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Behavior of forced asymmetric oscillators at resonance *

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Abstract

This article collects recent results concerning the behavior at resonance of forced oscillators driven by an asymmetric restoring force, with or without damping. This synthesis emphasizes the key role played by a function denoted by $\Phi_{\alpha,\beta,p}$, which is, up to a sign reversal of its argument, a correlation product of the forcing term p and of a function representing a free oscillation for theundamped equation. The theoretical results are accompanied by graphical representations illustrating the behavior of the damped and undamped oscillators. In particular, the damped oscillator is considered, with a forcing term whose frequency is close to the frequency of the free oscillations. For that problem, frequency-response curves are studied, both theoretically and through numerical computations, revealing a hysteresis phenomenon, when $\Phi_{\alpha,\beta,p}$ is of constant sign.

1 Introduction

The oscillators studied here are represented by the equation

$$x'' + \alpha x^+ - \beta x^- = p(t), \tag{1}$$

where $x^+ = \max\{x, 0\}, x^- = \max\{-x, 0\}$, α, β being positive numbers. We also consider the equation with damping

$$x'' + \varepsilon x' + \alpha x^+ - \beta x^- = p(t). \tag{2}$$

These equations provide a fairly natural generalization of the classical linear oscillator, the restoring force being here piecewise linear. The interest for such equations has been motivated in particular by models of suspension bridges [11, 12].

We will consider only periodic forcing terms; it is convenient to work with the period 2π. We then speak of resonance when the period 2π is the period of the free oscillations, i.e. of the solutions of the homogeneous equation

$$x'' + \alpha x^+ - \beta x^- = 0. \tag{3}$$

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It is easy to see that this occurs when

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n}\,, \tag{4}$$

for some integer n. The curve defined by (4), which passes through the point (n^2, n^2), is one of the so-called Fučík curves [9, 10] for this boundary value problem; we will denote it by C_n. When $(\alpha, \beta) \in C_n$, all solutions of (3) are 2π-periodic; we will denote by $\varphi_{\alpha,\beta}$ the particular solution corresponding to the initial conditions $x(0) = 0, x'(0) = 1$; it is easily computed that

$$\varphi_{\alpha,\beta}(t) = \begin{cases} \dfrac{1}{\sqrt{\alpha}} \sin\left(\sqrt{\alpha}\, t\right) & \text{for} \quad t \in \left[0, \dfrac{\pi}{\sqrt{\alpha}}\right], \\ -\dfrac{1}{\sqrt{\beta}} \sin\left(\sqrt{\beta}\left(t - \dfrac{\pi}{\sqrt{\alpha}})\right)\right) & \text{for} \quad t \in \left[\dfrac{\pi}{\sqrt{\alpha}}, \dfrac{2\pi}{n}\right], \end{cases} \tag{5}$$

$\varphi_{\alpha,\beta}$ being of minimal period $2\pi/n$.

Despite the simplicity of equations (1), (2), the behavior of the solutions turns out to be more complicated that could be expected at first sight. As will appear from the results recalled below, the global picture depends crucially on the fact that the function $\Phi_{\alpha,\beta,p}$, defined by

$$\Phi_{\alpha,\beta,p}(\theta) := \frac{1}{2\pi} \int_0^{2\pi} p(t)\varphi_{\alpha,\beta}(t+\theta)\, dt \quad (\theta \in \mathbb{R}), \tag{6}$$

vanishes or not at some point; that function has been introduced by Dancer [2, 3]. Notice that, if $(\alpha, \beta) \in C_n$, the function $\Phi_{\alpha,\beta,p}$, as $\varphi_{\alpha,\beta}$, is of period $2\pi/n$; its number of zeros in the interval $[0, 2\pi/n)$ will be determining for the behavior of the oscillator.

The paper is organized as follows. Section 2 is a short presentation of formulas concerning the function $\Phi_{\alpha,\beta,p}$, that will be needed later. In Section 3, we present results concerning the equation without damping (1), whereas Section 4 is devoted to the equation with damping (2). In Section 5, we also discuss the equation with damping, when the forcing term is no longer of period 2π, but of period close to 2π; we will study the variation of the amplitude of 2π-periodic solutions with respect to the frequency of the forcing term and draw frequency-response curves.

2 The function $\Phi_{\alpha,\beta,p}$

We list here, for later reference, some results about the function $\Phi_{\alpha,\beta,p}$. Remember that we are particularly interested in the number of zeros of $\Phi_{\alpha,\beta,p}$ in the interval $[0, 2\pi/n)$.

If (4) is satisfied with $\alpha = \beta = n$, the differential equation (3) is linear and we have $\varphi_{n,n}(t) = \sin(nt)/n$; the function $\Phi_{n,n,p}(\theta)$ is then, up to a factor $2/n$, made of the Fourier components of order n of the function p. Hence, $\Phi_{n,n,p}$ is of the form $A\cos(nt) + B\sin(nt)$; it is identically zero when the corresponding

Fourier coefficients A, B are equal to 0; otherwise, it has 2 zeros in the interval $[0, 2\pi/n)$. Clearly, the function $\Phi_{n,n,p}(\theta)$ cannot be of constant sign, a situation which, by contrast, can occur when $\alpha \neq \beta$.

