin{array}{c} u \in C^1([0,1]) : u \text{ admits exactly } (k-1) \text{ zeros in } (0,1), \\ \text{all are simple, } u(0) = u(1) = 0 \text{ and } u'(0) > 0 \end{array} \right\},$$

$S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. If $u \in C([0,1])$ is a real-valued function vanishing at x_1 and x_2 and not between them, (with $x_1 < x_2$) we call its restriction to the open interval (x_1, x_2) a bump of u. So, each function in S_k^+ has exactly k bumps such that the first one is positive, and any two consecutive bumps have opposite sign.

Let A_k^+ $(k \geq 1)$ be the subset of S_k^+ consisting of the functions u satisfying:

- Every bump of u is symmetrical about the center of the interval of its definition.
- Every positive (resp. negative) bump of u can be obtained by translating the first positive (resp. negative) bump.
- The derivative of each bump of u vanishes once and only once.

Let $A_k^- = -A_k^+$ and $A_k = A_k^+ \cup A_k^-$. Denote by $(\lambda_k)_{k \geq 1}$ the eigenvalues of the one dimensional p-Laplacian operator with Dirichlet boundary conditions,

$$-(\varphi_p(u'))' = \lambda \varphi_p(u) \quad \text{in } (0,1),$$
$$u(0) = u(1) = 0.$$

One has for each integer $k \geq 1$ and $p > 1$, $\lambda_k = k^p \lambda_1$ and

$$\lambda_1 = (p-1)(2\int_0^1 (1-t^p)^{-1/p} dt)^p = (p-1)(\frac{2\pi}{p\sin(\pi/p)})^p.$$

Now we define some constants we shall use them in the statement of the main results. For all $p,\ \alpha, \beta > 1$, let

$$J(p,\alpha,\beta) := \frac{(p')^{-1/p}}{(\alpha-\beta)}(\frac{\alpha}{\beta})^{(p-\alpha)/p(\alpha-\beta)} B(\frac{(p-\beta)}{p(\alpha-\beta)}, \frac{p-1}{p}), \qquad (2.1)$$

where $B(\cdot,\cdot)$ is the Beta function. Also, for all $\beta > \alpha > 1$, and $p > 1$, let

$$K(p,\alpha,\beta) = \int_0^1 \{(\frac{1-t^\alpha}{\alpha}) - (\frac{1-t^\beta}{\beta})\}^{-1/p} dt. \tag{2.2}$$

We shall prove (see Lemma 5.2) that : $K(p,\alpha,\beta) < +\infty$ if and only if $p > 2$.

For all $\lambda \in \mathbb{R}$, denote S_λ the solution set of problem (1.3). The main results of this paper read as follows:

Theorem 2.1 *Let $p, \alpha, \beta > 1$ and $\lambda > 0$.*

(A) *Assume that one of the following conditions holds:*

- **(a)** *$\alpha > p$ and $\beta > p$, or*
- **(b)** *$\alpha = p$ and $\beta > p$, or*
- **(c)** *$\alpha < p$ and $\beta < p$, or*
- **(d)** *$\alpha = p$ and $\beta < p$.*

Then, for each integer $k = 1, 2, \cdots$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(B) *Assume that one of the following conditions holds:*

- **(a)** *$\alpha > p$ and $\beta = p$, or*
- **(b)** *$\alpha < p$ and $\beta = p$.*

Then, for each integer $k = 1, 2, \cdots$,

- **(i)** *If $\lambda \geq \lambda_k$, $S_\lambda \cap A_k = \emptyset$.*
- **(ii)** *If $0 < \lambda < \lambda_k$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.*

(C) *Assume that $1 < \beta < p < \alpha$. Then for each integer $k = 1, 2, \cdots$, there exist a real number $\mu_k > 0$ such that*

- **(i)** *If $\lambda > \mu_k$, $S_\lambda \cap A_k = \emptyset$.*
- **(ii)** *If $\lambda = \mu_k$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.*
- **(iii)** *If $\lambda \in (0, \mu_k)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \neq v_k$ and $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.*

Theorem 2.2 *Let $p, \alpha, \beta > 1$ and $\lambda < 0$.*

(A) *Assume that one of the following conditions holds:*

- **(a)** *$p > 2$,and $1 < \alpha < p < \beta$, or*
- **(b)** *$p > 2$,and $1 < \alpha < \beta = p$, or*
- **(c)** *$p > 2$, and $1 < \alpha < \beta < p$.*

Then, there exists an increasing sequence of positive real numbers $(\mu_k)_{k\geq 1}$ *such that for each integer* $k = 1, 2, \cdots,$

(i) *If* $\lambda < -\mu_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) *If* $-\mu_k \leq \lambda < 0$, *there exists* $u_k \in A_k^+$ *such that* $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(B) *Assume that* $1 < \beta < p < \alpha$. *Then there exists an increasing sequence of positive real numbers* $(\mu_k)_{k\geq 1}$ *such that for each integer* $k = 1, 2, \cdots,$

(i) *If* $\lambda \leq -\mu_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) *If* $-\mu_k < \lambda < 0$, *there exists* $u_k \in A_k^+$ *such that* $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(C) *Assume that* $1 < p \leq 2$, $\lambda < 0$ *and one of the following conditions holds:*

(a) $1 < \alpha < p < \beta$, *or*

(b) $1 < \alpha < \beta = p$, *or*

(c) $1 < \alpha < \beta < p$, *or*

(d) $1 < p \leq \beta < \alpha$.

Then, for each integer $k = 1, 2, \cdots,$ *there exists* $u_k \in A_k^+$ *such that* $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(D) *Assume that* $1 < p \leq 2$, $1 < \alpha = p < \beta$, *and* $\lambda < 0$. *Then for each integer* $k = 1, 2, \cdots,$

(i) *If* $k \geq (\frac{p^2}{\lambda_1})^{1/p}$, $S_\lambda \cap A_k = \emptyset$.

(ii) *If* $k < (\frac{p^2}{\lambda_1})^{1/p}$, *there exists* $u_k \in A_k^+$ *such that* $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(E) *Assume that* $1 < \beta < \alpha = p$, *then for each integer* $k = 1, 2, \cdots,$

(i) *If* $k \leq (2J(p, \alpha = p, \beta))^{-1}$ *or* $k \geq (p^2/\lambda_1)^{1/p}$, $S_\lambda \cap A_k = \emptyset$.

(ii) *If* $(2J(p, \alpha = p, \beta))^{-1} < k < (p^2/\lambda_1)^{1/p}$, *there exists* $u_k \in A_k^+$ *such that* $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(F) *Assume that* $2 < p = \alpha < \beta$, *then for each integer* $k = 1, 2, \cdots,$

(i) *If* $k < (2K(p, \alpha, \beta))^{-1}$ *or* $k \geq (p^2/\lambda_1)^{1/p}$, $S_\lambda \cap A_k = \emptyset$.

(ii) *If* $(2K(p, \alpha, \beta))^{-1} \leq k < (p^2/\lambda_1)^{1/p}$, *there exists* $u_k \in A_k^+$ *such that* $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(G) *Assume that* $1 < p \leq 2$ *and* $p < \alpha < \beta$. *Then, there exists an increasing sequence* $(\mu_k)_{k\geq 1}$ *such that* $\lim_{k\to+\infty} \mu_k = 0$ *and*

(i) *If* $\lambda < \mu_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) *If* $\lambda = \mu_k$, *there exists* $u_k \in A_k^+$ *such that* $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.

(iii) *If* $\lambda \in (\mu_k, 0)$*, there exist* $u_k, v_k \in A_k^+$ *such that* $u_k \neq v_k$ *and* $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

(H) *Assume that one of the following conditions holds:*

(a) $2 < p < \alpha < \beta$ *or*

(b) $1 < \beta < \alpha < p$.

Then, there exist two increasing sequences $(\mu_k)_{k\geq 1}$ *and* $(\nu_k)_{k\geq 1}$ *such that for all* $k \geq 1$, $\mu_k < \nu_k < 0$, $\lim_{k\to+\infty} \mu_k = 0$ *and*

If $\lambda < \mu_k$, $S_\lambda \cap A_k = \emptyset$.

If $\lambda = \mu_k$*, there exists* $u_k \in A_k^+$ *such that* $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.

If $\lambda \in (\mu_k, \nu_k]$*, there exist* $u_k, v_k \in A_k^+$ *such that* $u_k \neq v_k$ *and* $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

If $\lambda \in (\nu_k, 0)$*, there exists* $u_k \in A_k^+$ *such that* $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.

3 Quadrature method

To obtain our main results, we use the well-known quadrature method (see for instance, [1]-[5], [12]). In order to keep this article self-contained we describe it here for the particular odd case. For this, some notations are needed. Assume that $g \in C(\mathbb{R}, \mathbb{R})$, g is odd, and $p > 1$. Consider the boundary-value problem

$$-\left(\varphi_p\left(u'\right)\right)' = g\left(u\right) \quad \text{in } (0,1)\,;\; u(0) = u(1) = 0. \tag{3.1}$$

Denote by $p' = p/(p-1)$ the conjugate exponent of p. $(1/p + 1/p' = 1)$. Define $G(s) = \int_0^s g(t)\,dt$. For any $E > 0$, let,

$$X(E) = \{s > 0 : E^p - p'G(\xi) > 0, \forall \xi, 0 < \xi < s\},$$

and

$$r(E) = \begin{cases} 0 & \text{if } X(E) = \emptyset \\ \sup X(E) & \text{otherwise.} \end{cases}$$

Let

$$\widetilde{D} = \{E \in (0, +\infty) : 0 < r(E) < +\infty \text{ and } \int_0^{r(E)} (E^p - p'G(t))^{-1/p} dt < +\infty\},$$

and define the time-map,

$$T(E) = \int_0^{r(E)} (E^p - p'G(t))^{-1/p} dt,\ E \in \widetilde{D}.$$

Due to the oddness of g, it follows that u is a solution to (3.1) in A_k^+ if and only if $(-u)$ is also a solution to (3.1) in A_k^-.

In our odd case, time maps approach reads as follows. Let $E > 0$, $k \in \mathbb{N}^*$. Then, problem (3.1) admits a solution $u \in A_k^+$ satisfying $u'(0) = E$ if and only if $E \in \widetilde{D}$ and $kT(E) = 1/2$, and in this case the solution is unique[1].

Remark 3.1 *In practice, we first study the variations of the real-valued function $s \mapsto E^p - p'G(s)$, then compute $X(E)$ and deduce $r(E)$. Next, we compute $\widetilde{D}$. To this end, we first compute the set*

$$D := \{E > 0 : 0 < r(E) < +\infty \text{ and } g(r(E)) > 0\}$$

and then we deduce $\widetilde{D}$, by observing that: $D \subset \widetilde{D} \subset \overline{D}$. After that, we define the time map on $\widetilde{D}$ and then compute its limits at the boundary points of $\widetilde{D}$. Next, we study (when possible) the variations of T on $\widetilde{D}$. We achieve our study by discussing the number of solutions to equations $kT(E) = 1/2$, for k being an integer and $E \in \widetilde{D}$.

4 Proof of Theorem 2.1

This section is organized as follows. We begin by some lemmas in the first subsection. The first lemma (Lemma 4.1) is used in order to define the time map, while in Lemma 4.2 we compute the limits and in Lemma 4.3 we study the variations of the time map. Next we dedicate a separate subsection to the proof of each assertion of Theorem 2.1.

Preliminary lemmas

According to the practical remark (Remark 3.1), we begin by the following technical Lemma.

