Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 71, pp. 1–24.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Nonclassical Sturm-Liouville problems and Schrödinger operators on radial trees *

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Abstract

Schrödinger operators on graphs with weighted edges may be defined using possibly infinite systems of ordinary differential operators. This work mainly considers radial trees, whose branching and edge lengths depend only on the distance from the root vertex. The analysis of operators with radial coefficients on radial trees is reduced, by a method analogous to separation of variables, to nonclassical boundary-value problems on the line with interior point conditions. This reduction is used to study self adjoint problems requiring boundary conditions 'at infinity'.

1 Introduction

The consideration of differential operators on graphs has old roots in physics and physical chemistry [11, 21, 22, 26, 34]. More recently there have been mathematical studies, some concerned with the interpretation of differential operators on graphs as limits of partial differential operators on thin domains [10, 12, 18, 33, 35], while others focus on novel problems of spectral or scattering theory [4, 5, 6, 14, 23]. Additional mathematical work includes applications to nerve impluse transmission [24], and the study of evolution equations on networks [2, 19]. There is also a large literature where discrete problems in probability, combinatorics, and group theory lead to difference operators on graphs [7, 17, 25, 31].

Given a formally self adjoint differential operator on a graph $\mathcal{G}$, one of the first problems is to describe the domains for which the operator is self adjoint on $L^2(\mathcal{G})$. As in the study of classical ordinary differential operators, the domain description will typically involve boundary conditions. When the graph has a finite set of edges the problem of characterizing self adjoint domains for Schrödinger differential operators may be interpreted as a classical boundary-value problem. The domain description is also fairly straightforward if the graph has infinitely many edges, but the set of edge lengths has a positive lower bound [6].

Mathematics Subject Classifications: 34B10, 47E05.
Key words: Schrödinger operators on graphs, graph spectral theory, boundary-value problems, interior point conditions.

Submitted June 23, 2000. Published November 28, 2000.

There is an added complication when more general infinite graphs are considered. The completion of the graph as a metric space may introduce additional points whose neighborhoods contain infinitely many vertices. In some cases boundary conditions at these new points are needed.

A simple example constructed using the interval $(0, 1)$ will illustrate the problem. Suppose the vertices of the graph are $x_j = 1 - 2^{-j}$ for $j = 0, 1, 2, \dots$, while the edges are (x_j, x_{j+1}). At 0 impose the boundary condition $f(0) = 0$. At other vertices impose the conditions

$$f(x_j^+) = f(x_j^-), \quad f'(x_j^+) = f'(x_j^-), \quad j = 1, 2, 3, \dots .$$

By classical theory the operator $-D^2$ is symmetric, but not essentially self adjoint, with the domain consisting of compactly supported smooth functions satisfying these boundary and interior point conditions. The self adjoint extensions are determined by an additional boundary condition of the form $c_1 f(1) + c_2 f'(1) = 0$.

To shed light on the problem of domain description and other problems of operator theory, this work provides a detailed analysis of Schrödinger operators $-D^2 + q$ and the associated eigenvalue equation

$$-y'' + qy = \lambda y \tag{1}$$

for certain highly symmetric trees which we call radial trees (see Figure 1). A radial tree will be a tree whose vertex degrees and edge lengths are functions of the distance in the graph from the root vertex. In addition the coefficient q will be assumed to be radial.

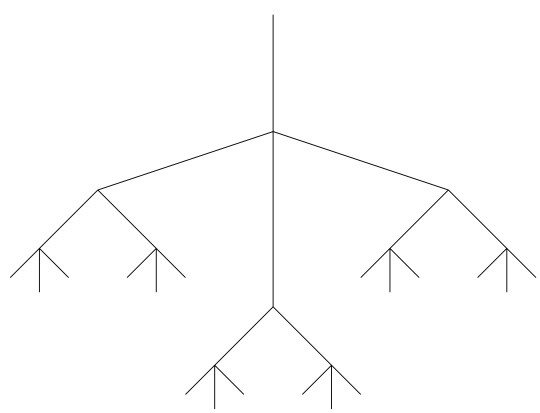

Figure 1: A radial tree

Since the radial trees are highly symmetric, one expects some corresponding simplification in the description of the invariant subspaces of radial operators $\mathcal{L} = -D^2 + q$. Roughly speaking, this simplification comes through a ‘separation of variables', which provides an orthogonal sum decomposition of L^2 for the tree into invariant subspaces $U_{v,k}$ for the Schrödinger operator. For each subspace $U_{v,k}$ there is an interval $\mathcal{I}_n \subset \mathbb{R}$ and an isometry taking $U_{v,k}$ onto a weighted Hilbert space $L^2(\mathcal{I}_n, w_n)$ which carries the restriction of $\mathcal{L}$ to a self adjoint operator $\mathcal{L}_n$ which is given by $-D^2 + q$ on its domain. Functions in the domain of $\mathcal{L}_n$ satisfy a sequence of jump conditions at interior points of the interval.

Having reduced the study of radial Schrödinger operators on the radial tree to a sequence of nonclassical boundary-value problems on intervals, these interval problems are then analyzed. This analysis is most detailed for trees of finite volume, where separated boundary conditions at the interval endpoints determine the domains of self adjoint operators, much as in the classical Sturm-Liouville problems. These operators have discrete spectrum. The eigenvalues may be identified with the roots of an entire function. Growth estimates for the entire function provide information about the distribution of eigenvalues.

The next section of the paper provides an overview of differential operators on graphs. Some results are established in a general setting for a graph $\mathcal{G}$ whose metric space completion is compact, or has finite volume. The third section uses subspaces of radial functions with support on subtrees of the initial radial tree $\mathcal{T}$, together with discrete Fourier transforms, to carry out the separation of variables reduction of the Schrödinger operators.

The fourth section uses product formulas to analyze solutions of (1) on intervals $\mathcal{I}_n \subset \mathbb{R}$, subject to jump conditions coming from the graph vertices. When the radial tree $\mathcal{T}$ has finite volume the asymptotic behaviour of the solutions as x increases has a fairly simple description. In particular one finds generalized boundary values at the right endpoint of the interval. Finally, in the fifth section the boundary behaviour of solutions to (1) is used to describe Sturm-Liouville type boundary-value problems giving rise to self adjoint operators.

The author gratefully acknowledges some improvements to the paper which were made by an anonymous referee.

2 Differential operators on graphs

In this work a graph $\mathcal{G}$ will have a countable vertex set and a countable edge set. Unless otherwise stated, graphs are assumed to be connected, and each vertex appears in only finitely many edges. Each edge has a positive weight (length) l_j.

A topological graph $\mathcal{G}$ may be constructed from this data [20, p. 190]. For each edge e_j let $[a_j, b_j]$ be a real interval of length l_j. Identify interval endpoints if the corresponding edge endpoints are the same vertex v. The Euclidean length on the intervals may be extended to paths consisting of finitely many nonoverlapping intervals by addition, and a metric $d(p_1, p_2)$ on $\mathcal{G}$ is defined as the infimum of the lengths of paths joining p_1 and p_2.

Several results from the theory of metric spaces will be used; [32, pp.139–170] may be consulted for the proofs. As a metric space $\mathcal{G}$ has a completion $\overline{\mathcal{G}}$. Recall that a metric space X is totally bounded if for every $\epsilon > 0$ there is a finite set $x_1, \dots, x_n \in X$ such that $\bigcup_k B(x_k, \epsilon)$ covers X. A metric space is compact if and only if it is complete and totally bounded. This gives a picture of graphs with compact completion.