In the examples considered below for $\alpha \neq \beta$, we will deal for instance with right-hand sides p of the form $p(t) = a + b\cos(kt)$. For such a function p, assuming that (α, β) satisfy (4) with $n = 1$ and that $\alpha \neq \beta$, it is computed that

$$\begin{aligned} \Phi_{\alpha,\beta,p}(\theta) &= \frac{a}{\pi}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \\ &+\frac{b}{2\pi}\,\frac{\beta-\alpha}{(k^2-\beta)(k^2-\alpha)}\left\{\left[1+\cos\left(\frac{k\pi}{\sqrt{\alpha}}\right)\right]\cos(kt)+\sin\left(\frac{k\pi}{\sqrt{\alpha}}\right)\sin(kt)\right\}. \end{aligned} \tag{7}$$

The function $\Phi_{\alpha,\beta,p}$ is of constant sign if

$$\frac{|a|}{|b|} > \frac{\alpha\beta}{|k^2-\beta||k^2-\alpha|}\cos\left(\frac{k\pi}{2\sqrt{\alpha}}\right), \tag{8}$$

the sign being that of $a(\beta - \alpha)$. It immediately results from the expression (7) that, when the inequality (8) is reversed, $\Phi_{\alpha,\beta,p}$ has $2k$ simple zeros in $[0, 2\pi)$.

3 Forced oscillator without damping

Concerning equation (1), with a forcing term p of period 2π, the comparison with the linear oscillator suggests the following two questions:

- For which p does the problem have 2π-periodic solutions?
- For which p does the problem have unbounded solutions?

A fairly complete answer to those questions is provided by the results recalled below. We start with sufficient conditions for the existence of 2π-periodic solutions.

Proposition 1 ([6]) *Assume that $(\alpha, \beta) \in C_n$, and that p is continuous and 2π-periodic. Then, if $\Phi_{\alpha,\beta,p}$ has $2z$ zeros in $[0, 2\pi/n)$, all zeros being simple, and if $z \neq 1$, equation (1) has (at least) one 2π-periodic solution.*

Notice that, since $\Phi_{\alpha,\beta,p}$ is $2\pi/n$-periodic, the number of zeros in $[0, 2\pi/n)$, if they are simple, must be even. The above result contains of course the case of a function $\Phi_{\alpha,\beta,p}$ of constant sign, a case already treated by Dancer [2, 3]. It also follows from Proposition 1 that the only situations where equation (1) can have no 2π-periodic solution, are the case where $\Phi_{\alpha,\beta,p}$ has 2 zeros in $[0, 2\pi/n)$, and the case where $\Phi_{\alpha,\beta,p}$ has multiple zeros. For the first case, it is possible to exhibit 2π-periodic forcing p such that equation (1) has no 2π-periodic solution (see [2, 3, 11, 17]). In particular, Wang Zaihong [17], extending results of Lazer and McKenna [11], has shown that, if $(\alpha, \beta) \in C_n$,

$$x'' + \alpha u^+ - \beta u^- = \cos nt$$

has no 2π-periodic solution. We do not know if functions p exist, for which $\Phi_{\alpha,\beta,p}$ has multiple zeros and for which equation (1) has no 2π-periodic solution.

On the other hand, it can be shown that, with $(\alpha, \beta) \in C_n$, the set of 2π-periodic solutions of equation (1) is bounded, unless $\Phi_{\alpha,\beta,p}$ has multiple zeros. Examples of unbounded sets of 2π-periodic solutions are however easy to construct. For instance, if ψ is a twice differentiable function such that $\psi(t)\varphi_{\alpha,\beta}(t) > 0$, for all t such that $\varphi_{\alpha,\beta}(t) \neq 0$, and if we define p by $p(t) = \psi''(t) + \alpha\psi^+(t) - \beta\psi^-(t)$, it is easy to verify that, for any $k > 0$, $\psi + k\varphi_{\alpha,\beta}$ is a solution of (1). With that choice of p, the function $\Phi_{\alpha,\beta,p}$ can be seen to be identically zero.

It has been observed by Ortega [15] that, because of a result of Massera [14], if equation (1) admits no 2π-periodic solutions, then all solutions must be unbounded. That condition is of course not necessary to have unbounded solutions. Indeed, a result of Alonso and Ortega [1] shows that, as soon as $\Phi_{\alpha,\beta,p}$ has zeros (assumed to be simple), equation (1) will admit unbounded solutions.

Proposition 2 *If the function $\Phi_{\alpha,\beta,p}$ takes both positive and negative values, and if all its zeros are simple, then there exists $R > 0$ such that every solution x of equation (1) with*

$$x(t_0)^2 + x'(t_0)^2 > R$$

for some $t_0 \in \mathbb{R}$, is such that

$$x(t)^2 + x'(t)^2 \to +\infty$$

as $t \to +\infty$ or $t \to -\infty$.