Lemma 4.1 *Consider the function defined on $\mathbb{R}^+$ by,*

$$s \longmapsto G(\lambda, E, s) := E^p - p'F(\lambda, s),$$

where $p, \alpha, \beta > 1, E > 0$ and $\lambda > 0$ are real parameters, and

$$F(\lambda, s) := \int_0^s f(\lambda, t)dt = \frac{1}{\alpha}s^\alpha + \frac{\lambda}{\beta}s^\beta, s \geq 0.$$

For all $\lambda > 0$ and $E > 0$ there exists a unique $s(\lambda, E) > 0$ such that the function $G(\lambda, E, \cdot)$ is strictly positive on $(0, s(\lambda, E))$, vanishes at $s(\lambda, E)$ and is strictly negative on $(s(\lambda, E), +\infty)$. Moreover,

(i) *The function $E \mapsto s(\lambda, E)$ is C^1 on $(0, +\infty)$, and*

$$\frac{\partial s}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{f(\lambda, s(\lambda, E))} > 0, \text{ for all } E > 0 \text{ and all } \lambda > 0.$$

(ii) $\lim_{E \to 0^+} s(\lambda, E) = 0$, $\lim_{E \to +\infty} s(\lambda, E) = +\infty$.

[1]This uniqueness means that if v is also a solution to (3.1) in A_k^+ and satisfying $v'(0) = E$ then $v \equiv u$.

Proof. For any fixed $p > 1$, $E > 0$ and $\lambda > 0$, consider the function

$$s \longmapsto G(\lambda, E, s) := E^p - p' F(\lambda, s),$$

defined on $[0, +\infty)$. One has

$$\frac{\partial G}{\partial s}(\lambda, E, s) = -p' f(\lambda, s).$$

Notice that,

$$f(\lambda, s) = s^{\alpha-1} + \lambda s^{\beta-1} > 0, \quad \text{for all } s > 0 \text{ and all } \lambda \geq 0. \tag{4.1}$$

Thus, the function $G(\lambda, E, \cdot)$ is strictly decreasing on $(0, +\infty)$. On the other hand, one has

$$G(\lambda, E, 0) = E^p > 0 \quad \text{ and } \quad \lim_{s \to +\infty} G(\lambda, E, s) = -\infty.$$

Therefore, $G(\lambda, E, \cdot)$ admits a unique positive zero, denoted by $s(\lambda, E)$, and it is strictly positive on $(0, s(\lambda, E))$ and is strictly negative on $(s(\lambda, E), +\infty)$.
Proof of (i). For any $p > 1$ and $\lambda \geq 0$, consider the real-valued function,

$$(E, s) \mapsto G(E, s) := E^p - p' F(\lambda, s),$$

defined on $\Omega = (0, +\infty)^2$. One has $G \in C^1(\Omega)$ and,

$$\frac{\partial G}{\partial s}(E, s) = -p' f(\lambda, s) \quad \text{in } \Omega,$$

hence, according to (4.1), it follows that

$$\frac{\partial G}{\partial s}(E, s) < 0, \quad \text{in } \Omega.$$

Observe that for all $E > 0$ and $\lambda \geq 0$, the couple $(E, s(\lambda, E))$ belongs to Ω and one has

$$G(E, s(\lambda, E)) = 0. \tag{4.2}$$

Thus, one can make use of the implicit function theorem to show that the function $E \mapsto s(\lambda, E)$ is $C^1(\mathbb{R}^+, \mathbb{R}^+)$ and to obtain the expression of $(\partial s/\partial E)(\lambda, E)$ given in **(i)**. Its sign is given by (4.1) and the fact that $s(\lambda, E) > 0$ for all $\lambda \geq 0$ and $E > 0$. Therefore, Assertion **(i)** is proved.
Proof of (ii). For any fixed $p > 1$ and $\lambda \geq 0$, Assertion **(i)** of the current lemma implies that the function defined on $(0, +\infty)$ by $E \mapsto s(\lambda, E)$ is strictly increasing. It is bounded from below by 0 and from above by $+\infty$. Thus, the limits $\lim_{E \to 0^+} s(\lambda, E) = l_0$ and $\lim_{E \to +\infty} s(\lambda, E) = l_+$ exist and satisfy

$$0 \leq l_0 < l_+ \leq +\infty.$$

Let us observe that for any fixed $p > 1$ and $\lambda \geq 0$, the function $(E, s) \mapsto G(E, s)$ is continuous on $[0, +\infty)^2$ and the function $E \mapsto s(\lambda, E)$ is continuous on $[0, +\infty)$ and satisfies (4.2). Thus, by passing to the limit in (4.2) as E tends to 0^+, one gets

$$0 = \lim_{E \to 0^+} G(E, s(\lambda, E)) = G(0, l_0).$$

Hence, l_0 is a zero, belonging to $[0, +\infty)$, to the equation in the variable s : $G(0, s) = 0$. By solving this equation one gets $l_0 = 0$.

Assume that $l_+ < +\infty$, then by passing to the limit in (4.2) as E tends to $+\infty$ one gets

$$+\infty = p' F(\lambda, l_+) < +\infty,$$

which is impossible. So, $l_+ = +\infty$. Therefore, Lemma 4.1 is proved. $\diamondsuit$

Now, for any $p > 1$, $\alpha, \beta > 1$, $\lambda > 0$ and $E > 0$, we compute $X(\lambda, E)$ as defined in Section 3. In fact, for all $E > 0$, $X(\lambda, E) = (0, s(\lambda, E))$, where $s(\lambda, E)$ is defined in Lemma 4.1. Then, $r(\lambda, E) = s(\lambda, E)$ for all $\lambda > 0$ and $E > 0$. Hence, for any $p > 1$, $\lambda > 0$,

$$0 < r(\lambda, E) < +\infty \text{ if and only if } E > 0.$$

Also, for all $E > 0$,

$$f(\lambda, r(\lambda, E)) = \varphi_\alpha(r(\lambda, E)) + \frac{\lambda}{\beta} \varphi_\beta(r(\lambda, E)) > 0.$$

So, $D(\lambda) = (0, +\infty)$ for all $\lambda > 0$. Therefore, $\widetilde{D}(\lambda) = (0, +\infty)$ for all $\lambda > 0$.

Before going further in the investigation, from Lemma 4.1, we deduce that for any fixed $p > 1$ and $\lambda > 0$

$$E^p = p' F(\lambda, r(\lambda, E)), \text{for all } E > 0 \tag{4.3}$$

$$\frac{\partial r}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{f(\lambda, r(\lambda, E))} > 0, \text{ for all } E > 0 \text{ and all } \lambda > 0. \tag{4.4}$$

$$\lim_{E \to 0^+} r(\lambda, E) = 0, \lim_{E \to +\infty} r(\lambda, E) = +\infty. \tag{4.5}$$

At present we define, for any $p > 1, \lambda > 0$ and $E > 0$, the time map

$$T(\lambda, E) = \int_0^{r(\lambda, E)} \{E^p - p' F(\lambda, \xi)\}^{-1/p} d\xi, E > 0.$$

By (4.3), it follows that

$$T(\lambda, E) = (p')^{-1/p} \int_0^{r(\lambda, E)} \{F(\lambda, r(\lambda, E)) - F(\lambda, \xi)\}^{-1/p} d\xi.$$

Furthermore, a simple change of variables and a substitution yield

$$T(\lambda,E)=(p')^{-1/p}\int_0^1\{\lambda r^{\beta-p}(\lambda,E)(\frac{1-t^\beta}{\beta})+r^{\alpha-p}(\lambda,E)(\frac{1-t^\alpha}{\alpha})\}^{-1/p}dt.$$

One may observe that

$$T(\lambda,E)=S(\lambda,r(\lambda,E)),\text{for all } \lambda>0,\ E>0,$$

where

$$S(\lambda,\rho)=(p')^{-1/p}\int_0^1\{\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta})+\rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha})\}^{-1/p}dt,$$

for all $\lambda>0$ and $\rho>0$.

Beacuase the function $E\mapsto r(\lambda,E)$ is an increasing C^1-diffeomorphism from $(0,+\infty)$ onto itself it follows that if we put, for all $\lambda>0$,

$$J_1(\lambda):=\{E\in(0,+\infty):T(\lambda,E)=1/2\},$$

and

$$J_2(\lambda):=\{\rho\in(0,+\infty):S(\lambda,\rho)=1/2\},$$

then

$$Card(J_1(\lambda))=Card(J_2(\lambda)),\ \text{for all } \lambda>0.$$

Hence, from now on, we will focus our attention in counting the number of solution(s) of the equation $S(\lambda,\rho)=1/2$ in the variable $\rho\in(0,+\infty)$, instead of the equation $T(\lambda,E)=1/2$ in the variable $E\in(0,+\infty)$. In the next lemma we shall compute the limits of $S(\lambda,\cdot)$ when $\lambda>0$.

Lemma 4.2 *For all $\lambda>0$, $p>1$,*

$$\lim_{\rho\to0^+}S(\lambda,\rho)=0,\quad \textit{if } \alpha<p$$

$$\lim_{\rho\to0^+}S(\lambda,\rho)=\begin{cases}\frac12\lambda_1^{1/p} & \textit{if } \alpha=p \textit{ and } \beta>p\\ 0 & \textit{if } \alpha=p \textit{ and } \beta<p\end{cases}$$

$$\lim_{\rho\to0^+}S(\lambda,\rho)=\begin{cases}+\infty & \textit{if } \alpha>p \textit{ and } \beta>p\\ \frac12(\frac{\lambda_1}{\lambda})^{1/p} & \textit{if } \alpha>p \textit{ and } \beta=p\\ 0 & \textit{if } \alpha>p \textit{ and } \beta<p\end{cases}$$

$$\lim_{\rho\to+\infty}S(\lambda,\rho)=\begin{cases}0 & \textit{if } \alpha<p \textit{ and } \beta>p\\ \frac12(\frac{\lambda_1}{\lambda})^{1/p} & \textit{if } \alpha<p \textit{ and } \beta=p\\ +\infty & \textit{if } \alpha<p \textit{ and } \beta<p\end{cases}$$

$$\lim_{\rho\to+\infty} S(\lambda,\rho) = \begin{cases} 0 & \text{if } \alpha = p \text{ and } \beta > p \\ \frac{1}{2}\lambda_1^{1/p} & \text{if } \alpha = p \text{ and } \beta < p \end{cases}$$

$$\lim_{\rho\to+\infty} S(\lambda,\rho) = 0 \quad \text{if } \alpha > p\,.$$

The proof of this lemma consists of easy computations.