Proposition 2.1. *A graph $\mathcal{G}$ has compact completion $\overline{\mathcal{G}}$ if and only if for every $\epsilon > 0$ there is a finite set of edges e_k, $k = 1, \dots, n$ such that for every $y \in \mathcal{G}$ there is a edge e_k and a point $x_k \in e_k$ such that $d(x_k, y) < \epsilon$.*

The identification of edges e_j with intervals facilitates the discussion of function spaces and differential operators. Let $L^2(\mathcal{G})$ denote the Hilbert space $\oplus_j L^2(e_j)$ with the inner product

$$\langle f, g\rangle = \int_{\mathcal{G}} f\overline{g} = \sum_j \int_{a_j}^{b_j} f_j(x)\overline{g_j(x)}\, dx, \quad f = (f_1, f_2, \dots).$$

A formal differential operator $L = -D^2 + q$ acts componentwise on functions $f \in L^2(\mathcal{G})$ in its domain. In our initial discussion the functions q are assumed to be real valued, measurable, and bounded. The boundedness requirement will be relaxed when radial operators are discussed.

In this paper the differential operators on $\mathcal{G}$ will have a common dense domain $\mathcal{D}_0$. To describe $\mathcal{D}_0$ we first distinguish interior vertices, which have more than one incident edge, from boundary vertices which have a single incident edge. Edges e_k incident on a vertex v are denoted $e_k \sim v$. The functions $f \in \mathcal{D}_0$ are C^∞ on each (closed) edge, vanish except on finitely many edges, vanish in a neighborhood of each boundary vertex, and satisfy the continuity and derivative conditions

$$f_j(v) = f_k(v), \quad e_j, e_k \sim v, \tag{2}$$

$$\sum_{e_k \sim v} f_k'(v) = 0.$$

The derivatives here are computed in local coordinates where v corresponds to the left endpoint of each edge interval.

The operator $L_0 = -D^2 + q$ with domain $\mathcal{D}_0$ is symmetric on $L^2(\mathcal{G})$. By essentially classical calculations [6] one may show that the adjoint operator L_0^* is $-D^2 + q$ acting on a domain $\mathcal{D}_1$. The domain $\mathcal{D}_1$ consists of those functions $f \in L^2(\mathcal{G})$ for which the components f_n and $f_n^{(1)}$ are continuous, $f_n^{(1)}$ is absolutely continuous on $[a_n, b_n]$, $Lf \in L^2(\mathcal{G})$, and the vertex conditions (2) are satisfied at interior vertices.

Since multiplication by q is a bounded self adjoint operator on $L^2(G)$, the operator $-D^2 + q$ will be self adjoint on a domain $\mathcal{D} \supset \mathcal{D}_0$ if and only if $-D^2$ is self adjoint on $\mathcal{D}$ [29, p. 162]. For now we restrict our attention to the operator $-D^2$. Integration by parts shows that $-D^2$ on the domain $\mathcal{D}_0$ has the associated positive quadratic form

$$Q(f, g) = \int_{\mathcal{G}} f'\overline{g'}.$$

Symmetric operators with positive forms always have self adjoint extensions (the Friedrich's extension).

It will be convenient to have criteria which insure that various self adjoint extensions of L_0 have compact resolvent, so that the spectrum will consist of a discrete set of eigenvalues of finite multiplicity. Some results in this direction may be achieved by employing the form Q. We start with a compactness result in the space $C(\overline{\mathcal{G}})$ of continuous functions on the metric completion of $\mathcal{G}$ with the sup norm.

Theorem 2.2. *Suppose that $\mathcal{G}$ is a connected graph which has a compact metric completion $\overline{\mathcal{G}}$. Let B denote the set of continuous functions on $\mathcal{G}$ which are absolutely continuous on each edge, and satisfy*

$$\int_{\mathcal{G}} |f|^2 + |f'|^2 \le 1.$$

Then each function $f \in B$ has a unique continuous extension to $\overline{\mathcal{G}}$, and the (extended) set B has compact closure in $C(\overline{\mathcal{G}})$.

Proof. Since $\mathcal{G}$ has a compact metric completion, it has a finite diameter L. There is a simple path $\gamma \subset \mathcal{G}$ of length at least $L/2$. For any function $f \in B$ the Cauchy-Schwarz inequality gives the integral bound

$$\int_\gamma |f| \le (\int_{\mathcal{G}} |f|^2)^{1/2} (L/2)^{1/2} \le (L/2)^{1/2}.$$

Thus there is a point $x_0 \in \mathcal{G}$ such that $|f(x_0)| \le (L/2)^{-1/2}$.

Pick any other point $x \in \mathcal{G}$ and connect x_0 and x by a simple path of length at most L. Integrate along the path (using the continuity of f across the vertices) to get

$$|f(x) - f(x_0)|^2 = |\int_{x_0}^x f'(t)\ dt|^2 \le d(x, x_0) \int_{x_0}^x |f'(t)|^2\ dt.$$

This gives a uniform bound for each $f \in B$. Replacing x_0 above by another point $y \in \mathcal{G}$ shows that the functions in B are uniformly equicontinuous.

By [32, p. 149] the functions in B extend by continuity to a uniformly equicontinuous family on the completion of $\mathcal{G}$. The Arzela-Ascoli theorem [32, p. 169] then gives the result. □

If $\mathcal{G}$ has finite volume then a uniformly convergent sequence also converges in L^2. Moreover the compactness of the set B will imply compactness of the resolvent for self adjoint extensions of L_0 whose associated quadratic form is Q [30, p. 245].

Corollary 2.3. *If $\mathcal{G}$ has finite volume then B has compact closure in L^2. If L is a self adjoint extension of L_0 whose associated quadratic form is*

$$\langle Lf, g \rangle = \int_{\mathcal{G}} f' \overline{g'},$$

then L has a compact resolvent.

When $\mathcal{G}$ has finite volume, explicit lower bounds on (nonconstant) eigenvalues may be obtained via the next lemma.

Proposition 2.4. *Suppose that f is real valued,*

$$\int_{\mathcal{G}} f^2 = 1$$

and $f(x) = 0$ for some $x \in \mathcal{G}$. Then

$$\int_{\mathcal{G}} (f')^2 \geq vol(\mathcal{G})^{-2}.$$

Proof. There is some point $y \in \mathcal{G}$ such that $f^2(y) \geq vol(\mathcal{G})^{-1}$. Connect y to x by a simple path γ. By the Cauchy-Schwarz inequality

$$vol(\mathcal{G})^{-1} \leq f(y)^2 = [f(y) - f(x)]^2 = [\int_{\gamma} f'(t)\ dt]^2 \leq vol(\mathcal{G}) \int_{\mathcal{G}} |f'|^2\ dV.$$

□

One may consider whether the finite volume hypothesis in Corollary 2.3 may be relaxed to the assumption that the diameter of $\mathcal{G}$ is finite, or that the completion of $\mathcal{G}$ is compact. We will sketch the construction of a counterexample. Start with the half open interval $(0, 1]$, and place vertices at the points $1/n$, $n \geq 2$. At each of these vertices attach K_n loops of length r_n, with $\lim_{n\to\infty} r_n = 0$. The resulting graph has finite diameter and compact completion. Next, construct smooth functions which have a constant value $c_n > 0$ on the loops at $1/n$, and which vanish at x if the distance from x to the set of loops at $1/n$ exceeds $\sigma_n > 0$. By a suitable selection of the constants K_n, r_n, c_n and σ_n, one finds symmetric operators L_0 which are bounded by any positive number ϵ on a subspace of infinite dimension. In particular no self adjoint extension can have compact resolvent.

3 Decomposing $L^2(\mathcal{T})$ of a radial tree

A graph is a tree if it is connected and simply connected. A weighted tree is a radial tree if there is a vertex R, the root, such that the degree of vertices and the lengths of edges are functions of the distance from R. A (formal) Schrödinger operator $-D^2 + q$ on a radial tree $\mathcal{T}$ will be called formally radial if q is a function of the distance from R.

Since the formally radial operator $L_0 = -D^2 + q$ with the domain $\mathcal{D}_0$ is symmetric and bounded below, it has self adjoint extensions. Such a self adjoint Schrödinger operator on a radial tree will be called radial if the domain is invariant under the automorphisms of the tree which fix R. One of the main goals of this work is to describe in detail some radial Schrödinger operators. In pursuit of this goal, the symmetries of the tree will be used to decompose $L^2(\mathcal{T})$. Similar decompositions appear in [23] and [31].