Combining Propositions 1 and 2, it clearly appears that, when $\alpha \neq \beta$, equation (1) can have both 2π-periodic and unbounded solutions. It is the case for instance for the equation

$$x'' + \alpha x^+ - \beta x^- = \cos(3t),$$

with $(\alpha, \beta) \in C_1, \alpha \neq \beta$. Indeed, it can be shown that, with $p(t) = \cos(3t)$, we have $\Phi_{\alpha,\beta,p}(t) = A_{\alpha,\beta}\cos(3t) + B_{\alpha,\beta}\sin(3t)$, with $A^2_{\alpha,\beta} + B^2_{\alpha,\beta} \neq 0, \forall(\alpha, \beta) \in C_1, \alpha \neq \beta$. Both Proposition 1 and Proposition 2 thus apply to that equation.

By contrast to Proposition 2, it has been proved by Liu Bin [13] that, when $\Phi_{\alpha,\beta,p}$ is of constant sign, and provided that p is sufficiently regular, all solutions of (1) are bounded.

Proposition 3 *If p is of class C^6, and if the function $\Phi_{\alpha,\beta,p}$ is of constant sign, then all solutions of equation (1) are such that*

$$\sup_{t\in\mathbb{R}}\{|x(t)| + |x'(t)|\} < \infty.$$

In order to gain some understanding of the behavior of solutions corresponding to large initial conditions, let us use the change of variables

$$x(t) = \rho u(t)\varphi_{\alpha,\beta}(t+\theta(t)), x'(t) = \rho u(t)\varphi'_{\alpha,\beta}(t+\theta(t))\,; \tag{9}$$

we will assume that $\rho > 0$ is "large", but fixed, and consider initial conditions like $u(0) = 1, \theta(0) = \theta_0$. That change of variables transforms (1) into

$$u' = \frac{1}{\rho}\, p(t)\varphi'_{\alpha,\beta}(t+\theta) \tag{10}$$

$$\theta' = \frac{1}{\rho u}\, p(t)\varphi_{\alpha,\beta}(t+\theta). \tag{11}$$

The factor $1/\rho$ being small, this can be considered a weakly nonlinear system to which the method of averaging can be applied. The averaged system, for which the variables will be denoted σ, τ, writes

$$\sigma' = \frac{1}{\rho}\, \Phi'_{\alpha,\beta,p}(\tau) \tag{12}$$

$$\tau' = -\frac{1}{\rho\sigma}\, \Phi_{\alpha,\beta,p}(\tau). \tag{13}$$

If we take the initial conditions $\sigma(0) = u(0), \tau(0) = \theta(0)$, the method of averaging (see [16]) ensures that $\sigma(t), \tau(t)$ are approximations of $u(t), \theta(t)$, with an error which is of the order of $1/\rho$, on an interval whose length is of the order of ρ. But,

$$\sigma(t)\Phi_{\alpha,\beta,p}(\tau(t)) = C \tag{14}$$

is a first integral for the system (12), (13). It then follows that, $u(t)\Phi_{\alpha,\beta,p}(\theta(t))$ will be close to a constant on an interval of length of the order of ρ. The following conclusions can be drawn from that observation.

1$^{\text{st}}$ CASE. We discuss first the case where $\Phi_{\alpha,\beta,p}$ is of constant sign. Since $u(t)\Phi_{\alpha,\beta,p}(\theta(t))$ remains close to a constant C on a interval of the order of ρ, $u(t)$ will, on such an interval, oscillate roughly between

$$\frac{|C|}{\max|\Phi_{\alpha,\beta,p}|} \text{ and } \frac{|C|}{\min|\Phi_{\alpha,\beta,p}|},$$

the constant C depending on the initial conditions. Since, by (9), ρu can be interpreted as the amplitude of the solutions, this means that a beating phenomenon is observed. Moreover, since u' is proportional to $1/\rho$ and since the amplitude is of the order of ρ, the amplitude fluctuations are slower for larger solutions. Figure 1 represents the actual behavior of a solution for the equation

$$x'' + \alpha x^+ - \beta x^- = -2 + \cos t,$$

with $\alpha = 2$, and $(\alpha, \beta) \in C_1$. Using formula (8) and the remark following it, it is easily seen that, for the choice of α, β, p made here, $\Phi_{\alpha,\beta,p}$ is positive.

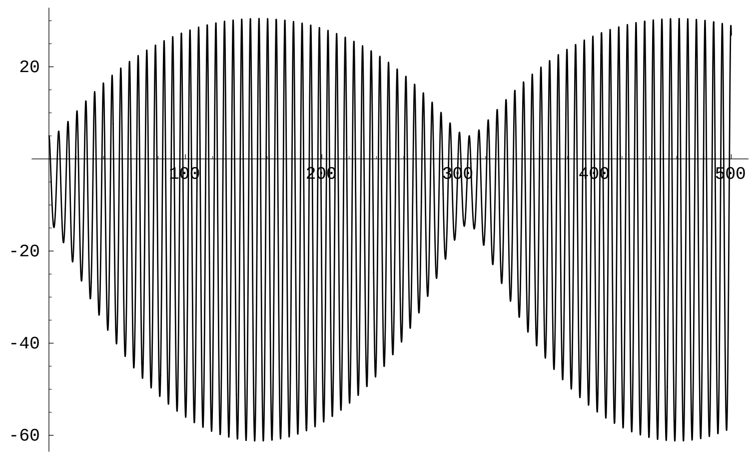

Figure 1: Beating phenomenon

The proof of Proposition 3 is based on a version of the so-called "twist theorem" of Kolmogorov-Arnold-Moser. To get an idea of the twist conditions involved, observe that, by (12), (13), (14) we have

$$\frac{2\pi}{\rho|C|}\min\Phi^2_{\alpha,\beta,p}\le|\tau(2\pi)-\tau(0)|\le\frac{2\pi}{\rho|C|}\max\Phi^2_{\alpha,\beta,p}\,.$$