Lemma 4.3 *For all $p > 1$, $\alpha, \beta > 1$ and $\lambda > 0$,*

1. *$S(\lambda,\cdot)$ is strictly decreasing on $(0,+\infty)$ provided that*
$$(\alpha > p \text{ and } \beta \geq p) \text{ or } (\alpha = p \text{ and } \beta > p).$$
2. *$S(\lambda,\cdot)$ is strictly increasing on $(0,+\infty)$ provided that*
$$(\alpha < p \text{ and } \beta \leq p) \text{ or } (\alpha = p \text{ and } \beta < p).$$
3. *$S(\lambda,\cdot)$ is strictly increasing on $(0,\rho_1(\lambda))$ and is strictly decreasing on $(\rho_2(\lambda),+\infty)$, provided that*
$$(\alpha > p \text{ and } \beta < p) \text{ or } (\alpha < p \text{ and } \beta > p),$$
where
$$\rho_1(\lambda) := ((-\lambda)\frac{(p-\beta)}{(p-\alpha)})^{1/(\alpha-\beta)} < \rho_2(\lambda) := ((-\lambda)\frac{(p-\beta)\alpha}{(p-\alpha)\beta})^{1/(\alpha-\beta)}.$$

Proof. For all $p > 1$, $\alpha, \beta > 1$ and $\lambda > 0$, easy computation yields

$$\frac{\partial S}{\partial \rho}(\lambda,\rho) = (p')^{-1/p}\int_0^\rho \frac{H(\lambda,\rho)-H(\lambda,u)}{p\rho(F(\lambda,\rho)-F(\lambda,u))^{1+(1/p)}}du,$$

where

$$H(\lambda,\rho) = pF(\lambda,\rho) - \rho f(\lambda,\rho) = (\frac{p-\alpha}{\alpha})\rho^\alpha + \lambda(\frac{p-\beta}{\beta})\rho^\beta.$$

It follows that,

$$\frac{\partial H}{\partial \rho}(\lambda,\rho) = (p-\alpha)\rho^{\alpha-1} + \lambda(p-\beta)\rho^{\beta-1}.$$

Thus, if ($\alpha > p$ and $\beta \geq p$) or ($\alpha = p$ and $\beta > p$), $H(\lambda,\cdot)$ is strictly decreasing on $(0,+\infty)$ and then, for all $\lambda > 0$ and $\rho > 0$

$$H(\lambda,\rho) - H(\lambda,u) < 0, \text{ for all } u \in (0,\rho),$$

and therefore,

$$\frac{\partial S}{\partial \rho}(\lambda,\rho) < 0, \quad \text{for all } \rho > 0.$$

If ($\alpha < p$ and $\beta \leq p$) or ($\alpha = p$ and $\beta < p$), $H(\lambda, \cdot)$ is strictly increasing on $(0, +\infty)$ and then for all $\rho > 0$, $\lambda > 0$

$$H(\lambda, \rho) - H(\lambda, u) > 0, \quad \text{for all } u \in (0, \rho),$$

and therefore,

$$\frac{\partial S}{\partial \rho}(\lambda, \rho) > 0, \quad \text{for all } \rho > 0.$$

If ($\alpha > p$ and $\beta < p$) or ($\alpha < p$ and $\beta > p$), $H(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda))$ and strictly decreasing on $(\rho_1(\lambda), +\infty)$. Moreover, $H(\lambda, \cdot)$ is strictly positive on $(0, \rho_2(\lambda))$ and strictly negative on $(\rho_2(\lambda), +\infty)$ and vanishes at 0 and at $\rho_2(\lambda)$. Therefore, for all $\lambda > 0$ and $\rho \in (0, \rho_1(\lambda))$

$$H(\lambda, \rho) - H(\lambda, u) > 0 \quad \text{for all } u \in (0, \rho)$$

and for all $\rho \in (\rho_2(\lambda), +\infty)$,

$$H(\lambda, \rho) - H(\lambda, u) < 0 \quad \text{for all } u \in (0, \rho).$$

That is, $S(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda))$ and is strictly decreasing on $(\rho_2(\lambda), +\infty)$.

Proof of Assertion A

Case (a). Assume that $\alpha > p$ and $\beta > p$. By Lemma 4.2, it follows that for all $\lambda > 0$,

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = +\infty, \text{ and } \lim_{\rho \to +\infty} S(\lambda, \rho) = 0,$$

and by Lemma 4.3, $S(\lambda, \cdot)$ is strictly decreasing on $(0, +\infty)$. Thus, for each integer $k = 1, 2, \cdots$, the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits a unique solution for all $\lambda > 0$. Therefore, for each integer $k = 1, 2, \cdots$, problem (1.3) admits a unique pair of solutions $\{u_k, v_k\}$ in A_k, for all $\lambda > 0$. Moreover $v_k = -u_k$.

Case (b). Assume that $\alpha = p$ and $\beta > p$. By Lemma 4.2, it follows that for all $\lambda > 0$,

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = \frac{1}{2}\lambda_1^{1/p}(p), \quad \text{and} \quad \lim_{\rho \to +\infty} S(\lambda, \rho) = 0,$$

and by Lemma 4.3, $S(\lambda, \cdot)$ is strictly decreasing on $(0, +\infty)$. Thus, for each integer $k = 1, 2, \cdots$, the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits at least a solution in $(0, +\infty)$ if and only if $(k/2)\lambda_1^{1/p}(p) > 1/2$, that is, if and only if,

$$\lambda_1(p) > k^{-p}, \tag{4.6}$$

and in this case, the solution is unique. Notice that for each integer $k = 1, 2, \cdots$, $k^{-p} \leq 1$. So, (4.6) holds provided that

$$\lambda_1(p) > 1, \text{ for all } p > 1\,. \tag{4.7}$$

In the appendix, we shall prove that (4.7) holds. Therefore, for each integer $k = 1, 2, \cdots$, problem (1.3) admits a unique pair of solutions $\{u_k, v_k\}$ in A_k, for all $\lambda > 0$. Moreover, $v_k = -u_k$.

The proofs of **Cases (c)** and **(d)** are similar and then omitted. Therefore, Assertion A is proved.

Proof of Assertion B

Assume that $\alpha > p$ (resp. $\alpha < p$) and $\beta = p$. By Lemma 4.2, it follows that for all $\lambda > 0$,

$$\begin{gathered}\lim_{\rho\to 0} S(\lambda,\rho) = \frac{1}{2}(\frac{\lambda_1}{\lambda})^{1/p} \text{ (resp. =0), and}\\ \lim_{\rho\to+\infty} S(\lambda,\rho) = 0 \text{ (resp. } = \frac{1}{2}(\frac{\lambda_1}{\lambda})^{1/p}),\end{gathered}$$

and by Lemma 4.3, $S(\lambda, \cdot)$ is strictly decreasing (resp. increasing) on $(0, +\infty)$. Thus, for each integer $k = 1, 2, \cdots$, the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits at least a solution in $(0, +\infty)$ if and only if $(k/2)(\lambda_1/\lambda)^{1/p} > 1/2$, that is, if and only if $0 < \lambda < k^p\lambda_1(p) = \lambda_k(p)$, and in this case, the solution is unique. Therefore, for each integer $k = 1, 2, \cdots$, problem (1.3) admits a unique pair of solutions $\{u_k, v_k\}$ (resp. admits no solution) in A_k, for all λ satisfying $0 < \lambda < \lambda_k(p)$ (resp. $\lambda \geq \lambda_k(p)$). Moreover, $u_k = -v_k$.

Proof of Assertion C

Assume that $1 < \beta < p < \alpha$. By Lemma 4.2, it follows that for all $\lambda > 0$,

$$\lim_{\rho\to 0^+} S(\lambda,\rho) = \lim_{\rho\to+\infty} S(\lambda,\rho) = 0.$$

Thus, for all $\lambda > 0$, there exists a unique $M(\lambda) > 0$ such that $M(\lambda) = \sup_{\rho\geq 0} S(\lambda, \rho)$.

Lemma 4.4 *Assume that* $1 < \beta < p < \alpha$. *Then,*

(a) $M(\cdot)$ *is continuous on* $(0, +\infty)$.

(b) $M(\cdot)$ *is strictly decreasing on* $(0, +\infty)$.

(c) $\lim_{\lambda\to 0^+} M(\lambda) = +\infty$, *and* $\lim_{\lambda\to+\infty} M(\lambda) = 0$.

Proof. Recall that for all $\lambda > 0$ and $\rho > 0$

$$S(\lambda,\rho) = (p')^{-1/p}\int_0^1 (\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha}))^{-1/p}dt.$$

For all $\lambda > 0$ and $\rho > 0$, let $\bar\rho = \bar\rho(\lambda,\rho) := \lambda^{1/(\beta-\alpha)}\rho$. Then, $\rho = \lambda^{1/(\alpha-\beta)}\bar\rho$ and a simple substitution yields:

$$S(\lambda,\rho) = \lambda^{(p-\alpha)/p(\alpha-\beta)}S(1,\bar\rho(\lambda,\rho)).$$

Thus,

$$M(\lambda) = \lambda^{(p-\alpha)/p(\alpha-\beta)}\sup_{\rho\geq 0} S(1,\bar\rho(\lambda,\rho)) = \lambda^{(p-\alpha)/p(\alpha-\beta)}\sup_{\bar\rho\geq 0} S(1,\bar\rho).$$

Therefore,

$$M(\lambda) = \lambda^{(p-\alpha)/p(\alpha-\beta)}M(1), \text{ for all } \lambda > 0. \tag{4.8}$$

Assertions **(a), (b)** and **(c)** are simple consequences of formula (4.8). Therefore, Lemma 4.4 is proved. ♢

By Lemma 4.4 (or by formula (4.8)), it follows that the function $M(\cdot)$ admits an inverse function $M^{-1}(\cdot)$ defined and strictly decreasing on $(0,+\infty)$ and satisfies:

$$\lim_{y\to 0} M^{-1}(y) = +\infty, \text{ and } \lim_{y\to+\infty} M^{-1}(y) = 0\,.$$

Therefore, for each integer $k = 1, 2, \cdots$, we define $L_k := M^{-1}(1/2k)$. Thus, $(L_k)_{k\geq 1}$ is a strictly increasing sequence and satisfies $\lim_{k\to+\infty} L_k = +\infty$, and

- $kM(\lambda) < 1/2$, for all $\lambda > L_k$,
- $kM(\lambda) = 1/2$, for all $\lambda = L_k$
- $kM(\lambda) > 1/2$, for all $\lambda \in (0, L_k)$.

Therefore, the equation $kS(\lambda,\rho) = 1/2$, in the variable $\rho > 0$,

- admits no solution for all $\lambda > L_k$,
- admits at least a solution for $\lambda = L_k$,
- admits at least two solutions for all $\lambda \in (0, L_k)$.

Thus, for each integer $k = 1, 2, \cdots$, problem (1.3),

- admits no solution in A_k, for all $\lambda > L_k$,
- admits at least one pair of solutions $\{u_k, v_k\}$ in A_k, for $\lambda = L_k$. Moreover, $u_k = -v_k$.

- admits at least two pairs of solutions $\{u_k, U_k\} \cup \{v_k, V_k\}$ in A_k, for all $\lambda \in (0, L_k)$. Moreover, $U_k = -u_k$ and $V_k = -v_k$.

At present, let us prove that for each integer $k = 1, 2, \cdots$, there exists $\varepsilon_k \in (0, L_k)$ such that the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits exactly two solutions for all $\lambda \in (0, \varepsilon_k)$. To this end, it suffices to prove that for each integer $k = 1, 2, \cdots$, there exists $\varepsilon_k \in (0, L_k)$ such that for all $\lambda \in (0, \varepsilon_k)$:

$$kS(\lambda, \rho) > 1/2, \text{ for all } \rho \in [\rho_1(\lambda), \rho_2(\lambda)], \tag{4.9}$$

where $\rho_i(\lambda)$, $i = 1, 2$, are defined in Lemma 4.3. In fact, assume that for all $\lambda \in (0, \varepsilon_k)$, (4.9) holds, then $kS(\lambda, \rho_1(\lambda)) > 1/2$ for all $\lambda > 0$, and by Lemma 4.2, $\lim_{\rho\to 0} S(\lambda, \rho) = 0$, and by Lemma 4.3, $kS(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda)]$. Thus, for all $\lambda > 0$ there is a unique solution in $(0, \rho_1(\lambda))$ to the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$. Also, by (4.9) it follows that $kS(\lambda, \rho_2(\lambda)) > 1/2$ for all $\lambda > 0$, and by Lemma 4.2, $\lim_{\rho\to+\infty} kS(\lambda, \rho) = 0$, and by Lemma 4.3, $kS(\lambda, \cdot)$ is strictly decreasing on $[\rho_2(\lambda), +\infty)$. Thus, for all $\lambda > 0$ there is a unique solution in $(\rho_2(\lambda), +\infty)$ to the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$. On the other hand, (4.9) implies that $kS(\lambda, \rho) \neq 1/2$, for all $\lambda > 0$ and all $\rho \in [\rho_1(\lambda), \rho_2(\lambda)]$. Thus, there is no solution in $[\rho_1(\lambda), \rho_2(\lambda)]$ to the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$.