Radial trees are closely associated to a class of abelian groups. Given a finite or infinite sequence of positive integers $\delta(0), \delta(1), \dots$, let $\mathbb{Z}_{\delta(i)}$ denote the additive group of integers modulo $\delta(j)$. The group $\mathcal{Z} = \oplus_i \mathbb{Z}_{\delta(i)}$ will be the complete direct sum of the groups $\mathbb{Z}_{\delta(i)}$, whose elements are sequences with i-th component from $\mathbb{Z}_{\delta(i)}$. Addition is performed componentwise in $\mathbb{Z}_{\delta(i)}$.

It will help to establish some notation for the tree (see Figure 2). If $u, w \in \mathcal{T}$, say that w is below u if the simple path from w to R contains u. Points $w \in \mathcal{T}$ have a metric depth, which is the distance from the root. The vertices v have a combinatorial depth j, which is the number of edges separating v from the root R. Below each vertex with combinatorial depth j will be $\delta(j)$ incident edges. The classical degree of the root is thus $\delta(0)$, while the degree of vertices with combinatorial depth $j > 0$ is $\delta(j)+1$. The vertices at combinatorial depth $j > 0$ may be identified with the elements of the group $\mathcal{Z}_j = \oplus_{i=0}^{j-1} \mathbb{Z}_{\delta(i)}$. Similarly, edges may be indexed by their vertex of greatest depth. The number of edges extending from depth $j-1$ to j is $N_j = \prod_{i=0}^{j-1} \delta(i)$. The full group $\mathcal{Z}$ may be identified with the set of all simple paths of maximal length starting at the root.

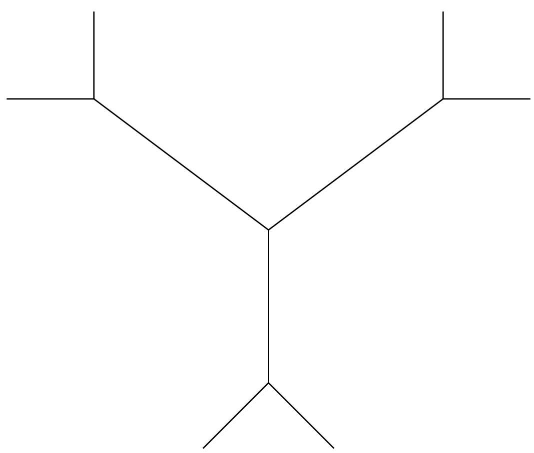

Figure 2: A radial tree with $\delta(0) = 3$, $\delta(1) = 2$

With this identification the group $\mathcal{Z}$ acts on the tree by permuting vertices and edges. In particular the components $(0, \dots, 0, \mathbb{Z}_{\delta(i)}, 0, \dots)$ rotate subtrees. Using this group action the space $L^2(\mathcal{T})$ will be decomposed into a countable orthogonal sum of invariant subspaces for the radial Schrödinger operators, with certain symmetries [31]. The reduced Schrödinger operators may then be interpreted as differential operators defined on intervals of $\mathbb{R}$.

Two types of subtrees will be associated with vertices v having incident edges below them. T_v will denote the subtree rooted at v and consisting of all vertices and edges below v. For $l \in \mathbb{Z}_{\delta(j)}$, let $S_{v,l}$ denote the tree rooted at v, but containing only the one edge (v, l) immediately below v and all vertices and edges below that edge.

Next we introduce a collection of subspaces $U_{v,k}$ of $L^2(\mathcal{T})$, defined for $k = 0,\dots,\delta(0)-1$ if $v = R$, and defined for $k = 1,\dots,\delta(j)-1$ if v has depth $j > 0$. To construct these subspaces, begin with the functions f which are radial on the tree $S_{v,0}$, and which vanish on the complement of T_v. For $k = 1,\dots,\delta(j)-1$ the subspace $U_{v,k}$ is the set of functions satisfying

$$f(t_l) = e^{2\pi ikl/\delta(j)} f(t_0), \quad t_l \in S_{v,l}, \tag{3}$$

$$l = 0,\dots,\delta(j)-1, \quad k = 1,\dots,\delta(j)-1,$$

where the points $t_l \in S_{v,l}$ have the same metric depth as t_0. In case $v = R$, the subspace $U_{R,0}$ consists of all radial functions on $\mathcal{T}$.

Theorem 3.1. *The distinct subspaces $U_{v,k}$ are orthogonal, and their linear span is dense in $L^2(\mathcal{T})$.*

Proof. For two distinct vertices v, w, the trees T_v and T_w are either disjoint, in which case the subspaces $U_{v,k}$ and $U_{w,m}$ are obviously orthogonal, or after a possible relabeling, v lies above w. Notice that each element f of $U_{v,k}$ is radial when restricted to T_w. If w has combinatorial depth j and $g \in U_{w,m}$ the calculation

$$\int_{T_w} f\bar{g} = \sum_{l=0}^{\delta(j)-1} \int_{S_{w,l}} f(x_l)\bar{g}(x_l)$$

$$= \sum_{l=0}^{\delta(j)-1} \int_{S_{w,0}} f(x_0)\bar{g}(x_0)e^{-2\pi ilm/\delta(j)} = 0, \quad m = 1,\dots,\delta(j)-1,$$

shows that $U_{w,m}$ is orthogonal to functions f which are radial on T_w.

If $k \ne m$ the orthogonality of functions $f \in U_{v,k}$ and $g \in U_{v,m}$ is established with a similar computation,

$$\int_{T_v} f\bar{g} = \sum_{l=0}^{\delta(j)-1} \int_{S_{v,0}} f(x_0)e^{2\pi ilk/\delta(j)}\bar{g}(x_0)e^{-2\pi ilm/\delta(j)}$$

$$= \int_{S_{v,0}} f(x_0)\bar{g}(x_0) \sum_{l=0}^{\delta(j)-1} e^{2\pi il(k-m)/\delta(j)} = 0, \quad k - m \ne 0 \mod \delta(j).$$

Turning to the denseness of the linear span of the spaces $U_{v,k}$, define

$$V_j = \oplus_{v,k} U_{v,k}, \quad \mathrm{depth}(v) \le j.$$

The main idea is to show that for each nonnegative integer j the subspace V_j includes all functions vanishing below the vertices with combinatorial depth $j+1$

and all functions which are radial on each $S_{v,l}$ if the combinatorial depth of v is j. The proof is by induction.

For $j = 0$ we want to show that any function f supported on an edge (R, m) incident on R may be written as a linear combination of functions $f_k \in U_{R,k}$, $k = 0, \dots \delta(0) - 1$. For t_0 in edge $(R, 0)$ and t_m in edge (R, m) at the same depth, define f_k by $f_k(t_0) = f(t_m)$. Another discrete Fourier transform calculation gives

$$\frac{1}{\delta(0)} \sum_{k=0}^{\delta(0)-1} e^{-2\pi ikm/\delta(0)} f_k(t_l) \tag{4}$$

$$= \frac{1}{\delta(0)} \sum_{k=0}^{\delta(0)-1} e^{-2\pi ikm/\delta(0)} e^{2\pi ikl/\delta(0)} f_k(t_0) = \begin{Bmatrix} f(t_m) & l = m \\ 0 & l \neq m \end{Bmatrix}.$$

Similarly, the linear combinations of functions $f_k \in U_{R,k}$ includes all functions which are radial on each $S_{R,l}$.

To complete the argument suppose the induction hypothesis is true for $i < j$. For a vertex v with combinatorial depth j the radial functions on T_v are in V_{j-1} by the induction hypothesis. With the addition of the subspaces $U_{v,k}$ for $k = 1, \dots, \delta(j) - 1$ the argument used for the root R may be adopted with trivial modifications to handle the general case.