Denoting τ_1,τ_2 solutions corresponding respectively to ρ_1,ρ_2, it is clearly possible to find ρ_1,ρ_2, with $\rho_2>\rho_1$, such that

$$|\tau_2(2\pi)-\tau_2(0)|\le|\tau_1(2\pi)-\tau_1(0)|,$$

independently of $\tau_1(0),\tau_2(0)$. This is, for the averaged system (12), (13), the desired twist effect: interpreting $(\rho\sigma,\tau)$ as pseudo-polar coordinates, it appears that the angular variation $|\tau(2\pi)-\tau(0)|$, on the interval $[0,2\pi]$, decreases when ρ is increased. From there, it can be shown, through the approximation of u,θ by σ,τ that, for ρ sufficiently large, the Poincaré map $(x(0),x'(0))\mapsto(x(2\pi),x'(2\pi))$, relative to equation (1) for the period 2π, also presents a twist effect for solutions of large amplitude.

The dynamics of the system can be illustrated by looking at Poincaré sections. Figure 2 shows Poincaré sections for the equation

$$x''+\alpha x^+-\beta x^-=2.5+\cos t,$$

with $\alpha=2$, and $(\alpha,\beta)\in C_1$. For that choice of α,β,p, the function $\Phi_{\alpha,\beta,p}$ is negative everywhere. One recognizes in Figure 2 invariant curves for the Poincaré map associated to equation (1), and the so-called "islands chains" typical of area-preserving maps, when a twist condition is satisfied.

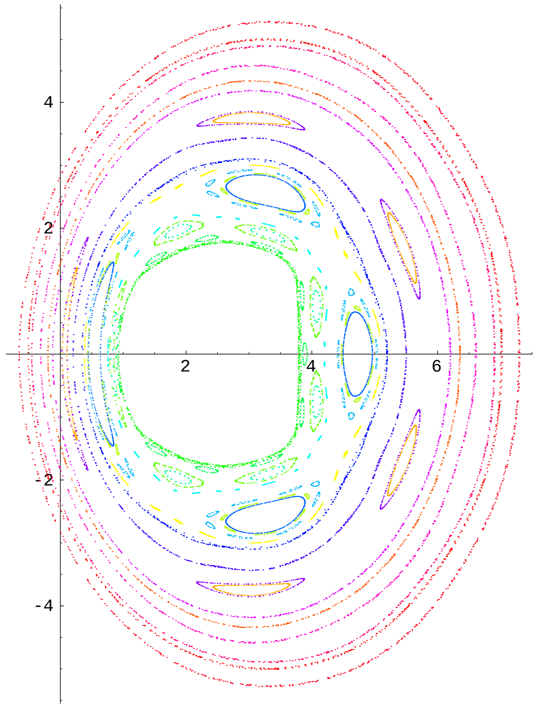

Figure 2: Poincaré sections for $x'' + 2x^+ - \beta x^- = 2.5 + \cos t$, with $(2, \beta) \in C_1$.

2^nd^ CASE. Suppose that $\Phi_{\alpha,\beta,p}$ has zeros, all zeros being simple. Assuming $C \neq 0$, the relation $\sigma(t)\Phi_{\alpha,\beta,p}(\tau(t)) = C$ prevents $\Phi_{\alpha,\beta,p}(\tau(t))$ from changing sign. By (13), we see that τ' will be of constant sign. For the sake of definiteness, assume for instance that τ is decreasing. If τ^* is the nearest zero of $\Phi_{\alpha,\beta,p}$ at the left of $\tau(0)$, it is fairly clear that we must have

$$\lim_{t\to\infty} \tau(t) = \tau^*.$$

By (14), we then have

$$\lim_{t\to\infty} \sigma(t) = +\infty.$$

and, by (12),

$$\lim_{t\to\infty} \sigma'(t) = \frac{1}{\rho}\, \Phi'_{\alpha,\beta,p}(\tau^*),$$

showing that, asymptotically, the growth of σ is linear (τ^* is assumed to be a simple zero of $\Phi_{\alpha,\beta,p}$). Since $\sigma(t)$ is close to $u(t)$ on large intervals, the same conclusions will roughly hold for the amplitude $u(t)$. The argument can be turned into a proof of Proposition 2 (see [8]).