Now, let us prove that for each integer $k = 1, 2, \cdots$, there exists $\varepsilon_k \in (0, L_k)$ such that for all $\lambda > 0$ (4.9) holds. Simple computation shows that for all $\lambda > 0$

$$\begin{aligned} S(\lambda, \rho_1(\lambda)) &= (p')^{-1/p} \lambda^{(p-\alpha)/p(\alpha-\beta)} \int_0^1 \{ (\frac{p-\beta}{\alpha-p})^{(\beta-p)/(\alpha-\beta)} (\frac{1-t^\beta}{\beta}) \\ &\quad + (\frac{p-\beta}{\alpha-p})^{(\alpha-p)/(\alpha-\beta)} (\frac{1-t^\alpha}{\alpha}) \}^{-1/p} dt. \end{aligned}$$

It follows that the function $\lambda \mapsto S(\lambda, \rho_1(\lambda))$ is continuous and strictly decreasing on $(0, +\infty)$, and

$$\lim_{\lambda\to 0} S(\lambda, \rho_1(\lambda)) = +\infty, \quad \text{and} \quad \lim_{\lambda\to+\infty} S(\lambda, \rho_1(\lambda)) = 0\,.$$

Thus, for each integer $k = 1, 2, \cdots$, there exists a unique $\mu_k > 0$ such that

$$kS(\mu_k, \rho_1(\mu_k)) = \frac{1}{2}\,.$$

Furthermore, the sequence $(\mu_k)_{k\geq 1}$ is strictly increasing and $\lim_{k\to+\infty} \mu_k = +\infty$. It is easy to prove that for each integer $k = 1, 2, \cdots$

$$\mu_k \leq L_k. \tag{4.10}$$

In fact, if the contrary holds, using the fact that the function $\lambda \mapsto kS(\lambda, \rho_1(\lambda))$ is strictly decreasing on $(0, +\infty)$, it follows that

$$\frac{1}{2} = kS(\mu_k, \rho_1(\mu_k)) < kS(L_k, \rho_1(L_k))\,.$$

But, for each integer $k = 1, 2, \cdots$,

$$kS(L_k, \rho_1(L_k)) \le k \sup_{\rho \ge 0} S(L_k, \rho) = kM(L_k) = \frac{1}{2}\,,$$

a contradiction which proves (4.10).

On the other hand, the function $\lambda \mapsto \rho_2(\lambda)$ is continuous and strictly increasing on $(0, +\infty)$ and

$$\lim_{\lambda \to 0} \rho_2(\lambda) = 0, \text{ and } \lim_{\lambda \to +\infty} \rho_2(\lambda) = +\infty,$$

then, for each integer $k = 1, 2, \cdots$, there exists a unique $\varepsilon_k > 0$ such that

$$\rho_2(\varepsilon_k) = \rho_1(\mu_k)\,.$$

Using the fact that ρ_1, ρ_2 and $(\mu_k)_{k \ge 1}$ are strictly increasing it follows that $(\varepsilon_k)_{k \ge 1}$ is also strictly increasing. Also, using the fact that

$$\lim_{\mu \to +\infty} \rho_1(\mu) = \lim_{\mu \to +\infty} \rho_2(\mu) = \lim_{k \to +\infty} \mu_k = +\infty,$$

it follows that: $\lim_{k \to +\infty} \varepsilon_k = +\infty$.

Furthermore, notice that using the fact that ρ_1 is strictly increasing on $(0, +\infty)$ and $\rho_1 < \rho_2$ on $(0, +\infty)$, it follows that for each integer $k = 1, 2, \cdots$, $\varepsilon_k \in (0, \mu_k)$, and by (4.10), it follows that

$$0 < \varepsilon_k < L_k, \text{ for each integer } k = 1, 2, \cdots \tag{4.11}$$

Now, we believe that for each integer $k = 1, 2, \cdots$, and all $\lambda \in (0, \varepsilon_k)$, (4.9) holds. In fact, let $k = 1, 2, \cdots$, be fixed and $\lambda_0 \in (0, \varepsilon_k)$ and $\bar\rho \in [\rho_1(\lambda_0), \rho_2(\lambda_0)]$.

The variations of ρ_1 and ρ_2 and the fact that $\rho_1 < \rho_2$ on $(0, +\infty)$ imply that there exists $\bar\lambda \in [\lambda_0, \mu_k)$ such that $\rho_1(\bar\lambda) = \bar\rho$. Thus, $kS(\lambda_0, \bar\rho) = kS(\lambda_0, \rho_1(\bar\lambda))$. But $S(\cdot, \rho)$ is decreasing on $(0, +\infty)$. Thus, $kS(\lambda_0, \rho_1(\bar\lambda)) \ge kS(\bar\lambda, \rho_1(\bar\lambda))$.

On the other hand, the function $\lambda \mapsto S(\lambda, \rho_1(\lambda))$ is strictly decreasing on $(0, +\infty)$, thus,

$$kS(\bar\lambda, \rho_1(\bar\lambda)) > kS(\mu_k, \rho_1(\mu_k))\,.$$

Therefore,

$$kS(\lambda_0, \bar\rho) = kS(\lambda_0, \rho_1(\bar\lambda)) \ge kS(\bar\lambda, \rho_1(\bar\lambda)) > kS(\mu_k, \rho_1(\mu_k)) = \frac{1}{2},$$

and hence, (4.9) holds. Therefore, a p-laplacian version of Villegas result can be stated at this point. In fact; we can state that if $1 < \beta < p < \alpha$. Then, for each integer $k = 1, 2, \cdots$, there exist two real numbers ε_k and L_k such that $\varepsilon_k \le L_k$ and

(i) If $\lambda > L_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $\lambda = L_k$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) \supset \{u_k, -u_k\}$.

(iii) If $\lambda \in [\varepsilon_k, L_k)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \neq v_k$ and $(S_\lambda \cap A_k) \supset \{u_k, v_k, -u_k, -v_k\}$.

(iv) If $\lambda \in (0, \varepsilon_k)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \neq v_k$ and $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

Let us summarize. At this point, we have shown that when $1 < \beta < p < \alpha$, then for all $\lambda > 0$,

$$\lim_{\rho\to 0} S(\lambda,\rho) = \lim_{\rho\to+\infty} S(\lambda,\rho) = 0,$$

hence for all $\lambda > 0$, $S(\lambda,\cdot)$ admits at least a critical point; a maximum in $(0,+\infty)$. Next, it was proved that for all $\lambda > 0$, there exist $\rho_1(\lambda)$ and $\rho_2(\lambda)$ such that $0 < \rho_1(\lambda) < \rho_2(\lambda)$ and $\frac{\partial S}{\partial\rho}(\lambda,\rho) > 0$ on $(0,\rho_1(\lambda)]$ and $\frac{\partial S}{\partial\rho}(\lambda,\rho) < 0$ on $[\rho_2(\lambda),+\infty)$. So, the critical point belongs necessarily to $(\rho_1(\lambda),\rho_2(\lambda))$. Also, it was proved that the function $\lambda \mapsto M(\lambda) := \sup_{0<\rho<+\infty} S(\lambda,\rho)$, is continuous, strictly decreasing on $(0,+\infty)$ and

$$\lim_{\lambda\to 0^+} M(\lambda) = +\infty, \text{ and } \lim_{\lambda\to+\infty} M(\lambda) = 0\,.$$

Thus, to complete the proof of Assertion (**C**), it remains to prove that for all $\lambda > 0$, $S(\lambda,\cdot)$ admits *at most one* critical point in $(\rho_1(\lambda),\rho_2(\lambda))$. To this end we shall prove that $S(\lambda,\cdot)$ is concave on $(\rho_1(\lambda),\rho_2(\lambda))$ for all $\lambda > 0$. Similar idea was previously used by the authors Addou and Benmezaï in [3, Lemma 7, (iii)].

The derivative of $S(\lambda,\cdot)$ is given by

$$\frac{\partial S}{\partial\rho}(\lambda,\rho) = (p')^{-1/p}\int_0^1 \frac{H(\lambda,\rho)-H(\lambda,u)}{p\rho(F(\lambda,\rho)-F(\lambda,u))^{(p+1)/p}}du$$

where $F(\lambda,\rho) = \int_0^\rho f(\lambda,t)dt = \frac{1}{\alpha}u^\alpha + \frac{\lambda}{\beta}u^\beta$, and

$$H(\lambda,\rho) = pF(\lambda,\rho) - \rho f(\lambda,\rho) = (\frac{p-\alpha}{\alpha})\rho^\alpha + \lambda(\frac{p-\beta}{\beta})\rho^\beta.$$

Easy computations show that for all $\rho > 0$ and $\lambda > 0$

$$\begin{aligned}(p')^{1/p}\frac{\partial^2 S}{\partial\rho^2}(\lambda,\rho) &= \int_0^1 \frac{(p+1)(H(\lambda,\rho)-H(\lambda,\rho\xi))^2}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho\xi))^{(2p+1)/p}}d\xi \\ &\quad + \int_0^1 \frac{p(\Psi(\lambda,\rho)-\Psi(\lambda,\rho\xi))(F(\lambda,\rho)-F(\lambda,\rho\xi))}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho\xi))^{(2p+1)/p}}d\xi,\end{aligned}$$

where

$$\begin{aligned}\Psi(\lambda,\rho) &= -p(p+1)F(\lambda,\rho) + 2p\rho f(\lambda,\rho) - \rho^2 f'_\rho(\lambda,\rho) \\ &= \frac{(p-\alpha)(\alpha-(p+1))}{\alpha}\rho^a + \lambda\frac{(p-\beta)(\beta-(p+1))}{\beta}\rho^\beta\,.\end{aligned}$$

Some substitutions yield

$$(p')^{1/p}\frac{\partial^2 S}{\partial\rho^2}(\lambda,\rho)=\int_0^1\frac{(1-\xi^\beta)^2P(X(\xi))}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho\xi))^{(2p+1)/p}}d\xi,$$

where $X(1)=\frac{\alpha}{\beta}$ and $X(\xi)=\frac{1-\xi^\alpha}{1-\xi^\beta}$ if $\xi\in[0,1)$, and P is the second degree polynomial function

$$\begin{aligned}P(X) &= \frac{(\alpha-p)\rho^{2\alpha}}{\alpha}X^2-(\frac{p(\alpha-\beta)^2+p(\alpha+\beta)-2\alpha\beta}{\alpha\beta})\lambda\rho^{\alpha+\beta}X\\ &\quad+\frac{(\beta-p)}{\beta}\lambda^2\rho^{2\beta}\,.\end{aligned}$$

It can easily be verified that $X(\xi)\in[1,\alpha/\beta]$ for all $\xi\in[0,1]$. In fact, $X(0)=1$ and $\lim_{\xi\to1}X(\xi)=\alpha/\beta$ and X is strictly increasing on $(0,1)$ since, $X'(\xi)=\xi^{\beta-1}(1-\xi^\beta)^{-2}(\beta-\alpha\xi^{\alpha-\beta}+(\alpha-\beta)\xi^\alpha)$ and $\beta-\alpha\xi^{\alpha-\beta}>\beta-\alpha>0$, for all $\xi\in(0,1)$. Thus $X'(\xi)>0$ for all $\xi\in(0,1)$.