□

Consider next how the subspaces $U_{v,k}$ may be used to reduce certain differential operators on the tree $\mathcal{T}$ to a sequence of differential operators with interior point conditions on intervals $\mathcal{I}_n$. Take a vertex v with combinatorial depth n and metric depth x_n. Let l_{j+1} be the length of the edges joining vertices at combinatorial depth j to vertices at combinatorial depth $j+1$. For $j \geq n$ define a sequence of real numbers x_j by $x_{j+1} = x_j + l_{j+1}$, and take $\mathcal{I}_n = \cup_j [x_j, x_{j+1}]$.

For $x \in \mathcal{I}_n$, the mapping which sends $f \in U_{v,k}$ to its value $f(t)$ at a point $t \in S_{v,0}$ with metric depth x in $\mathcal{T}$ is an isometric bijection from $U_{v,k}$ to a weighted space $L^2(\mathcal{I}_n, w_n)$. The weight function $w_n(x)$ is equal to $N_n^{-1} N_{j+1}$ on the interval $[x_j, x_{j+1})$ where as before $N_j = \prod_{k=0}^{j-1} \delta(k)$. The weighted inner product is

$$\langle f, g \rangle = \sum_{j \geq n} \int_{x_j}^{x_{j+1}} w_n(x) f(x) \bar{g}(x) \; dx.$$

Self adjoint operators $\mathcal{L} = -D^2 + q$ on $L^2(\mathcal{T})$ may be constructed in the following manner. For the given radial potential q, find self adjoint operators $\mathcal{L}_n = -D^2 + q$ on $L^2(\mathcal{I}_n, w_n)$. Use the identification of $L^2(\mathcal{I}_n, w_n)$ with the spaces $U_{v,k}$ to map f in the domain of $\mathcal{L}_n$ into $L^2(\mathcal{T})$, and similarly identify $\mathcal{L}_n f$ with $\mathcal{L} f$. To satisfy the vertex conditions (2) the functions in the domain of $\mathcal{L}_n$ must satisfy the jump conditions

$$f(x_j^-) = f(x_j^+), \quad f'(x_j^-) = \delta(j) f'(x_j^+), \quad j > n.$$

In addition there are vertex conditions at v which must be satisfied. For vertices v other than the root, functions in $U_{v,k}$ vanish in the complement of T_v, so we must have the boundary condition $f(x_n) = 0$. The required vanishing of the sum of the derivatives at v is always satisfied in $U_{v,k}$ since

$$\sum_{l=0}^{\delta(n)-1} e^{2\pi ikl/\delta(n)} = 0, \quad k = 1, \dots, \delta(n) - 1.$$

The same considerations apply at the root for the spaces $U_{R,k}$ if $k \neq 0$. When $k = 0$ there are two cases to consider. If $\delta(0) = 1$ then any of the classical Sturm-Liouville conditions $a_1 f(x_0) + b_1 f(x_0) = 0$ with $a_1, b_1 \in \mathbb{R}$ may be imposed. If $\delta(0) > 1$ the interior vertex conditions (2) must be satisfied at the root. This can be achieved for the subspace $U_{R,0}$ by imposing the condition $f'(x_0) = 0$.

4 Solving $-y'' + qy = \lambda y$ with jump conditions

The symmetries of a radial tree have provided a decomposition of $L^2(\mathcal{T})$ into orthogonal subspaces $U_{v,k}$ which may be identified with a weighted Hilbert space $L^2(\mathcal{I}_n, w_n)$ on a real interval $\mathcal{I}_n$. By means of this identification, certain self adjoint operators $\mathcal{L}_n = -D^2 + q$ on $L^2(\mathcal{I}_n, w_n)$ may be used to construct self adjoint operators $\mathcal{L} = -D^2 + q$ on $L^2(\mathcal{T})$. Functions in the domain of $\mathcal{L}_n$ are required to satisfy the interior point jump conditions

$$f(x_j^-) = f(x_j^+), \quad f'(x_j^-) = \delta(j) f'(x_j^+), \quad j > n. \tag{5}$$

In addition, one of the left endpoint boundary conditions $a_1 f(x_n) + b_1 f(x_n) = 0$ is imposed.

This section will provide an analysis of solutions to (1) satisfying (5). It is convenient to define $x_\infty = \lim_j x_j$; the value will be $+\infty$ when $\sum_j l_j = \infty$. The behaviour of solutions to (1) as $x \to x_\infty$ has implications for the explicit description of domains for the operators $\mathcal{L}_n$ in terms of generalized boundary conditions at x_∞. The growth of solutions as $|\lambda| \to \infty$ will be used to analyze the distribution of eigenvalues. In the discussion of operators $\mathcal{L}_n = -D^2 + q$ on $L^2(\mathcal{I}_n, w_n)$ the functions q are still assumed to be real valued and measurable, but the previous boundedness requirement will be relaxed.

For notational convenience the sequence $x_n, x_{n+1}, \dots$ in $\mathcal{I}_n$ will be reindexed as $x_0, x_1, \dots$. The same reindexing will apply to interval lengths $l_j = x_j - x_{j-1}$, the branching numbers $\delta(j)$, and the edge counts N_j. The weight w_n will simply be denoted $w(x)$.

A piecewise linear rescaling of variables converts the operator $-D^2 + q(x)$ subject to the jump conditions (5) into a more conventional form. Define

$$\xi = \frac{x}{N_{j+1}} + \sum_{k=1}^{j} \frac{x_k}{N_k}, \quad x_j \le x < x_{j+1}, \quad N_j = \prod_{i=0}^{j-1} \delta(i). \tag{6}$$

Let $\xi_j = \xi(x_j)$ and observe that $\xi_{j+1} - \xi_j = (x_{j+1} - x_j)/N_{j+1}$. If f satisfies the jump conditions (5) then $F(\xi) = f(x)$ is continuous with a continuous derivative on $[\xi_0, \xi_\infty)$.

Similarly, if $Y(\xi) = y(x)$, $Q(\xi) = q(x)$ and

$$W(\xi) = N_{j+1}, \quad \xi_j \le x < \xi_{j+1},$$

then the equation

$$-y'' + q(x)y = \lambda y$$

becomes

$$-Y'' + W(\xi)^2 Q(\xi) Y = \lambda W(\xi)^2 Y. \tag{7}$$

The usual reduction to an integral equation and the method of successive approximations may be applied to the equation in this form. If $z_1 = Y$, $z_2 = Y'$ then the integral equation is

$$\begin{pmatrix} z_1(\xi) \\ z_2(\xi) \end{pmatrix} = \begin{pmatrix} z_1(\xi_0) \\ z_2(\xi_0) \end{pmatrix} + \int_{\xi_0}^{\xi} \begin{pmatrix} 0 & 1 \\ W(s)^2[Q(s) - \lambda] & 0 \end{pmatrix} \begin{pmatrix} z_1(s) \\ z_2(s) \end{pmatrix} \, ds.$$

Notice in particular that

$$\int_{\xi_0}^{\xi_j} W(s)^2 \, ds = \sum_{i=0}^{j-1} N_{i+1}^2 (\xi_{i+1} - \xi_i) = \sum_{i=0}^{j-1} N_{i+1}(x_{i+1} - x_i) = \sum_{i=0}^{j-1} N_{i+1} l_{i+1},$$

so that the function W^2 will be integrable on $[\xi_0, \xi_\infty)$ if the graph has finite volume. If in addition $W(\xi)^2 Q(\xi)$ is integrable on $[\xi_0, \xi_\infty)$ the usual Picard iteration method yields a sequence of successive approximations which converge uniformly to the desired solution on $[\xi_0, \xi_\infty)$. [8, p. 97–98]

4.1 Basic description of solutions to (1)

The jump conditions (5) determine the initial data $y(x_j^+), y'(x_j^+)$ from the data $y(x_j^-), y'(x_j^-)$, so solutions on one subinterval $[x_j, x_{j+1}]$ have a unique continuation to $\mathcal{I}_n$. In particular this shows that the space of solutions to (1) on $\mathcal{I}_n$ satisfying (5) has dimension 2 as a complex vector space.