An example of solutions whose amplitude grow exactly linearly is provided by the equation

$$x'' + \alpha x^+ - \beta x^- = 2\varphi'_{\alpha,\beta}(t),$$

with $(\alpha, \beta) \in C_1$. Indeed, it is easy to check that $x(t) = (t - t_0)\varphi_{\alpha,\beta}(t)$ is a solution of that equation for $t \geq t_0$.

4 Forced oscillator with damping

We consider now the equation with damping

$$x'' + \varepsilon x' + \alpha x^+ - \beta x^- = p(t). \tag{15}$$

We will assume that the damping coefficient is small and will study the following questions:

- For which p does equation (15) have 2π-periodic solutions whose amplitude grows to infinity when ε goes to 0?
- In the opposite case, what is the behavior of solutions with large initial conditions?

The following proposition [6, 7] shows basically that large amplitude 2π-periodic solutions are present when $\Phi_{\alpha,\beta,p}$ has simple zeros, whereas the set of 2π-periodic solutions is bounded, independently of ε close to 0, when $\Phi_{\alpha,\beta,p}$ is of constant sign.

Proposition 4 *Let α, β satisfy (4).*
(i) Assume that θ^* is a simple zero of $\Phi_{\alpha,\beta,p}$. Let $u^* := 2|\Phi'_{\alpha,\beta,p}(\theta^*)|$. Then, for $\varepsilon\Phi'_{\alpha,\beta,p}(\theta^*) > 0$, with ε small enough, the equation (15) has a 2π-periodic solution x_ε such that

$$(x_\varepsilon(t), x'_\varepsilon(t)) = \frac{1}{|\varepsilon|} u_\varepsilon(t)(\varphi_{\alpha,\beta}(t + \theta_\varepsilon(t)), \varphi'_{\alpha,\beta}(t + \theta_\varepsilon(t)))\,,$$

the functions $u_\varepsilon, \theta_\varepsilon$ being 2π-periodic and such that

$$\lim_{\varepsilon\to 0} u_\varepsilon(t) = u^*\ , \lim_{\varepsilon\to 0} \theta_\varepsilon(t) = \theta^*\ ,$$

uniformly in t. Moreover, the solution x_ε is asymptotically stable if $\varepsilon > 0$, unstable if $\varepsilon < 0$.
(ii) If $\Phi_{\alpha,\beta,p}$ has $2z$ zeros in $[0, 2\pi/n)$, all simple, there exists $\varepsilon_0 > 0$ and $R > 0$ such that, if $0 < |\varepsilon| \leq \varepsilon_0$, there are exactly z 2π-periodic solutions of (15) having $\|\cdot\|_\infty$-norm larger than R. If $z = 1$ and if problem (1) has no 2π-periodic solution, problem (15) has a unique 2π-periodic solution for $0 < |\varepsilon| \leq \varepsilon_0$.
(iii) If $\Phi_{\alpha,\beta,p}$ is of constant sign (and does not vanish), the set of 2π-periodic solutions of (15) is bounded, independently of ε in some neighborhood of 0.

When $\Phi_{\alpha,\beta,p}$ is of constant sign, the information provided by the above theorem, concerning the 2π-periodic solutions, can be completed by a result concerning all solutions of (15). Indeed, as stated in the next proposition [4], when $\Phi_{\alpha,\beta,p}$ is of constant sign, all solutions end up ultimately in a set whose size is $o(\varepsilon)$ for ε going to 0 by positive values.

Proposition 5 *Let α, β satisfy (4). If $\Phi_{\alpha,\beta,p}$ of constant sign (and does not vanish), and if x_ε denotes any solution of (2),*

$$\lim_{\varepsilon\to 0_+} \limsup_{t\to\infty} \varepsilon(|x_\varepsilon(t)| + |x'_\varepsilon(t)|) = 0.$$

In Figure 3, we show the asymptotic behavior of the Poincaré sections for the damped oscillator represented by the equation

$$x'' + \varepsilon x' + \alpha x^+ - \beta x^- = 2.5 + \cos t,$$

with $\alpha = 2, (\alpha, \beta) \in C_1, \varepsilon = 0.001$. As indicated above, the function $\Phi_{\alpha,\beta,p}$ is then negative everywhere. An analysis of the numerical results shows the presence of four periodic solutions, three of them being subharmonic solutions of order $3, 4$ and 7 respectively; all solutions tend towards those periodic solutions when $t \to \infty$. This damped oscillator thus apparently has several asymptotically stable periodic solutions.