Therefore, we are interested in the sign of $P(X)$ when $X\in[1,\alpha/\beta]$. Its discriminant is

$$d=(Y^2-4Z)\frac{\lambda^2\rho^{2(\alpha+\beta)}}{(\alpha\beta)^2}\,,$$

where

$$\begin{aligned}Y &= p(\alpha-\beta)^2+p(\alpha+\beta)-2\alpha\beta \qquad (4.12)\\ &= p\alpha^2+(-2(p+1)\beta+p)\alpha+p\beta(\beta+1)\\ Z &= \alpha\beta(\alpha-p)(\beta-p)\,.\end{aligned}$$

Notice that by our hypothesis $1<\beta<p<\alpha$, it follows that $Z<0$, so that $d>0$. The roots of P are given by

$$X_1(\lambda,\rho)=\lambda\frac{Y-\sqrt{Y^2-4Z}}{2\beta(\alpha-p)\rho^{\alpha-\beta}}\le0,\ \text{and}\ X_2(\lambda,\rho)=\lambda\frac{Y+\sqrt{Y^2-4Z}}{2\beta(\alpha-p)\rho^{\alpha-\beta}}\,.$$

Notice that the function $\rho\to X_2(\lambda,\rho)$ is strictly decreasing on $(0,+\infty)$. So, it would be perfect if

$$X_2(\lambda,\rho_2(\lambda))>\frac{\alpha}{\beta}\quad\text{for all }\lambda>0\,. \qquad (4.13)$$

In fact, it follows therefore that,

$$[1,\frac{\alpha}{\beta}]\subset(X_1(\lambda,\rho),X_2(\lambda,\rho)),\ \forall\lambda>0,\forall\rho\in(\rho_1(\lambda),\rho_2(\lambda)).$$

Hence, $P(X(\xi))<0$, for all $\xi\in[0,1]$, so,

$$\frac{\partial^2 S}{\partial\rho^2}(\lambda,\rho)<0,\forall\lambda>0,\forall\rho\in(\rho_1(\lambda),\rho_2(\lambda)),$$

which will prove the uniqueness of the critical point of $S(\lambda,\cdot)$, and therefore, Theorem 2.1.

Let us prove the estimates (4.13). Notice that

$$X_2(\lambda,\rho_2(\lambda)) = \frac{Y+\sqrt{Y^2-4Z}}{2\alpha(p-\beta)} > \frac{\alpha}{\beta} \Longleftrightarrow \beta\sqrt{Y^2-4Z} > 2\alpha^2(p-\beta)-\beta Y\,.$$

By taking the square of each member side we get

$$X_2(\lambda,\rho_2(\lambda)) > \frac{\alpha}{\beta} \Longleftrightarrow 4\alpha^2\beta(p-\beta)Y-4\beta^2Z-4\alpha^4(p-\beta)^2>0\,.$$

Next, we substitute Y and Z as in (4.12) we get

$$X_2(\lambda,\rho_2(\lambda)) > \frac{\alpha}{\beta} \Longleftrightarrow 4\alpha(p-\beta)Q_{\beta,p}(\alpha)>0,$$

where $Q_{\beta,p}(\alpha)$ is the third degree polynomial function defined by

$$Q_{\beta,p}(\alpha)=\alpha^3(\beta(p+1)-p)+\alpha^2\beta(-2(p+1)\beta+p)+\alpha\beta^2(\beta(p+1)+p)-p\beta^3\,.$$

It remains to show that $Q_{\beta,p}(\alpha)>0$ for all $1<\beta<p<\alpha$. One can remark that β is a root of $Q_{\beta,p}(\cdot)$ of multiplicity at least twice. That is to say, $Q_{\beta,p}(\beta)=Q'_{\beta,p}(\beta)=0$. Thus there exist two constants A and B such that $Q_{\beta,p}(\alpha)=(\alpha-\beta)^2(A\alpha+B)$. Immediate identification yields: $A=\beta(p+1)-p$ and $B=-p\beta$. It remains to prove that $1<\beta<p<\alpha \Longrightarrow A\alpha+B>0$. Notice that $1<\beta<p<\alpha$ implies that $(\beta-1)p+\beta>\beta$ which implies that $\beta((\beta-1)p+\beta)^{-1}<1$ and then $-BA^{-1}<p$ and by $1<\beta<p<\alpha$ it follows that $A\alpha+B>0$, which completes the proof of (4.13). Therefore, Theorem 2.1 is proved.

5 Proof of Theorem 2.2

This section is organized as the previous one.

Preliminary lemmas

Lemma 5.1 *Consider the function defined on* $\mathbb{R}^+$ *by*

$$s \longmapsto N(\lambda,E,s) := E^p - p'F(\lambda,s),$$

where $p,\alpha,\beta>1$, $\lambda<0$ *and* $E\in(0,+\infty)$, *are real parameters and*

$$F(\lambda,s)=\int_0^s f(\lambda,t)dt=\frac{1}{\alpha}s^\alpha+\frac{1}{\beta}s^\beta,\ \ s\geq 0\,.$$

Assume that $(\alpha-\beta)>0$, *then for all* $\lambda<0$ *and* $E>0$, *the function* $N(\lambda,E,\cdot)$ *admits a unique positive zero,* $s(\lambda,E)$, *and is strictly positive on* $(0,s(\lambda,E))$.

Assume that $(\alpha-\beta)<0$, *then*

(a) *If* $E > E_* := (p'(-\lambda)^{\alpha/(\alpha-\beta)}(\frac{\beta-\alpha}{\alpha\beta}))^{1/p}$, $N(\lambda,E,\cdot)$ *is strictly positive on* $(0,+\infty)$.

(b) *If* $E = E_*$, $N(\lambda,E,\cdot)$ *vanishes at* $s = (-\lambda)^{1/(\alpha-\beta)}$ *and is strictly positive on* $(0,(-\lambda)^{1/(\alpha-\beta)}) \cup ((-\lambda)^{1/(\alpha-\beta)},+\infty)$.

(c) *If* $0 < E < E_*$, $N(\lambda,E,\cdot)$ *admits a first positive zero,* $s(\lambda,E) > 0$*, and is strictly positive on* $(0,s(\lambda,E))$.

Moreover, if $(\alpha-\beta) > 0$ *(resp.* $(\alpha-\beta) < 0$*),*

(i) *the function* $E \mapsto s(\lambda,E)$ *is* C^1 *on* $(0,+\infty)$ *(resp.* $(0,E_*(\lambda))$*) and*

$$\frac{\partial s}{\partial E}(\lambda,E) = \frac{(p-1)E^{p-1}}{f(\lambda,s(\lambda,E))} > 0, \text{ for all } \lambda < 0 \text{ and } E > 0$$

(resp. $E \in (0,E_*(\lambda))$*).*

(ii) $\lim_{E\to 0} s(\lambda,E) = ((-\lambda)\frac{\alpha}{\beta})^{1/(\alpha-\beta)}$, *(resp.* $\lim_{E\to 0} s(\lambda,E) = 0$*).*

(iii) $\lim_{E\to+\infty} s(\lambda,E) = +\infty$, *(resp.* $\lim_{E\to E_*} s(\lambda,E) = (-\lambda)^{1/(\alpha-\beta)}$*).*

The proof is the same as that of Lemma 4.1 and then it is omitted.

Assume that $\alpha-\beta > 0$ (resp. $\alpha-\beta < 0$) then for any $p > 1$ and $E \in (0,+\infty)$ (resp. $E \in (0,E_*(\lambda))$ we compute $X(\lambda,E)$ as defined in Section 3. We derive from Lemma 5.1, for the case where $\alpha-\beta > 0$, $X(\lambda,E) = (0,s(\lambda,E))$ and for the case where $\alpha-\beta < 0$

$$X(\lambda,E) = \begin{cases} (0,+\infty) & \text{if } E > E_*(\lambda) \\ (0,(-\lambda)^{1/(\alpha-\beta)}) & \text{if } E = E_*(\lambda) \\ (0,s(\lambda,E)) & \text{if } 0 < E < E_*(\lambda), \end{cases}$$

where $s(\lambda,E)$ is defined in Lemma 5.1. Then, for $\alpha-\beta > 0$

$$r(\lambda,E) := \sup X(\lambda,E) = s(\lambda,E), \quad \text{for all } \lambda < 0 \text{ and } E > 0$$

and for $\alpha-\beta < 0$,

$$r(\lambda,E) = \begin{cases} +\infty & \text{if } E > E_*(\lambda) \\ (-\lambda)^{1/(\alpha-\beta)} & \text{if } E = E_*(\lambda) \\ s(\lambda,E) & \text{if } 0 < E < E_*(\lambda)\,. \end{cases}$$

Hence, for $p > 1$ and $\lambda < 0$,

$$0 < r(\lambda,E) < +\infty \text{ if and only if } \begin{cases} E \in (0,+\infty), & \text{if } \alpha-\beta > 0, \\ 0 < E \le E_*(\lambda) & \text{if } \alpha-\beta < 0. \end{cases}$$

Also,

$$f(\lambda,r(\lambda,E)) > 0 \text{ if and only if } \begin{cases} E \in (0,+\infty), & \text{if } \alpha-\beta > 0, \\ 0 < E < E_*(\lambda) & \text{if } \alpha-\beta < 0. \end{cases}$$

So, for all $\lambda < 0$,

$$D(\lambda) = \begin{cases} (0,+\infty) & \text{if } \alpha-\beta>0, \\ (0,E_*(\lambda)) & \text{if } \alpha-\beta<0. \end{cases}$$

By $\widetilde{D} \subset (0,+\infty)$ and $\widetilde{D} \subset \overline{D}$, it follows that for $\alpha-\beta>0$, $\widetilde{D} = D = (0,+\infty)$ and for $\alpha-\beta<0$, $\widetilde{D} = (0,E_*(\lambda)]$ if $\int_0^{r(E_*(\lambda))}(E^p - p'G(t))^{-1/p}dt < +\infty$ and $\widetilde{D} = D = (0,E_*(\lambda))$ otherwise.

Before going further in the investigation, we deduce from Lemma 5.1 that for any $p>1$ and $\lambda<0$,

$$\begin{gathered} E^p = p'F(\lambda, r(\lambda,E)), \text{ for all } E \in \widetilde{D}(\lambda) \qquad (5.1) \\ \frac{\partial r}{\partial E}(\lambda,E) = \frac{(p-1)E^{p-1}}{f(\lambda,r(\lambda,E))} > 0, \text{ for all } E \in D(\lambda) \\ \lim_{E\to0^+} r(\lambda,E) = \begin{cases} ((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)} & \text{if } \alpha-\beta>0 \\ 0 & \text{if } \alpha-\beta<0\,. \end{cases} \\ \lim_{E\to+\infty} r(\lambda,E) = +\infty \text{ if } \alpha-\beta>0\,, \\ \lim_{E\to E_*} r(\lambda,E) = (-\lambda)^{1/(\alpha-\beta)} \text{ if } \alpha-\beta<0\,. \end{gathered}$$

At present, we define for any $p>1$, $\lambda<0$ the time-map

$$T(\lambda,E) = \int_0^{r(\lambda,E)} \{E^p - p'F(\lambda,\xi)\}^{-1/p} d\xi,\ E \in \widetilde{D}(\lambda)\,.$$

By (5.1), it follows that

$$T(\lambda,E) = (p')^{-1/p}\int_0^{r(\lambda,E)} \{F(\lambda,r(\lambda,E)) - F(\lambda,\xi)\}^{-1/p} d\xi,\ E \in \widetilde{D}(\lambda)\,.$$

Furthermore, a simple change of variable and a substitution yield

$$T(\lambda,E) = (p')^{-1/p}\int_0^1 \{\lambda r^{\beta-p}(\lambda,E)(\frac{1-t^\beta}{\beta}) + r^{\alpha-p}(\lambda,E)(\frac{1-t^\alpha}{\alpha})\}^{-1/p}dt.$$

One may observe that

$$T(\lambda,E) = S(\lambda, r(\lambda,E)), \quad \text{for all } \lambda<0 \text{ and } E \in \widetilde{D}(\lambda),$$

where

$$S(\lambda,\rho) = (p')^{-1/p}\int_0^1 \{\lambda \rho^{\beta-p}(\lambda,E)(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\lambda,E)(\frac{1-t^\alpha}{\alpha})\}^{-1/p}dt,$$

for all $\lambda<0$ and all $\rho \in R(\lambda)$ where, for all $\lambda<0$, $R(\lambda)$ is the range of the function $E \mapsto r(\lambda,E)$, defined on $\widetilde{D}(\lambda)$, that is, for $\alpha-\beta>0$, $R(\lambda) :=$

$(((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)}, +\infty)$, and for $\alpha-\beta<0$, $R(\lambda) := (0, (-\lambda)^{1/(\alpha-\beta)}]$ if $S(\lambda, r(\lambda, E_*(\lambda)) < +\infty$ and $R(\lambda) := (0, (-\lambda)^{1/(\alpha-\beta)})$ otherwise.