On each interval $[x_j, x_{j+1}]$ the space of solutions of (1) has a basis $c(x, x_j, \lambda)$, $s(x, x_j, \lambda)$ satisfying

$$\begin{aligned} c(x_j, x_j, \lambda) = 1, &\quad s(x_j, x_j, \lambda) = 0, \\ c'(x_j, x_j, \lambda) = 0, &\quad s'(x_j, x_j, \lambda) = 1. \end{aligned} \tag{8}$$

In addition a basis $c(x, \lambda), s(x, \lambda)$ may be obtained by continuation of the basis $c(x, x_0, \lambda), s(x, x_0, \lambda)$ to the entire interval $\mathcal{I}_n$.

The continuation of solutions of (1) from $[x_0, x_1]$ to subsequent intervals $[x_j, x_{j+1}]$ may be described using a sequence of transition matrices

$$\tau_j = \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1}(j) \end{pmatrix}.$$

At x_1^- the values and derivatives for the functions c and s are the columns of the 2×2 matrix

$$\begin{pmatrix} c(x_1, x_0, \lambda) & s(x_1, x_0, \lambda) \\ c'(x_1, x_0, \lambda) & s'(x_1, x_0, \lambda) \end{pmatrix}.$$

The matrix τ_1 takes the vector of initial data at x_1^- to that at x_1^+ so that the jump conditions (5) are satisfied:

$$\begin{pmatrix} y(x_1^+) \\ y'(x_1^+) \end{pmatrix} = \tau_1 \begin{pmatrix} y(x_1^-) \\ y'(x_1^-) \end{pmatrix}.$$

The solutions $c(x, \lambda)$ and $s(x, \lambda)$ on the interval $x_1 \le x < x_2$ are given by

$$(c(x, x_1, \lambda), s(x, x_1, \lambda))\tau_1 \begin{pmatrix} c(x_1, x_0, \lambda) & s(x_1, x_0, \lambda) \\ c'(x_1, x_0, \lambda) & s'(x_1, x_0, \lambda) \end{pmatrix}.$$

By induction the next result is established.

Lemma 4.1. *On the interval $x_j \le x < x_{j+1}$ the solution matrix (c, s) for* (1) *has the form*

$$(c(x, \lambda), s(x, \lambda))$$

$$= (c(x, x_j, \lambda), s(x, x_j, \lambda)) \prod_{i=1}^{j} \tau_i \begin{pmatrix} c(x_i, x_{i-1}, \lambda) & s(x_i, x_{i-1}, \lambda) \\ c'(x_i, x_{i-1}, \lambda) & s'(x_i i, x_{i-1}, \lambda) \end{pmatrix}.$$

Consistent with the usage in this lemma, matrix products are assumed to have factors whose indices decrease from left to right.

The solutions $c(x, x_j, \lambda), s(x, x_j, \lambda)$ may be compared in a standard way ([13], [27, p. 13]) to the elementary functions $\cos(\omega[x - x_j]), \omega^{-1} \sin(\omega[x - x_j])$, where $\omega = \sqrt{\lambda}$. Let $\Im(\omega)$ denote the imaginary part of ω. Usually these estimates emphasize the λ dependence, but we will also need to make the x dependence explicit. For this reason a sketch of the proof is provided.

Lemma 4.2. *Define*

$$C_q(x) = \exp(\int_{x_j}^{x} |q(t)| \, dt) - 1, \quad x_j \le x \le x_{j+1}.$$

If $|x - x_j| \le 1$ the solutions $c(x, x_j, \lambda), s(x, x_j, \lambda)$ of (1) *satisfy*

$$|c(x, x_j, \lambda) - \cos(\omega[x - x_j])| \le |\omega^{-1}| e^{|\Im\omega|[x - x_j]} C_q(x),$$

$$|c'(x, x_j, \lambda) + \omega \sin(\omega[x - x_j])| \le e^{|\Im\omega|[x - x_j]} C_q(x),$$

$$|s(x, x_j, \lambda) - \omega^{-1} \sin(\omega[x - x_j])| \le |\omega^{-2}| e^{|\Im\omega|[x - x_j]} C_q(x),$$

$$|s'(x, x_j, \lambda) - \cos(\omega[x - x_j])| \le |\omega^{-1}| e^{|\Im\omega|[x - x_j]} C_q(x).$$

Proof. There is no loss of generality if we take $x_j = 0$. By using the variation of parameters formula, a solution of (1) satisfying $y(0,\lambda) = \alpha$, $y'(0,\lambda) = \beta$, with $\alpha, \beta \in \mathbb{C}$, may be written as a solution of the integral equation

$$y(x,\lambda) = \cos(\omega x)\alpha + \frac{\sin(\omega x)}{\omega}\beta + \int_0^x \frac{\sin(\omega[x-t])}{\omega} q(t)y(t,\lambda)\ dt. \tag{9}$$

Differentiation with respect to x gives

$$y'(x,\lambda) = -\omega\sin(\omega x)\alpha + \cos(\omega x)\beta + \int_0^x \cos(\omega[x-t])q(t)y(t,\lambda)\ dt. \tag{10}$$

Start with the elementary estimates

$$|\sin(\omega x)|, |\cos(\omega x)| \le e^{|\Im\omega|x}, \quad |\omega^{-1}\sin(\omega x)| = |\int_0^x \cos(\omega t)\ dt| \le xe^{|\Im\omega|x}.$$

For $c(x,0,\lambda)$ the integral equation (9) and the assumption $|x| \le 1$ give

$$|e^{-|\Im\omega|x}c(x,0,\lambda)| \le 1 + \int_0^x |q(t)|e^{-|\Im\omega|t}|c(t,0,\lambda)|\ dt.$$

By Gronwall's inequality [15, p. 24]

$$|e^{-|\Im\omega|x}c(x,0,\lambda)| \le \exp(\int_0^x |q(t)|\ dt).$$

Thus (9) implies that

$$|c(x,0,\lambda) - \cos(\omega x)| \le |\omega^{-1}|e^{|\Im\omega|x}\int_0^x |q(t)|\exp(\int_0^t |q(s)|\ ds)\ dt$$

$$= |\omega^{-1}|e^{|\Im\omega|x}[\exp(\int_0^x |q(t)|\ dt) - 1].$$

There is a similar inequality for $s(x,0,\lambda)$ and (10) leads to the inequalities for $|y'|$. □

Elements of $\mathbb{C}^2$ are given the Euclidean norm, and 2×2 matrices A will have the standard operator norm

$$\|A\| = \sup_{\|z\|\le 1} \|Az\|, \quad z \in \mathbb{C}^2.$$

Introduce the matrix

$$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}.$$

If q is integrable and the interval $\mathcal{I}_n$ has finite length, then Lemma 4.2 implies

$$\|\Omega^{-1}\begin{pmatrix} c(x,x_j,\lambda) & s(x,x_j,\lambda) \\ c'(x,x_j,\lambda) & s'(x,x_j,\lambda)\end{pmatrix}\Omega - \begin{pmatrix} \cos(\omega[x-x_j]) & \sin(\omega[x-x_j]) \\ -\sin(\omega[x-x_j]) & \cos(\omega[x-x_j])\end{pmatrix}\|$$

$$= O(\omega^{-1}\int_{x_j}^{x_{j+1}} |q(x)|\ dx), \tag{11}$$

the estimates holding uniformly for λ bounded.

4.2 Asymptotics for $c(x,\lambda)$ and $s(x,\lambda)$

The products arising in Lemma 4.1 may be simplified. For brevity define

$$R_i(\omega) = \Omega^{-1}\begin{pmatrix} c(x_i,x_{i-1},\lambda) & s(x_i,x_{i-1},\lambda) \\ c'(x_i,x_{i-1},\lambda) & s'(x_i,x_{i-1},\lambda)\end{pmatrix}\Omega.$$

Since the matrices τ_j and Ω commute, we find that

$$\prod_{i=1}^{j}\tau_i\begin{pmatrix} c(x_i,x_{i-1},\lambda) & s(x_i,x_{i-1},\lambda) \\ c'(x_i,x_{i-1},\lambda) & s'(x_i,x_{i-1},\lambda)\end{pmatrix} = \Omega\Big[\prod_{i=1}^{j}\tau_i R_i(\omega)\Big]\Omega^{-1}.$$

It will help to see that the matrix products have a limit as $j\to\infty$. The first result addresses the case when the tree $\mathcal{T}$ has finite metric depth.