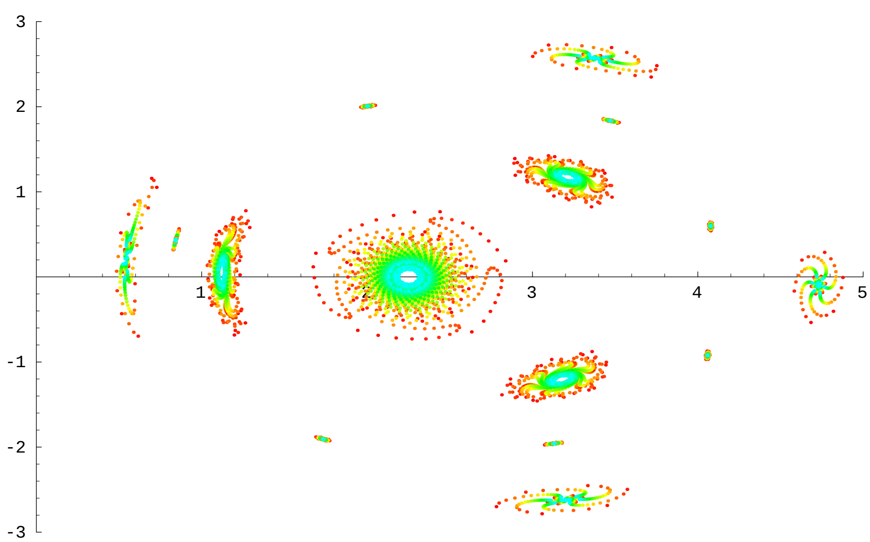

Figure 3: Poincaré sections for $x'' + 0.001x' + 2x^+ - \beta x^- = 2.5 + \cos t$, with $(2, \beta) \in C_1$.

5 Frequency-response curves

In this section, we study the damped equation, with a forcing term whose frequency is close, but not equal to the frequency of the free oscillations, i.e. we assume that (α,β) satisfy (4) and consider the equation

$$x'' + \varepsilon x' + \alpha x^+ - \beta x^- = p(t)\,,$$

with ε "small" and p of period T, with $T = 2\pi + O(\varepsilon)$, for $\varepsilon \to 0$. By means of a change of time scale, it is basically equivalent to consider the problem

$$x'' + \varepsilon x' + (1+\varepsilon k)(\alpha x^+ - \beta x^-) = p(t), \tag{16}$$

where (α,β) still satisfy (4), p is of period 2π and k is a given constant. That equation has been studied in [5]; we recall here the main results.

It is shown in [5] that, when 2π-periodic solutions having an amplitude of the order of $1/\varepsilon$ are present, they can be written under the form

$$(x_\varepsilon(t), x'_\varepsilon(t)) = \frac{1}{|\varepsilon|} u_\varepsilon(t)(\varphi(t+\theta_\varepsilon(t)), \varphi'(t+\theta_\varepsilon(t)),$$

with

$$\lim_{\varepsilon\to 0} \theta_\varepsilon(t) = \theta^*, \lim_{\varepsilon\to 0} u_\varepsilon(t) = u^* = 2|\Phi'_{\alpha,\beta,p}(\theta^*)|, \tag{17}$$

θ^* being a solution of

$$\varepsilon\Phi'_{\alpha,\beta,p}(\theta^*) > 0 \quad , \qquad k\Phi'_{\alpha,\beta,p}(\theta^*) - \Phi_{\alpha,\beta,p}(\theta^*) = 0. \tag{18}$$

Conditions (18) are easily studied by looking, in the xy-plane, at the intersection of the line $y = x/k$ with the curve parametrized by

$$\theta \mapsto (\Phi_{\alpha,\beta,p}(\theta), \Phi'_{\alpha,\beta,p}(\theta)).$$

The fact that $\Phi_{\alpha,\beta,p}$ admits zeros or not again makes a difference; it is obvious that, when $\Phi_{\alpha,\beta,p}$ admits zeros, all of them being simple, a value of θ^* can always be found such that conditions (18) are satisfied. This observation is the basis of the next proposition, which is limited, for the sake of simplicity, to the case $n = 1$.

Proposition 6 *Assume that (4) is satisfied with $n = 1$. Let the sign of ε be fixed. If $\Phi_{\alpha,\beta,p}$ has z simple zeros $\theta_1,\dots,\theta_z \in [0,2\pi)$ with $\varepsilon\Phi'_{\alpha,\beta,p}(\theta_i) > 0$ for $i = 1,\dots,z$, then, for any $k \in \mathbb{R}$, problem (16) has for ε sufficiently small, at least z periodic solutions of period 2π, whose amplitude is $O(1/\varepsilon)$ for $\varepsilon \to 0$. Moreover, if $\varepsilon > 0$ and if*

$$k\Phi'_{\alpha,\beta,p}(\theta) - \Phi_{\alpha,\beta,p}(\theta) = 0 \Longrightarrow k\Phi''_{\alpha,\beta,p}(\theta) - \Phi'_{\alpha,\beta,p}(\theta) \neq 0, \tag{19}$$

those z solutions are asymptotically stable.

By contrast to the above case, when $\Phi_{\alpha,\beta,p}$ is of constant sign, the existence of solutions for (18) and hence, the presence of 2π-periodic solutions having an amplitude of the order of $1/\varepsilon$ for (16), will depend on the value of k. It is fairly immediate that, for $\varepsilon > 0$, a value of θ^* satisfying conditions (18) can be found only if $sk \geq k_{\mathrm{crit}}$, where s is the sign of $\Phi_{\alpha,\beta,p}$ and

$$k_{\mathrm{crit}} = 1 \Big/ \max\left\{ \frac{\Phi'_{\alpha,\beta,p}(\theta)}{|\Phi_{\alpha,\beta,p}(\theta)|} \mid \theta \in [0, 2\pi/n) \right\}.$$

This explains the following proposition.