Due to the fact that the function $E \mapsto r(\lambda, E)$ is an increasing and continuous function from $\widetilde{D}(\lambda)$ onto $R(\lambda)$, it follows that if we put

$$J_1(\lambda) := \{E \in \widetilde{D}(\lambda) : T(\lambda, E) = \frac{1}{2}\}, \lambda < 0, \text{ and}$$

$$J_2(\lambda) := \{\rho \in R(\lambda) : S(\lambda, E) = \frac{1}{2}\}, \lambda < 0,$$

then,

$$\mathrm{Card} J_1(\lambda) = \mathrm{Card} J_2(\lambda), \text{ for all } \lambda < 0\,.$$

Hence, from now on, we will focus our attention on counting the number of solution(s) of the equation $S(\lambda, \rho) = 1/2$ in the variable $\rho \in R(\lambda)$ instead of the equation $T(\lambda, E) = 1/2$ in the variable $E \in \widetilde{D}(\lambda)$.

Lemma 5.2 *Assume that* $\lambda < 0$. *Then*

(a) *If* $\alpha - \beta > 0$,

$$\lim_{\rho \to ((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)}} S(\lambda, \rho) = \begin{cases} (-\lambda)^{\frac{(p-\alpha)}{p(\alpha-\beta)}} J(p, \alpha, \beta) & \text{if } 1 < \beta < p \\ +\infty & \text{if } p \le \beta. \end{cases}$$

where $J(p, \alpha, \beta)$ *is defined in (2.1).*

(b) *If* $\alpha - \beta > 0$,

$$\lim_{\rho \to +\infty} S(\lambda, \rho) = \begin{cases} 0 & \text{if } p \le \beta \\ \frac{1}{2}(\frac{\lambda_1}{p^2})^{1/p} & \text{if } p > \beta \text{ and } \alpha = p \\ 0 & \text{if } p > \beta \text{ and } \alpha > p \\ +\infty & \text{if } p > \beta \text{ and } \alpha < p \end{cases}$$

(c) *If* $\alpha - \beta < 0$,

$$\lim_{\rho \to 0} S(\lambda, \rho) = \begin{cases} 0 & \text{if } p > \alpha \\ \frac{1}{2}(\frac{\lambda_1}{p^2})^{1/p} & \text{if } p = \alpha \\ +\infty & \text{if } p < \alpha \end{cases}$$

(d) *If* $\alpha - \beta < 0$,

$$\lim_{\rho \to (-\lambda)^{1/(\alpha-\beta)}} S(\lambda, \rho) = (-\lambda)^{\frac{(p-\alpha)}{p(\alpha-\beta)}} \int_0^1 \{(\frac{1-t^\alpha}{\alpha}) - (\frac{1-t^\beta}{\beta})\}^{-1/p} dt$$

and

$$\int_0^1 \{(\frac{1-t^\alpha}{\alpha}) - (\frac{1-t^\beta}{\beta})\}^{-1/p} dt < +\infty \Longleftrightarrow p > 2.$$

Proof. Assume that $\alpha-\beta>0$. Easy computation yields

$$\begin{aligned}\lim_{\rho\to((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)}} S(\lambda,\rho) &= (p')^{-1/p}(\frac{-\lambda}{\beta})^{(p-\alpha)/p(\alpha-\beta)}\alpha^{(p-\beta)/p(\alpha-\beta)}\times \\ &\quad \times\int_0^1 t^{-\beta/p}(1-t^{\alpha-\beta})^{-1/p}dt.\end{aligned}$$

It can be shown that for all $p>1$ and $1<\beta<\alpha$

$$\int_0^1 t^{-\beta/p}(1-t^{\alpha-\beta})^{-1/p}dt=\begin{cases}+\infty & \text{if } \beta\geq p\\ \frac{1}{\alpha-\beta}B(\frac{(p-\beta)}{p(\alpha-\beta)},\frac{p-1}{p}) & \text{if } 1<\beta<p.\end{cases}$$

In fact, in the case where $\beta\geq p$, we use the estimates

$$t^{-\beta/p}(1-t^{\alpha-\beta})^{-1/p}\geq t^{-\beta/p},\ \text{for all } t\in(0,1),$$

and in the case where $1<\beta<p$, we use the change of variable $x=t^{\alpha-\beta}$, as in Lavrentiev and Chabat [13, pp. 595-596]. Therefore Assertion **(a)** is proved.

Also, in the case where $\alpha-\beta>0$ and $\beta-p\geq 0$

$$\begin{aligned}\lim_{\rho\to+\infty}(\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta})+\rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha}))^{-1/p} &= \\ \lim_{\rho\to+\infty}\rho^{(p-\beta)/p}(\lambda(\frac{1-t^\beta}{\beta})+\rho^{\alpha-\beta}(\frac{1-t^\alpha}{\alpha}))^{-1/p} &= 0\cdot 0=0\,.\end{aligned}$$

and an easy discussion shows that for $\alpha-\beta>0$,

$$\begin{aligned}&\lim_{\rho\to+\infty}(\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta})+\rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha}))^{-1/p}\\ &=\begin{cases}0 & \text{if } \beta\geq p\\ 0 & \text{if } \beta<p \text{ and } \alpha>p\\ +\infty & \text{if } \beta<p \text{ and } \alpha<p\\ ((1-t^p)/p)^{-1/p} & \text{if } \beta<p \text{ and } \alpha=p\end{cases}\end{aligned}$$

Therefore, Assertion **(b)** is proved.

In the case where $\alpha-\beta<0$ and $\lambda<0$,

$$\begin{aligned}&\lim_{\rho\to 0}\int_0^1(\lambda\rho^{\beta-p}((1-t^\beta)/\beta)+\rho^{\alpha-p}((1-t^\alpha)/\alpha))^{-1/p}dt\\ &=\lim_{\rho\to 0}\rho^{(p-\alpha)/p}\int_0^1(\lambda\rho^{\beta-\alpha}(1-t^\beta)/\beta+(1-t^\alpha)/\alpha)^{-1/p}dt\\ &=\lim_{\rho\to 0}\rho^{(p-\alpha)/p}\int_0^1((1-t^\alpha)/\alpha)^{-1/p}dt\,.\end{aligned}$$

Notice that for all $\alpha>1$

$$\int_0^1(1-t^\alpha)^{-1/p}dt=\frac{1}{\alpha}B(\frac{1}{\alpha},\frac{p-1}{p})<+\infty\,.$$

This follows by making use of the change of variable $x = t^\alpha$, see Lavrentiev and Chabat [13, pp. 595-596]. Assertion **(c)** follows.

Also, in the case $\alpha - \beta < 0$ and $\lambda < 0$, easy computation shows

$$\lim_{\rho \to (-\lambda)^{1/(\alpha-\beta)}} S(\lambda, \rho) = (p')^{-1/p} (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} \times \int_0^1 ((1-t^\alpha)/\alpha) - ((1-t^\beta)/\beta))^{-1/p} dt \,.$$

By making use of L'Hopital's rule two times, we compute

$$\lim_{t \to 1^-} \frac{(1-t^\alpha)/\alpha - (1-t^\beta)/\beta}{(1-t)^2} = \frac{\beta - \alpha}{2} > 0 \,.$$

So, the integral $\int_0^1 ((1-t^\alpha)/\alpha) - ((1-t^\beta)/\beta))^{-1/p} dt$ is convergent if and only if the integral $\int_0^1 (1-t)^{-2/p} dt$ does so. Therefore, Assertion **(d)** follows from the well known fact that

$$\int_0^1 (1-t)^{-2/p} dt < +\infty \text{ if and only if } p > 2 \,.$$

Therefore, Lemma 5.2 is proved.

Lemma 5.3 *Assume that $p > 1$, $\lambda < 0$, and $\alpha \neq \beta$, $\alpha, \beta > 1$.*

1. *If one of the following conditions holds:*

   (a) $\beta < p \leq \alpha$

   (b) $\beta = p < \alpha$

   (c) $p < \beta < \alpha$

   then, $S(\lambda, \cdot)$ is strictly decreasing on $R(\lambda)$.

2. *If one of the following conditions holds:*

   (a) $\alpha \leq p < \beta$

   (b) $\alpha < p = \beta$

   (c) $\alpha < \beta < p$

   then, $S(\lambda, \cdot)$ is strictly increasing on $R(\lambda)$.

3. *If one of the following conditions holds:*

   (a) $p < \alpha < \beta$

   (b) $\beta < \alpha < p < \alpha + \beta$

   then, $S(\lambda, \cdot)$ is strictly decreasing on $(\inf R(\lambda), \rho_1]$ and is strictly increasing on $[\rho_2, \sup R(\lambda))$, where $\rho_1(\lambda)$ and $\rho_2(\lambda)$ are defined in Lemma 4.3.

4. If $\beta < \alpha < p$ and $\alpha + \beta \le p$, then $S(\lambda, \cdot)$ is strictly increasing on $[\rho_2, +\infty)$.

This Lemma follows by a similar discussion as that in the proof of Lemma 4.3. So, the proof of this Lemma is omitted.

Note that the third and the fourth assertions of Lemma 5.3 above do not provide the exact variations of the map $S(\lambda, \cdot)$ over its entire definition domain, which are necessary for the process of showing the exactness part in the main result. They are the aim of the following pioneer lemma.

Lemma 5.4 *Assume that one of the following conditions holds:*

(c1) $1 < p \le 2$, *and* $p < \alpha < \beta$,

(c2) $2 < p < \alpha < \beta$,

(c3) $1 < \beta < \alpha < p$.

Then, for all $\lambda < 0$, there exists an interior point $\rho^*(\lambda) \in int(R(\lambda))$ such that $S(\lambda, \cdot)$ is strictly decreasing on $(\inf R(\lambda), \rho^*(\lambda))$ and then strictly increasing on $(\rho^*(\lambda), \sup R(\lambda))$.

We shall prove Lemma 5.4 in two steps.

Step 1: Existence If

$$p < \alpha < \beta \text{ or } 1 < \beta < \alpha < p < \alpha + \beta \tag{5.2}$$

the existence of $\rho^*(\lambda)$ follows immediatly from Lemma 5.3, Assertion 3.