Lemma 4.3. *Suppose that q is integrable and $\sum_j l_j < \infty$. As $j\to\infty$ the product $\prod_{i=1}^{j}\tau_i R_i(\omega)$ converges uniformly on compact subsets of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ to a meromorphic matrix function $M_1(\omega)$. If $\delta(j) > 1$ for infinitely many j, then for each ω we have* $\det(M_1(\omega)) = 0$.

Proof. Let K be a compact subset of $\mathbb{C}^*$, and write $R_i(\omega)$ as a perturbation of the identity, $R_i(\omega) = I + E_i(\omega)$. Now consider a product

$$\prod_{i=k}^{l}\tau_i R_i(\omega) = \prod_{i=k}^{l}[\tau_i + \tau_i E_i].$$

Expand the product as a sum, with each summand the product of $l-k+1$ matrices τ_i or $\tau_i E_i i$, and the first term being $\prod_{i=k}^{l}\tau_i$.

Using the fact that the matrix norm is subadditive and submultiplicative, the norm

$$\|\prod_{i=k}^{l}[\tau_i + \tau_i E_i] - \prod_{i=k}^{l}\tau_i\|$$

is bounded by the sum of the product of norms of the factors in the terms of the expanded sum. Noting that $\|\tau_i\|\le 1$, these terms are individually no greater than the corresponding terms in the expansion of $\prod_{i=k}^{l}(1+\|E_i\|)-1$. These observations lead to the estimate

$$\|\prod_{i=k}^{l}[\tau_i + \tau_i E_i] - \prod_{i=k}^{l}\tau_i\| \le \prod_{i=k}^{l}(1+\|E_i\|)-1, \quad k\le l \tag{12}$$

The estimate of (11) implies that

$$E_i = R_i - I = O(l_i) + O(\int_{x_{j-1}}^{x_j}|q|)$$

for $\omega \in K$. Since $\sum l_i < \infty$ and q is integrable, the sum $\sum_i \|E_i(\omega)\|$ converges uniformly for $\omega \in K$. This implies [1, p. 190] convergence of the products

$$\lim_{l \to \infty} \prod_{i=k}^{l} [1 + \|E_i(\omega)\|],$$

again uniformly for $\omega \in K$.

Based on these observations, (12) shows that the products $\prod_{i=k}^{l} [\tau_i + \tau_i E_i]$ are bounded independent of $l \geq k$, and moreover the difference

$$\prod_{i=k}^{l} [\tau_i + \tau_i E_i] - \prod_{i=k}^{l} \tau_i \tag{13}$$

goes to 0 as $k \to \infty$ independent of l as long as $l \geq k$. Notice that $\tau_i = I$ if $\delta(i) = 1$, while

$$\lim_{j \to \infty} \prod_{i=1}^{j-1} \tau_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{if} \quad \delta(i) > 1 \quad \text{infinitely often.}$$

The convergence argument is completed by considering

$$\| \prod_{i=1}^{l} [\tau_i + \tau_i E_i] - \prod_{i=1}^{k} [\tau_i + \tau_i E_i] \|$$

$$\leq \| \prod_{i=\lceil k \rceil/2+1}^{l} [\tau_i + \tau_i E_i] - \prod_{i=\lceil k \rceil/2+1}^{k} [\tau_i + \tau_i E_i] \| \, \| \prod_{i=1}^{\lceil k \rceil/2} [\tau_i + \tau_i E_i] \|.$$

The factor $\| \prod_{i=1}^{\lceil k \rceil/2} [\tau_i + \tau_i E_i] \|$ is bounded independent of k, and the first factor on the right of the inequality goes to 0 with k by (13). The products thus form a Cauchy sequence of analytic functions uniformly for $\omega \in K$.

Since the determinant is continuous from 2×2 matrices to $\mathbb{C}$, the limit matrix has determinant 0 if $\delta(i) > 1$ infinitely often. □

Notice that each of the matrix products $\Omega \Big[\prod_{i=1}^{j} \tau_i R_i(\omega) \Big] \Omega^{-1}$ arising in Lemma 4.1 is an entire function of λ. It follows from Lemma 4.3 that these products converge uniformly on any circle of positive radius centered at 0, so by the maximum principle they converge uniformly on any compact set in $\mathbb{C}$. This establishes the next corollary.

Lemma 4.4. *If q is integrable and $\sum_j l_j < \infty$, the products*

$$\prod_{i=1}^{j} \tau_i \begin{pmatrix} c(x_i, x_{i-1}, \lambda) & s(x_i, x_{i-1}, \lambda) \\ c'(x_i, x_{i-1}, \lambda) & s'(x_i, x_{i-1}, \lambda) \end{pmatrix} = \Omega \Big[\prod_{i=1}^{j} \tau_i R_i(\omega) \Big] \Omega^{-1}.$$

converge to an entire matrix function $M(\lambda)$ as $j \to \infty$.

Lemma 4.1 and Lemma 4.4 imply

$$\lim_{j\to\infty}\begin{pmatrix} c(x_j^+,\lambda) & s(x_j^+,\lambda) \\ c'(x_j^+,\lambda) & s'(x_j^+,\lambda)\end{pmatrix} = M(\lambda). \tag{14}$$

In case $\sum_{i=1}^{\infty} l_i N_i < \infty$ and $W^2(\xi)Q(\xi)$ is integrable on $[\xi_0,\xi\infty)$, we may take advantage of the change of variables $x \to \xi$ discussed at the beginning of this section. Let $C(\xi,\lambda)$ and $S(\xi,\lambda)$ be solutions of (7) satisfying

$$\begin{pmatrix} C(\xi_0,\lambda) & S(\xi_0,\lambda) \\ C'(\xi_0,\lambda) & S'(\xi_0,\lambda)\end{pmatrix} = I.$$

Because (7) is essentially regular on a finite interval, the matrix

$$\begin{pmatrix} C(\xi_\infty,\lambda) & S(\xi_\infty,\lambda) \\ C'(\xi_\infty,\lambda) & S'(\xi_\infty,\lambda)\end{pmatrix}$$

will be nonsingular. Consequently, either $\lim_{j\to\infty} c(x_j,\lambda)$ or $\lim_{j\to\infty} s(x_j,\lambda)$ will be nonzero, and $M(\lambda)$ is not the zero function. If $q = 0$ and $\sum_{i=1}^{\infty} l_i < \infty$ the same conclusion may be established by direct computation of $M(0)$.

Theorem 4.5. *Suppose that $\sum_j l_j < \infty$ and q is integrable. Then every solution of $-y'' + q(x)y = \lambda y$ on $[x_0, x_\infty)$ satisfying the jump conditions (5) is bounded. If in addition $M(\lambda)$ is not the zero function, then except possibly for a discrete set of $\lambda \in \mathbb{C}$ there are linearly independent solutions $y_1(x,\lambda), y_2(x,\lambda)$ satisfying*

$$\lim_{x\to x_\infty} y_1(x,\lambda) = \beta \neq 0, \quad \lim_{x\to x_\infty} y_2(x,\lambda) = 0.$$

If $\sum_j l_j < \infty$ and $\delta(j) > 1$ infinitely often, then every solution satisfies

$$\lim_{j\to\infty} y'(x_j^-,\lambda) = \lim_{j\to\infty} y'(x_j^+,\lambda) = 0,$$

and

$$\lim_{x\to x_\infty} y'(x,\lambda) = 0, \quad x \notin \{x_j\}.$$

Proof. By virtue of (14), for every $\lambda \in \mathbb{C}$ the functions $c(x,\lambda)$, $s(x,\lambda)$, $c'(x,\lambda)$, and $s'(x,\lambda)$ are bounded. Since $\lim_{j\to\infty} l_j = 0$, we find that

$$\lim_{x\to x_\infty}(c(x,\lambda),s(x,\lambda)) = \lim_{j\to\infty}(c(x_j^+,\lambda),s(x_j^+,\lambda)) = (M_{11}(\lambda),M_{12}(\lambda)).$$