Proposition 7 *Let $\varepsilon > 0$ and let $\Phi_{\alpha,\beta,p}$ be nonconstant, but of constant sign s $(= \pm 1)$ (and not vanishing) on $[0, 2\pi)$. Then,*

- *for $sk < k_{crit}$, the set of 2π-periodic solutions of (16) is bounded, independently of ε, for $\varepsilon \to 0_+$;*
- *for $sk > k_{crit}$, equation (16) has, for ε sufficiently small, at least two 2π-periodic solutions having an amplitude $O(1/\varepsilon)$ for $\varepsilon \to 0_+$; moreover, if (19) is satisfied, there is at least one asymptotically stable 2π-periodic solution and one unstable 2π-periodic solution of amplitude $O(1/\varepsilon)$ for $\varepsilon \to 0_+$.*

The amplitude of the solutions of the 2π-periodic solutions can be estimated by (17). This has been done in [5] for $p(t) = a + \cos t$. With $n = 1$, the function $\Phi_{\alpha,\beta,p}$ is then of the form

$$\Phi_{\alpha,\beta,p} = ac_0 + c_1 \cos(\theta - \theta_0) \text{ for some } \theta_0 \,;$$

c_0 and c_1 can be computed from (7), which gives

$$\begin{aligned} c_0 &= \frac{1}{\pi}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right), \\ c_1 &= \frac{1}{\pi}\left|\cos\left(\frac{\pi}{2\sqrt{\alpha}}\right)\right| \frac{|\beta - \alpha|}{|\beta - 1|\,|\alpha - 1|}. \end{aligned}$$

From (18), it is easy to show that the value of u^*, which determines the amplitude, satisfies the relation

$$\left(\frac{u^*}{2}\right)^2 + \left(sk\frac{u^*}{2} - ac_0\right)^2 = c_1^2,$$

(we assume $\varepsilon > 0$).

Figure 4 represents, for $n = 1, \alpha = 2, (\alpha, \beta) \in C_1, \varepsilon > 0$, the amplitude u^* as a function of k, for $a = 1$ and for $a = 2$. The last value corresponds to a function $\Phi_{\alpha,\beta,p}$ of constant (negative) sign, whereas the former leads to a function $\Phi_{\alpha,\beta,p}$

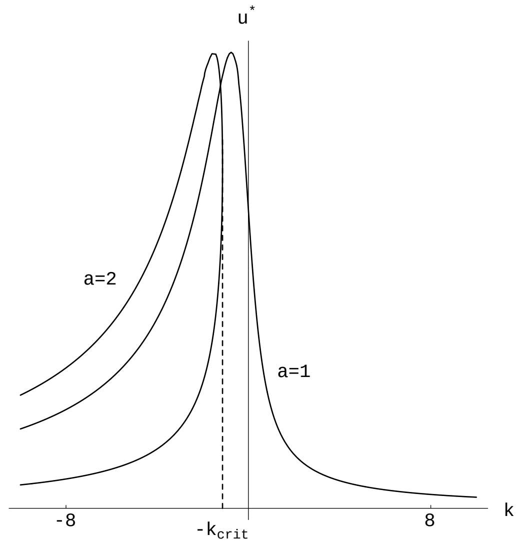

Figure 4: Large periodic solutions of $x'' + \varepsilon x' + (1+\varepsilon k)(\alpha x^+ - \beta x^-) = a + \cos t$

having zeros. It follows from Proposition 7 that for $k > -k_{\text{crit}}$, the set of 2π-periodic solutions is bounded, independently of ε, for ε small. The value k_{crit} is easily computed to be

$$k_{\text{crit}} = \frac{\sqrt{a^2 c_0^2 - c_1^2}}{c_1} \simeq 1.13\,.$$

The diagram in Figure 4 shows the amplitude in the limiting situation $\varepsilon \to 0_+$. It must be pointed out that the limit need not be uniform in k, and consequently, the asymptotic behavior suggested by that figure for $k \to \pm\infty$, might not correspond to what is observed for a particular value of ε. It is therefore interesting to look at actual frequency-response curves. Such curves are represented in Figures 5 and 6. The equation considered is

$$x'' + 0.1x' + \alpha x^+ - \beta x^- = a + \cos \omega t, \text{ with } \alpha = 2, (\alpha, \beta) \in C_1.$$

The curves in the two diagrams show, as a function of ω, the norm of the vector of initial conditions for 2π-periodic solutions, respectively when $a = 0.5$ and when $a = 5$. In Figure 5, the frequency-response curve is similar to that of a linear oscillator whereas, in the second case, where the function $\Phi_{\alpha,\beta,p}$ is of constant (negative) sign, a "foldover" of the curve is observed. The dashed part in the frequency-response curve corresponds to an unstable periodic solution, meaning that an hysteresis phenomenon is present, as in Duffing's equation with a forcing term. When the pulsation ω is decreased, starting from a value $\omega_0 > 1.5$, for instance, the amplitude of the response will increase, until the value