Also, if

$$1 < \beta < \alpha < \alpha + \beta \le p, \tag{5.3}$$

then, according to Lemma 5.3, Assertion 4, existence follows after proving that for all $\lambda < 0$, $S(\lambda, \cdot)$ is strictly decreasing on a right neighborhood of $\inf R(\lambda)$. (Notice that in this case $\rho_1(\lambda) \le \inf R(\lambda)$). In fact, we shall prove:

Lemma 5.5 *If $1 < \beta < \alpha < \alpha + \beta \le p$ and $\lambda < 0$ then*

$$\frac{\partial S}{\partial \rho}\Big(\lambda, \Big(\frac{-\lambda\alpha}{\beta}\Big)^{1/(\alpha-\beta)}\Big) = -\infty.$$

Proof. The derivative of $S(\lambda, \cdot)$ is given by

$$\frac{\partial S}{\partial \rho}(\lambda, \rho) = (p')^{-1/p} \int_0^1 \frac{H(\lambda, \rho) - H(\lambda, \rho u)}{p\rho(F(\lambda, \rho) - F(\lambda, \rho u))^{1+\frac{1}{p}}} du.$$

where

$$F(\lambda, \rho) = \frac{1}{\alpha}\rho^\alpha + \frac{\lambda}{\beta}\rho^\beta, \rho > 0 \quad \text{and} \quad H(\lambda, \rho) = \Big(\frac{p-\alpha}{\alpha}\Big)\rho^\alpha + \lambda\Big(\frac{p-\beta}{\beta}\Big)\rho^\beta, \rho > 0.$$

Simple computations yield

$$\begin{aligned}\frac{\partial S}{\partial \rho}(\lambda,(-\lambda\frac{\alpha}{\beta})^{1/(\alpha-\beta)}) &= \frac{(p')^{-1/p}}{p}(\frac{\beta}{-\lambda})^{\frac{p+\alpha}{p(\alpha-\beta)}}\alpha^{\frac{p+\beta}{p(\beta-\alpha)}}\\ &\quad\times\int_0^1 \frac{(p-\alpha)(1-u^\alpha)-(p-\beta)(1-u^\beta)}{(u^\beta-u^\alpha)^{1+\frac{1}{p}}}du.\end{aligned}$$

The improper integral has two singularities; at 0 and at 1. Then we write

$$\frac{\partial S}{\partial \rho}(-1,(-\lambda\frac{\alpha}{\beta})^{1/(\alpha-\beta)}) = \frac{(p')^{-1/p}}{p}(\frac{\beta}{-\lambda})^{\frac{p+\alpha}{p(\alpha-\beta)}}\alpha^{\frac{p+\beta}{p(\beta-\alpha)}}(I_0+I_1)$$

where

$$I_0 = \int_0^{1/2} \frac{(p-\alpha)(1-u^\alpha)-(p-\beta)(1-u^\beta)}{(u^\beta-u^\alpha)^{1+\frac{1}{p}}}du,$$

$$I_1 = \int_{1/2}^{1} \frac{(p-\alpha)(1-u^\alpha)-(p-\beta)(1-u^\beta)}{(u^\beta-u^\alpha)^{1+\frac{1}{p}}}du.$$

In what follows we shall prove that $I_0=-\infty$ and $I_1\in\mathbb{R}$.

First, we write I_0 as follows:

$$I_0 = \int_0^{1/2} \frac{-(\alpha-\beta)-[(p-\alpha)u^\alpha-(p-\beta)u^\beta]}{(1-u^{\alpha-\beta})^{1+\frac{1}{p}}u^{\beta(1+\frac{1}{p})}}du.$$

One may observe that in a right neighborhood of 0,

$$\frac{-(\alpha-\beta)-[(p-\alpha)u^\alpha-(p-\beta)u^\beta]}{(1-u^{\alpha-\beta})^{1+\frac{1}{p}}u^{\beta(1+\frac{1}{p})}} \simeq \frac{-(\alpha-\beta)}{u^{\beta(1+\frac{1}{p})}},$$

and by (5.3), it follows that $\beta(1+\frac{1}{p})>1$ and $-(\alpha-\beta)<0$, so

$$\int_0^{1/2}\frac{-(\alpha-\beta)}{u^{\beta(1+\frac{1}{p})}}du=-\infty,$$

and therefore $I_0=-\infty$.

Next, we write I_1 as follows:

$$I_1 = \int_{1/2}^{1} \frac{(p-\alpha)(1-u^\alpha)-(p-\beta)(1-u^\beta)}{u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}}}du\,.$$

Applying Taylor's theorem to the function N, defined by

$$N(u)=(p-\alpha)(1-u^\alpha)-(p-\beta)(1-u^\beta),$$

it follows that

$$\begin{aligned} I_1 &= \int_{1/2}^{1} \frac{[(p-\alpha)\alpha-(p-\beta)\beta](1-u)}{u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}}}du \\ &+ \int_{1/2}^{1} \frac{-\frac{1}{2}[(p-\alpha)\alpha(\alpha-1)-(p-\beta)\beta(\beta-1)](1-u)^2+o((1-u)^2)}{u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}}}du. \end{aligned}$$

Notice that $\lim_{u\to 1^-} \frac{1-u^{\alpha-\beta}}{1-u} = (\alpha-\beta) \in \mathbb{R}$ then, in a left neighborhood of 1,

$$u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}} \simeq (\alpha-\beta)(1-u)^{1+\frac{1}{p}}.$$

Next, one has to distinguish two cases.

Case $[(p-\alpha)\alpha-(p-\beta)\beta] \neq 0$. In this case the integrand function in I_1 is equivalent in a left neighborhood of 1 to the function $u \mapsto \frac{(p-\alpha)\alpha-(p-\beta)\beta}{(\alpha-\beta)(1-u)^{1/p}}$ and since $p > 1$ it follows that $\int_0^{1/2} \frac{(p-\alpha)\alpha-(p-\beta)\beta}{(\alpha-\beta)(1-u)^{1/p}}du \in \mathbb{R}$ and therefore $I_1 \in \mathbb{R}$.

Case $[(p-\alpha)\alpha-(p-\beta)\beta] = 0$. In this case the integrand function in I_1 is equivalent in a left neighborhood of 1 to the function $u \mapsto (-\frac{1}{2})(p-\alpha)\alpha(1-u)^{1-\frac{1}{p}}$ which is a continuous function on the compact interval $[\frac{1}{2}, 1]$ then $\int_{1/2}^1(-\frac{1}{2})(p-\alpha)\alpha(1-u)^{1-\frac{1}{p}}du \in \mathbb{R}$. Then, in this case $I_1 \in \mathbb{R}$ too. Therefore Lemma 5.5 is proved.

Step2: Uniqueness First, we point out that Step 1 shows a little bit general result than the existence. In fact, for all $\lambda < 0$, it was proved that $\rho^*(\lambda)$ exists and belongs necessarily to $(\rho_1(\lambda), \rho_2(\lambda))$ if (5.2) holds and belongs to $(\inf R(\lambda), \rho_2(\lambda))$ if (5.3) holds. So, to prove uniqueness, we shall restrict ourselves to $(\rho_1(\lambda), \rho_2(\lambda))$ (resp. to $(\inf R(\lambda), \rho_2(\lambda))$). That is, we shall prove that $S(\lambda, \cdot)$ admits *at most one* critical point in $(\rho_1(\lambda), \rho_2(\lambda))$ (resp. in $(\inf R(\lambda), \rho_2(\lambda))$). To this end we shall prove that for all $\lambda < 0$, $S(\lambda, \cdot)$ is convex in a neighborhood of each of its critical points lying in $(\rho_1(\lambda), \rho_2(\lambda))$ (resp. in $(\inf R(\lambda), \rho_2(\lambda))$). Similar idea was previously used in [5].

This step follows by two lemmas. The first one is technical but the second one is the heart of this step.

Lemma 5.6 *Let $p, \alpha, \beta > 1$. If $\alpha \neq \beta$, then $(\frac{\beta}{\alpha})^{1/(\alpha-\beta)} < 1$. If one of the following conditions holds: (a) $p < \alpha < \beta$ or (b) $\beta < \alpha < p$, then for all $\lambda < 0$, the function defined on the interval $[0, \rho_2(\lambda)]$ by $\rho \mapsto \Psi(\lambda, \rho) := (p-\alpha)\rho^\alpha + \lambda(p-\beta)\rho^\beta$, is strictly decreasing on $[0, \rho_1(\lambda)(\frac{\beta}{\alpha})^{1/(\alpha-\beta)}]$ and is strictly increasing on $[\rho_1(\lambda)(\frac{\beta}{\alpha})^{1/(\alpha-\beta)}, \rho_2(\lambda)]$. Moreover, $\Psi(\lambda, 0) = \Psi(\lambda, \rho_1(\lambda)) = 0$, for all $\lambda < 0$.*

The proof is very simple and therefore omitted. For all $\lambda < 0$, let $\rho_*(\lambda) := \max\{\rho_1(\lambda), \inf R(\lambda)\}$.

Lemma 5.7 *Let $p, \alpha, \beta > 1$. Assume that one of the following conditions holds: (a) $p < \alpha < \beta$, or (b) $\beta < \alpha < p$. Then for all $\lambda < 0$,*

$$\frac{\partial^2 S}{\partial \rho^2}(\lambda,\rho) + (\frac{p+1}{\rho})\frac{\partial S}{\partial \rho}(\lambda,\rho) > 0, \text{ for all } \rho \in (\rho_*(\lambda), \rho_2(\lambda)) .$$

Proof. Notice that for all $\lambda < 0$, $(\rho_*(\lambda), \rho_2(\lambda)) \subset R(\lambda) =: domS(\lambda, \cdot)$. The second derivative of $S(\lambda, \cdot)$ is given by

$$\begin{aligned} \frac{\partial^2 S}{\partial \rho^2}(\lambda,\rho) &= (p')^{-1/p}\int_0^1 \frac{(p+1)(H(\lambda,\rho)-H(\lambda,\rho u))^2}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho u))^{(2p+1)/p}}du \\ &\quad +(p')^{-1/p}\int_0^1 \frac{\Phi(\lambda,\rho)-\Phi(\lambda,\rho u)}{p\rho(F(\lambda,\rho)-F(\lambda,\rho u))^{(p+1)/p}}du, \end{aligned}$$

where

$$\begin{aligned} \Phi(\lambda,\rho) &= -p(p+1)F(\lambda,\rho) + 2p\rho f(\lambda,\rho) - \rho^2 f'_\rho(\lambda,\rho) \\ &= \frac{(p-\alpha)(\alpha-(p+1))}{\alpha}\rho^a + \lambda\frac{(p-\beta)(\beta-(p+1))}{\beta}\rho^\beta . \end{aligned}$$

It follows that

$$\begin{aligned} &(p')^{1/p}p\rho\{\rho\frac{\partial^2 S}{\partial \rho^2}(\lambda,\rho) + (p+1)\frac{\partial S}{\partial \rho}(\lambda,\rho)\} \\ &= \int_0^\rho \frac{\Psi(\lambda,\rho)-\Psi(\lambda,\xi)}{(F(\lambda,\rho)-F(\lambda,\xi))^{(p+1)/p}}d\xi \\ &\quad +(\frac{p+1}{p})\int_0^\rho \frac{(H(\lambda,\rho)-H(\lambda,\xi))^2}{(F(\lambda,\rho)-F(\lambda,\xi))^{(2p+1)/p}}du , \end{aligned}$$

where

$$\Psi(\lambda,\rho) := \Phi(\lambda,\rho) + (p+1)H(\lambda,\rho) = \rho\frac{\partial H}{\partial \rho}(\lambda,\rho) = (p-\alpha)\rho^\alpha + \lambda(p-\beta)\rho^\beta .$$

By Lemma 5.6, it follows that for all $\lambda < 0$ and all $\rho \in (\rho_*(\lambda), \rho_2(\lambda))$,

$$\Psi(\lambda,\rho) - \Psi(\lambda,\xi) > 0, \text{ for all } \xi \in (0,\rho) .$$

Therefore

$$\int_0^\rho \frac{\Psi(\lambda,\rho)-\Psi(\lambda,\xi)}{(F(\lambda,\rho)-F(\lambda,\xi))^{(p+1)/p}}d\xi > 0, \text{ for all } \lambda < 0 \text{ and all } \rho \in (\rho_*(\lambda), \rho_2(\lambda))$$

and Lemma 5.7 is proved, which ends the proof of Lemma 5.4. ♢

Notice that if one of the hypothesis of Assertions A through F holds then $S(\lambda, \cdot)$ is monotonic, the proofs follow by an elementary discussion as in Assertion A or B of Theorem 2.1. Therefore the proofs of Assertions A, through F are omitted.