Write

$$c'(x_{j+1}^-) - c'(x_j^+) = \int_{x_j^+}^{x_{j+1}^-} c''(t)dt = \int_{x_j^+}^{x_{j+1}^-}[q(t)-\lambda]c(t)dt. \tag{15}$$

Since $c(t,\lambda)$ is bounded and $q(t)$ is integrable, the condition $\sum_j l_j < \infty$, implies that

$$\lim_{j\to\infty} |c'(x_{j+1}^-) - c'(x_j^+)| = 0.$$

The jump condition gives $c'(x_j^+) = c'(x_j^-)/\delta(j)$, or

$$\lim_{j\to\infty} c'(x_j^-)/\delta(j) = \lim_{j\to\infty} c'(x_j^+) = \lim_{j\to\infty} c'(x_{j+1}^-) = \lim_{j\to\infty} c'(x_j^-),$$

which forces $\lim_{j\to\infty} c'(x_j^-), = 0$ if $\delta(j) > 1$ infinitely often, and consequently $\lim_{x\to x_\infty} c'(x) = 0$. The argument is the same for $s(x,\lambda)$, and so $\lim_{x\to x_\infty} y'(x,\lambda) = 0$ for any solution y.

Thus $M_{21}(\lambda) = 0 = M_{22}(\lambda)$. By assumption $M(\lambda)$ is not identically 0. If, for instance, $M_{11}(0) \ne 0$, the function $c(x,\lambda)$ satisfies

$$\lim_{x\to x_\infty} c(x,\lambda) = M_{11}(\lambda) \ne 0,$$

except possibly for a discrete set of $\lambda \in \mathbb{C}$. Thus we may take $y_1(x,\lambda) = c(x,\lambda)$, and $y_2(x,\lambda)$ may be selected from the null space of the functional $y(x_\infty)$. □

An additional growth estimate will be useful when the distribution of eigenvalues is considered.

Theorem 4.6. *Suppose that $\sum l_j < \infty$ and q is integrable. Then the matrix function $M(\lambda)$ is entire of order $1/2$.*

Proof. It will suffice to establish the desired estimate for the function $M_1(\omega) = \prod \tau_j R_j(\omega)$. Let

$$F_j = \begin{pmatrix} \cos(\omega l_j) & \sin(\omega l_j) \\ -\sin(\omega l_j) & \cos(\omega l_j) \end{pmatrix}, \quad l_j = x_j - x_{j-1},$$

and define $G_j = R_j - F_j$. Then we have

$$\|M_1(\omega)\| \le \prod \|\tau_j R_j(\omega)\| \le \prod \|R_j(\omega)\| \le \prod [\|F_j\| + \|G_j\|].$$

Notice that the matrix F_j is normal, with orthonormal eigenvectors

$$\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix},$$

and eigenvalues $\exp(\pm i\omega l_j)$. Thus

$$\|F_j\| = e^{|\Im \omega l_j|},$$

while the estimates of Lemma 4.2 give

$$\|G_j\| \le |\omega|^{-1}[\exp(\int_{x_{j-1}}^{x_j} |q|) - 1]e^{|\Im \omega l_j|}.$$

It follows that

$$\prod [\|F_j\| + \|G_j\|] \le e^{|\Im \omega \sum_j l_j|} \prod [1 + |\omega|^{-1} (\exp(\int_{x_{j-1}}^{x_j} |q|) - 1)],$$

and the product on the right is convergent uniformly for $|\omega| \ge 1$ since

$$|\exp(\int_{x_{j-1}}^{x_j} |q|) - 1| = O(\int_{x_{j-1}}^{x_j} |q|),$$

which is summable. As desired, there is a constant C_1 such that

$$\|M_1(\omega)\| \le C_1 e^{|\Im \omega \sum_j l_j|}.$$

□

5 Operator Theory

5.1 Deficiency Indices

This section is concerned with the identification of self adjoint boundary-value problems for $-D^2 + q$ on the interval $\mathcal{I}_n$. By means of the separation of variables results this will also provide self adjoint operators on the tree. When the function q is bounded the theory of deficiency indices [9] is helpful. This approach is used first. More singular cases are then treated for trees with finite volume.

In case q is bounded, consider the symmetric operator $S = -D^2 + q$ whose domain consists of smooth functions on $\mathcal{I}_n$ satisfying the jump conditions (5), having support in a finite set of intervals, and vanishing, along with their derivatives, at x_0 and x_∞. Recall that the dimensions of the deficiency subspaces $N(S^* - \lambda I)$ are constant for λ with positive, respectively negative imaginary part.

As one can see using the ideas in [6], elements of the deficiency subspaces must be classical solutions of the differential equation $-y'' + qy = \lambda y$ on each subinterval $[x_k, x_{k+1}]$ satisfying the jump conditions (5), hence the dimension of each deficiency subspace is no bigger than 2. Since S is bounded below, the deficiency indices are the same.

The operator S may be extended to a symmetric operator S_1 by replacing the requirement that functions and their derivatives vanish at x_0 with the classical boundary condition

$$af(0) + bf'(0) = 0, \quad a, b \in \mathbb{R}, \quad a^2 + b^2 > 0.$$

Since S_1 is a proper symmetric extension of the closure of S, the deficiency indices of S_1 must be either $(1, 1)$ or $(0, 0)$. To determine whether the operator is essentially self adjoint, or requires an additional boundary condition 'at ∞', it is necessary to consider bounds on the solutions to (1).

The fact that solutions of the equation $-y'' + qy = \lambda y$ have limits as $x \to x_\infty$ allows us to determine the deficiency indices of S_1. First of all, in the finite volume case $\sum N_j l_j < \infty$, all solutions of this equation are square integrable. This means that the deficiency indices of S are $(2,2)$, and those of S_1 are $(1,1)$. Now consider the case $\sum l_j < \infty$, $\sum N_j l_j = \infty$. If $q = 0$ then $M(\lambda)$ is not the zero function, and Theorem 4.5 says that for all but a discrete set of $\lambda \in \mathbb{C}$ there is a solution of (1) which has a nonzero limit at x_∞. Such a solution cannot be in $L^2(\mathcal{I}_n, w)$, so the deficiency indices of S in this case must be either $(0,0)$ or $(1,1)$. Since S_1 is a proper symmetric extension of S, it must be essentially self adjoint. The addition of the bounded operator multiplication by q does not change the self adjointness. We summarize with the next result.

Theorem 5.1. *Suppose that q is bounded, and the edge lengths l_j satisfy $\sum l_j < \infty$. If $\sum N_j l_j < \infty$ the deficiency indices of S are $(2,2)$. If $\sum N_j l_j = \infty$ the deficiency indices of S are $(1,1)$. In case $\sum N_j l_j = \infty$ the operator $\mathcal{L}_n = -D^2 + q$ whose domain is the set of functions in the domain of S^* satisfying the boundary conditions*

$$af(x_0) + bf'(x_0) = 0, \quad a, b \in \mathbb{R}, \quad a^2 + b^2 > 0,$$

is a self adjoint operator on $L^2(\mathcal{I}_n, w)$.

5.2 Trees with finite volume

For trees with finite volume it is particularly convenient to use the change of variables $x \to \xi$ of (6). This change of variables provides a Hilbert space isometry from $L^2(I_n, w)$ onto $L^2([\xi_0, \xi_\infty), W^2(\xi))$ since

$$\int_{I_n} f(x)\overline{g(x)}w(x)\ dx = \int_{\xi_0}^{\xi_\infty} F(\xi)\overline{G(\xi)}W^2(\xi)\ d\xi.$$

The quadratic form for the operator S becomes

$$\int_{I_n} (|f'|^2 + q(x)|f|^2)w(x)\ dx = \int_{\xi_0}^{\xi_\infty} (W(\xi)^{-2}|F'(\xi)|^2 + Q(\xi)|F(\xi)|^2)W^2(\xi)\ d\xi.$$

If $q(x)$ is merely integrable rather than being bounded, the description of operator domains becomes more delicate. The quadratic form approach for singular ordinary differential operators may be found in [16, p. 343].