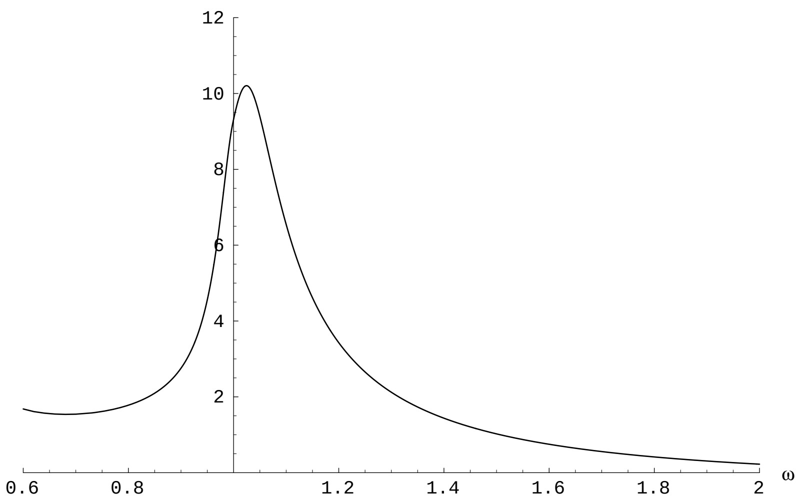

Figure 5: Frequency-response curve for $x'' + 0.1x' + \alpha x^+ - \beta x^- = 0.5 + \cos \omega t$

ω_1 is reached; at that point, the amplitude will jump abruptly to a smaller value. This phenomenon is illustrated by Figure 7, where a solution of the equation

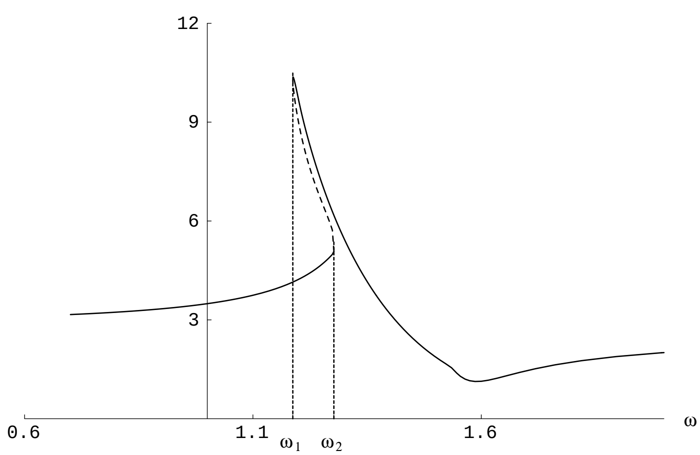

Figure 6: Frequency-response curve for $x'' + 0.1x' + \alpha x^+ - \beta x^- = 5 + \cos \omega t$

$$x'' + 0.1x' + \alpha x^+ - \beta x^- = 5 + \cos(1.6t - t^2/1500)$$

is plotted; as before, we assume that (α, β) belongs to C_1, with $\alpha = 2$. On the contrary, if the pulsation is increased starting from a value $\omega < \omega_1$, the largest amplitudes are not reached.

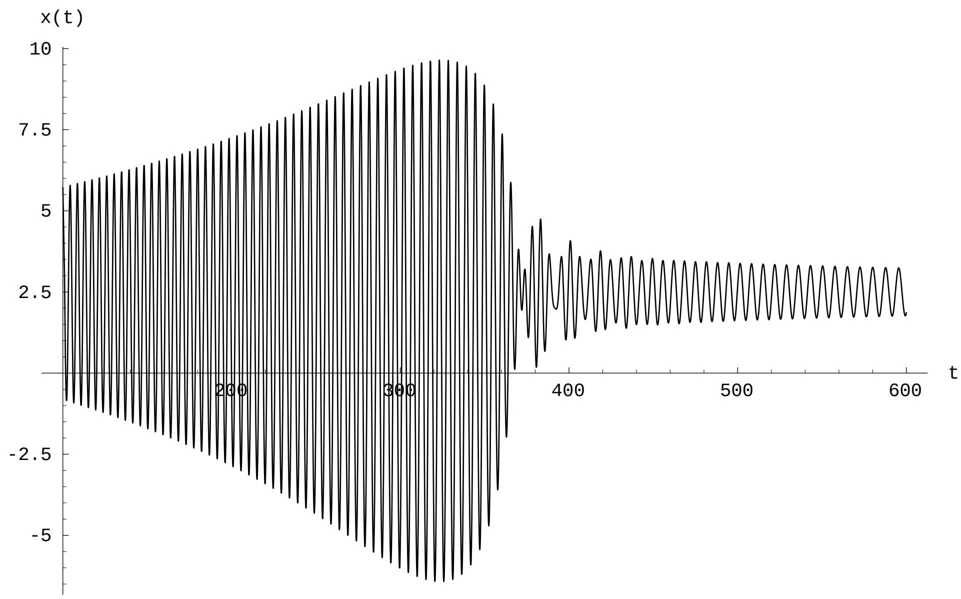

Figure 7: A solution of $x'' + 0.1x' + \alpha x^+ - \beta x^- = 5 + \cos(1.6t - t^2/1500)$

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