Concerning the remaining assertions, the same ideas performed for Assertion C of Theorem 2.1 apply. For this, it suffices to use Lemma 5.4 and the following

Lemma 5.8 *By Lemma 5.4 let* $m(\lambda) := \inf_{\rho\in R(\lambda)} S(\lambda,\rho)$, $\forall\lambda < 0$. *Then,*

(a) $m(\cdot)$ *is continuous on* $(-\infty, 0)$

(b) $m(\cdot)$ *is strictly decreasing on* $(-\infty, 0)$

(c) $\lim_{\lambda\to-\infty} m(\lambda) = +\infty$ *and* $\lim_{\lambda\to 0} m(\lambda) = 0$.

Lemma 5.9 *The function* $\lambda \mapsto \ell(\lambda) := \lim_{\rho\mapsto\inf R(\lambda)} S(\lambda,\rho)$ *is either infinite on the whole set* $(-\infty, 0)$ *or satisfies Assertions (a), (b), and (c) of Lemma 5.8.*

Lemma 5.10 *The function* $\lambda \mapsto L(\lambda) := \lim_{\rho\mapsto\sup R(\lambda)} S(\lambda,\rho)$ *is (independently of* $\ell(\lambda)$*) either infinite on the whole set* $(-\infty, 0)$ *or satisfies Assertions (a), (b), and (c) of Lemma 5.8.*

Proof of Lemma 5.8. Recall that for all $\lambda < 0$ and $\rho \in R(\lambda)$,

$$S(\lambda,\rho) = (p')^{-1/p} \int_0^1 \{\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha})\}^{-1/p} dt\,.$$

For all $\lambda < 0$ and $\rho \in R(\lambda)$, let $\bar\rho = \bar\rho(\lambda,\rho) := (-\lambda)^{1/(\beta-\alpha)}\rho$. Then, $\rho = (-\lambda)^{1/(\alpha-\beta)}\bar\rho$ and a simple substitution yields

$$S(\lambda,\rho) = (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} S(-1, \bar\rho(\lambda,\rho))\,.$$

Thus,

$$\begin{aligned} m(\lambda) &= (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} \inf_{\rho\in R(\lambda)} S(-1,\bar\rho(\lambda,\rho)) \\ &= (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} \inf_{\rho\in R(-1)} S(-1,\rho) \end{aligned}$$

So,

$$m(\lambda) = (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} m(-1), \forall\lambda < 0\,.$$

Therefore, Lemma 5.8 is proved. ♢

Proof of Lemma 5.9. By Lemma 5.2 it follows in the case where $1 < \beta < \alpha < \alpha + \beta \le p$ that $\lim_{\rho\to\inf R(\lambda)} S(\lambda,\rho) = (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} J(p,\alpha,\beta) < +\infty$, where $J(p,\alpha,\beta)$ is defined in (2.1). Hence, Lemma 5.9 is proved in this case. The other cases are similar or easier.

The proof of Lemma 5.10 is similar to that of Lemma 5.9.

6 Remarks

By the study of Problem (1.3) with $\lambda \in \mathbb{R}$, it can be deduced the structure of the solution set of a quite general problem

$$\begin{gathered} -(\varphi_p(u'))' = \mu\varphi_\alpha(u) + \lambda\varphi_\beta(u) \quad \text{in } (0,1) \\ u(0) = u(1) = 0\,, \end{gathered} \tag{6.1}$$

when $p, \alpha, \beta > 1$, $\mu > 0$ and $\lambda \in \mathbb{R}$. In fact, if v is a solution of (1.3) for $\lambda = \lambda_0 \in \mathbb{R}$ and $\alpha \neq p$, then for all $\mu_0 > 0$, the function $u := \mu_0^{1/(p-\alpha)} v$ is a solution to (6.1) with $\mu = \mu_0$ and $\lambda = \lambda_0 \mu_0^{(\beta-p)/(\alpha-p)}$. (If $\alpha = p$ and $\beta \neq p$, similar change of variable works). Conversely, if u is a solution of (6.1) with $\mu = \mu_0 > 0$, $\lambda = \lambda_0 \in \mathbb{R}$ and $\alpha \neq p$, then $v := \mu_0^{1/(\alpha-p)} u$ is a solution to (1.3) with $\lambda = \lambda_0 \mu_0^{(\beta-p)/(\alpha-p)}$.

The structure of the solution set of problem (6.1) when $\mu < 0$ and $\lambda \in \mathbb{R}$ can be deduced from that of the problem

$$\begin{gathered} -(\varphi_p(u'))' = -\varphi_\alpha(u) + \lambda\varphi_\beta(u) \quad \text{in } (0,1) \\ u(0) = u(1) = 0, \end{gathered}$$

which is not treated here. However, upon completing our paper, the work [11] by Díaz and Hernández appeared. Positive solutions to problem (6.1) with $\mu < 0$, $\lambda > 0$ and $1 < \alpha < \beta \leq p$ are treated there.

After completing this work, e-mail correspondence between the first author (I. Addou) and Professor Pedro Ubilla from Chile, revels that simultaneously and independently of the present authors, Professors J. Sánchez and P. Ubilla from Chile, were studying problem (1.3) with $\lambda > 0$. That is to say, they resolved the p-Laplacian version of the Ambrosetti-Brezis-Cerami problem. To do so, they provide essentially the same proof as that of Theorem 2.1 above by making use of the same idea performed in [3, Lemma 7, (iii)]. Their work was presented, under the title: "The exact number of positive solutions for an elliptic equation with concave and convex nonlinearities" by Professor P. Ubilla at the "USA-Chile Workshop on Nonlinear Analysis" meeting which held in Valparaíso in Chile on 17-21, January 2000. Also, it was published in this volume of this journal, see [15].

Also, after submitting this work for publication, e-mail correspondence between the first author (I. Addou) and Professor Shin-Hwa Wang from Taiwan (R. O. China) revels that he has write (independently of J. Sanchez and P. Ubilla and independently of the present authors) a paper [19] in which he resolves (for $p = 2$) the Ambrosetti-Brezis-Cerami problem [7, Sect. 6, (d)] (among many other interesting things), by making use of the quadrature technique. To deal with the difficult step (uniqueness of the maximum of the time map), he used an interesting argument which is comparable to that of [3, Lemma 7, (iii)] and used by him previously in [18, Proof of Theorem 7]. (See, also [16]).

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7 Appendix

In the process of our proofs, we have used the fact that

$$\lambda_1(p) = (p-1)(\frac{2\pi}{p\sin(\pi/p)})^p > 1, \forall p > 1.$$

In this appendix we shall prove the following:

(A1) $\lambda_1(p) > 1, \forall p \in (1,2)$.

(A2) $\lambda_1(2) = \pi^2$.

(A3) $\lambda_1(p) > 4, \forall p \in (2,+\infty)$.

(A4) $\lim_{p\to 1^+} \lambda_1(p) = 2$, and $\lim_{p\to+\infty} \lambda_1(p) = +\infty$.

Proof of (A1). Observe that for all $p \in (1,2)$, $\lambda_1(p)$ can be written as

$$\lambda_1(p) = (\frac{2}{p})^p (p-1)^{(1-p)} (\frac{\pi}{\frac{\sin(\pi/p)-\sin(\pi/1)}{p-1}})^p\,. \tag{7.1}$$

The function $p \mapsto \theta(p) := (\frac{2}{p})^p$ is strictly decreasing on $(1,2]$ and $\theta(2) = 1$. Thus,

$$(\frac{2}{p})^p > 1, \quad \forall p \in (1,2). \tag{7.2}$$

The function $p \mapsto K(p) := (p-1)^{(1-p)}$ is strictly increasing on $(1, 1+\exp(-1)]$ and is strictly decreasing on $[1+\exp(-1), 2)$. Thus, $K(p) > \min\{K(2), \lim_{p\to 1^+} K(p)\} = 1$, for all $p \in (1,2)$. Therefore,

$$(p-1)^{(1-p)} > 1, \forall p \in (1,2). \tag{7.3}$$

Notice that for all $c \in (1,2) : c^2 > 1 > -\cos(\pi/c) > 0$. Then, $\frac{\pi}{-(\pi^2/c)\cos(\pi/c)} > 1$, for all $c \in (1,2)$. Therefore,

$$(\frac{\pi}{-(\pi^2/c)\cos(\pi/c)})^p > 1,\ \forall (p,c) \in (1,2)^2\,. \tag{7.4}$$

On the other hand, for all $p \in (1,2)$, there exists $c = c_p \in (1,p) \subset (1,2)$, such that

$$\frac{\sin(\pi/p) - \sin(\pi/1)}{p-1} = -\frac{\pi}{c^2}\cos\frac{\pi}{c}\,.$$

Therefore, for all $p \in (1,2)$, there exists $c = c_p \in (1,p) \subset (1,2)$, such that

$$(\frac{\pi}{\frac{\sin(\pi/p)-\sin(\pi/1)}{p-1}})^p = (\frac{\pi}{-(\pi^2/c)\cos(\pi/c)})^p$$

and by (7.4), it follows that

$$(\frac{\pi}{\frac{\sin(\pi/p)-\sin(\pi/1)}{p-1}})^p > 1, \ \forall p \in (1,2). \tag{7.5}$$

Now, by (7.2), (7.3), (7.5), and (7.1), Assertion **(A1)** follows.
Proof of (A2). Simple computation.
Proof of (A3). Observe that for all $p > 2$, $\lambda_1(p)$ can be written as

$$\lambda_1(p) = (p-1) \cdot 2^p \cdot (\frac{\sin(\pi/p) - \sin(0)}{(\pi/p) - 0})^{-p}\,. \tag{7.6}$$

It is clear that for all $p > 2$

$$(p-1) > 1 \text{ and } 2^p > 2^2 = 4\,. \tag{7.7}$$

On the other hand, for all $p > 2$, there exists $c = c_p$ such that

$$0 < \frac{\sin(\pi/p) - \sin(0)}{(\pi/p) - 0} < \cos c < 1\,.$$

Therefore,

$$(\frac{\sin(\pi/p) - \sin(0)}{(\pi/p) - 0})^{-p} > 1, \forall p > 2\,. \tag{7.8}$$

Now, by (7.7), (7.8), and (7.6), Assertion **(A3)** follows.
Proof of (A4). It is clear that

$$\lim_{p\to 1^+} \theta(p) = 2, \ \lim_{p\to 1^+} K(p) = 1, \ \lim_{p\to 1^+} \pi^p = \pi, \text{ and}$$

$$\lim_{p\to 1}(\frac{\sin(\pi/p) - \sin(0)}{p-1})^p = (-\frac{\pi}{1^2}\cos\frac{\pi}{1})^1 = \pi\,.$$

Therefore, using expression (7.1) of $\lambda_1(p)$, it follows that $\lim_{p\to 1^+} \lambda_1(p) = 2$. The computation of the second limit is straightforward, which completes the proof of Assertion **(A4)**.

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