For our purposes it will be convenient to directly construct the Green's function for the boundary-value problem

$$-Y'' + W(\xi)^2[Q(\xi) - \lambda]Y = W(\xi)^2 F(\xi), \tag{16}$$

$$a_1 Y(\xi_0) + b_1 Y'(\xi_0) = 0, \quad a_2 Y(\xi_\infty) + b_2 Y'(\xi_\infty) = 0.$$

where $a_i, b_i \in \mathbb{R}$ and $a_i^2 + b_i^2 > 0$.

Let $\mathcal{D}$ denote the set of functions $G \in L^2([\xi_0, \xi_\infty), W^2(\xi))$ which are continuous, with absolutely continuous derivative, and such that $[W(\xi)^{-2}D^2 + Q]G \in L^2([\xi_0, \xi_\infty), W^2(\xi))$.

Theorem 5.2. *Assume that* $\sum N_j l_j < \infty$ *and that* $W^2(\xi)Q(\xi)$ *is integrable on* $[\xi_0, \xi_\infty)$. *The functions in* $\mathcal{D}$ *which also satisfy a set of boundary conditions in* (16) *is a domain on which the operator* $\mathcal{L}_n = W(\xi)^{-2}D^2 + Q$ *is self adjoint with compact resolvent on* $L^2([\xi_0, \xi_\infty), W^2(\xi))$.

Proof. Since the argument is straightforward the proof is merely outlined. As noted earlier, (7) may be treated as a regular problem on the finite interval $[\xi_0, \xi_\infty)$. If $U(\xi, \lambda)$ and $V(\xi, \lambda)$ are nontrivial solutions of (7) satisfying the boundary conditions at ξ_0 and ξ_∞ respectively, then the solution $Y(\xi, \lambda)$ of (16) may be written as [3, p.309]

$$Y(\xi, \lambda) = \int_{\xi_0}^{\xi_\infty} G(\xi, \eta, \lambda) W^2(\eta) F(\eta) \; d\eta, \tag{17}$$

with

$$G(\xi, \eta, \lambda) = \begin{Bmatrix} U(\xi)V(\eta)/\sigma, & \xi_0 \le \xi \le \eta, \\ U(\eta)V(\xi)/\sigma, & \eta \le \xi \le \xi_\infty, \end{Bmatrix} \quad \sigma = VU' - UV'.$$

As in the classical case eigenvalues of (16) must be real, and are the roots of a nontrivial entire function. Except at the eigenvalues $\sigma \neq 0$, and the functions $U(\xi, \lambda)$ and $V(\xi, \lambda)$ are bounded on $[\xi_0, \xi_\infty)$. The condition $\sum N_j l_j < \infty$ implies that bounded measurable functions are in $L^2([\xi_0, \xi_\infty), W^2(\xi))$. If $F \in L^2([\xi_0, \xi_\infty), W^2(\xi))$ then $Y(\xi)$ given by (17) is bounded by the Cauchy-Schwartz inequality. That is, except at eigenvalues of (16) the integral operator $\mathcal{G}(\lambda)$ defined by (17) is bounded on $L^2([\xi_0, \xi_\infty), W^2(\xi))$, and is self adjoint for $\lambda \in \mathbb{R}$. The range of $\mathcal{G}(\lambda)$ defines a domain on which $W(\xi)^{-2}D^2 + Q$ is self adjoint, and $\mathcal{G}(\lambda)$ is its resolvent, which is compact since the spectrum is discrete. □

Suppose the hypotheses of Theorem 5.2 hold. The explicit formula shows that the resolvents for the boundary-value problems on $[\xi_0, \xi_\infty)$ are the strong limits of the resolvents [28, pp. 284–290] obtained by imposing the right end-point conditions at ξ_j, and taking the limit as $j \to \infty$. This gives the sense in which radial Schrödinger operators on infinite trees are the limits of finite tree operators.

To characterize the distribution of eigenvalues, let $n(r)$ be the number of eigenvalues λ_m with $|\lambda_m| \le r$.

Theorem 5.3. *The eigenvalues* λ_m *of an operator* $\mathcal{L}_n$ *as described in Theorem 5.2, counted with multiplicity, satisfy*

$$n(r) \le O(r^{1/2+\epsilon})$$

for every $\epsilon > 0$.

Proof. The function $a_2 z(\xi_\infty, \lambda) + b_2 z'(\xi_\infty, \lambda)$ whose roots are the eigenvalues of $\mathcal{L}$, is entire of order 1/2 by Theorem 4.6. Since each eigenvalue has multiplicity at most 2, the result follows from the analogous result for the roots of an entire function of order 1/2, [36, p. 64]. □

Eigenvalue distributions for a wide variety of weighted Laplacians on trees are studied in [23].

Except for the implicit consequences of Proposition 2.4, we have not obtained any description of the dependence of the eigenvalues of $\mathcal{L}_n$ on n. In some special cases there is a simple spectral mapping relating the eigenvalues of $\mathcal{L}_n = -D^2$ and $\mathcal{L}_{n+k} = -D^2$. Suppose that for some $k > 0$ and $0 < r < 1$ the branching indices and lengths satisfy

$$\delta(n+k) = \delta(n), \quad l_{n+k} = rl_n, \quad n = 1, 2, \dots,$$

and that $\sum l_j N_j < \infty$. Assume that functions f in the domain of $\mathcal{L}_n$ satisfy the left endpoint boundary condition $f(x_n) = 0$. The conditions at x_∞, chosen independent of n, are either $f(x_\infty) = 0$ or $\lim_{j \to \infty} N_j f'(x_j) = 0$. The mapping

$$Y(x) = y(x_{n+k} + r(x - x_n))$$

takes functions y in the domain of $\mathcal{L}_{n+k}$ to Y in the domain of $\mathcal{L}_n$. If y is an eigenfunction for $\mathcal{L}_n = -D^2$ with eigenvalue λ, then Y is an eigenfunction for $\mathcal{L}_{n+k} = -D^2$ with eigenvalue $r^2\lambda$, and conversely. For these cases we obtain the relation

$$\mathrm{spec}(\mathcal{L}_n) = r^2 \mathrm{spec}(\mathcal{L}_{n+k}).$$

5.3 Mixed boundary conditions on the tree

In closing it is interesting to note that the techniques which have been developed here to reduce radial operators $\mathcal{L}$ on $L^2(\mathcal{T})$ may also be employed to consider operators with mixed boundary conditions 'at infinity'. As before the tree $\mathcal{T}$ and the potential q are radial. Suppose that $v(i)$, for $i = 1, \dots, N_j$, are the distinct vertices with combinatorial depth $j > 0$. Assume that $\sum N_j l_j < \infty$, and that for each i a boundary condition

$$a_i F(\xi_\infty) + b_i F'(\xi_\infty) = 0, \quad a_i, b_i \in \mathbb{R}$$

is given.

The determination of a domain for this operator begins with a change of the interior vertex conditions at the vertices $v(i)$. We impose the Dirichlet conditions $f(v(i)) = 0$ rather than the conditions (2). This new set of conditions decouples the tree into a finite collection of $N_j + 1$ subtrees, each of which is a radial tree with roots $v(i)$ or R. The operator $-D^2 + q$ is now radial on each subtree, so the previous analysis can be employed to identify self adjoint domains.

To recover the operator with mixed boundary conditions 'at infinity', first take the orthogonal direct sum of self adjoint operators from the subtrees. Now replace the Dirichlet boundary conditions at the vertices $v(i)$ with the original interior vertex conditions. The two operators are finite symmetric extensions of a common operator, and the techniques of [16, p. 188] show that they are both self adjoint.

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