

## A PENALTY METHOD FOR APPROXIMATIONS OF THE STATIONARY POWER-LAW STOKES PROBLEM

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ABSTRACT. We study approximations of the steady state Stokes problem governed by the power-law model for viscous incompressible non-Newtonian flow using the penalty formulation. We establish convergence and find error estimates.

### 1. INTRODUCTION

Let  $\Omega$  be a convex bounded domain in  $\mathbb{R}^d$ ,  $d \geq 2$ . We consider the steady state flow of a fluid in  $\Omega$ , where  $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \dots, u_d(\mathbf{x}))$  denotes the velocity of a fluid particle at  $\mathbf{x} = (x_1, \dots, x_d) \in \Omega$ . Let  $\boldsymbol{\sigma} \in \mathbb{R}^d \times \mathbb{R}^d$  denote the stress tensor for the fluid. The momentum equations for an isothermal steady state flow are

$$\rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \quad \text{in } \Omega, \quad (1.1)$$

where  $\rho$  is the density of the fluid,  $\mathbf{f} = (f_1, \dots, f_d)$  the body force, and  $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ . The  $j^{\text{th}}$  component of  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  is  $\sum_{i=1}^d u_i \frac{\partial u_j}{\partial x_i}$  and  $\nabla \cdot \boldsymbol{\sigma}$  is obtained by applying the divergence operator, defined by  $\nabla \cdot \mathbf{u} = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}$ , to each row of  $\boldsymbol{\sigma}$ . We further assume that the fluid is incompressible so it satisfies the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega. \quad (1.2)$$

The rate of deformation tensor  $\mathbf{D}(\mathbf{u}) \in \mathbb{R}^d \times \mathbb{R}^d$  is the symmetric gradient of  $\mathbf{u}$  with components  $D_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ . For incompressible fluids, the second invariant of  $\mathbf{D}(\mathbf{u})$  denoted  $\Pi_{\mathbf{D}}(\mathbf{u})$  satisfies

$$-2\Pi_{\mathbf{D}}(\mathbf{u}) = \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) = \sum_{i,j=1}^d D_{ij}(\mathbf{u})^2 = |\mathbf{D}(\mathbf{u})|^2,$$

where  $|\cdot|$  denotes the Euclidean matrix norm; that is for  $\mathbf{K}$ , a  $d \times d$  real matrix,  $|\mathbf{K}| = [\sum_{i,j=1}^d (k_{ij})^2]^{\frac{1}{2}}$ . In the power-law model for non-Newtonian fluid flows, it is

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assumed that viscosity  $\eta$  varies as a power of the shear-strain rate, giving a stress tensor  $\boldsymbol{\sigma}$  of the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + \eta(\Pi_{\mathbf{D}}(\mathbf{u}))\mathbf{D}(\mathbf{u}), \quad (1.3)$$

where  $p$  is the scalar pressure,  $\mathbf{I} \in \mathbb{R}^d \times \mathbb{R}^d$  is the  $d$  dimensional identity matrix, and

$$\eta(z) = \eta_0 |z|^{(r-2)/2}, \quad z \in \mathbb{R}.$$

Here we assume  $1 < r < +\infty$ , and  $\eta_0 > 0$ . Substituting (1.3) into (1.1), we obtain the steady state power-law Navier-Stokes equation

$$-k\nabla \cdot (|\mathbf{D}(\mathbf{u})|^{r-2}\mathbf{D}(\mathbf{u})) + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad (1.4)$$

where  $k = \eta_0/2^{\frac{r-2}{2}}$ .

The *power-law Stokes equation* is obtained by neglecting the inertial term  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  in (1.4). This model of non-Newtonian flow is very popular in chemical engineering [12] as well as in geophysics [42]. It has also been used in applications for the design of extrusion dies [11], [33], and for the study of the lithosphere [18], [19], [20]. To make the Stokes problem well posed, we assume that the solution satisfies the continuity equation (1.2) and, for simplicity, a homogeneous boundary condition of Dirichlet type. The resulting problem is

$$\begin{aligned} -k\nabla \cdot (|\mathbf{D}(\mathbf{u})|^{r-2}\mathbf{D}(\mathbf{u})) + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \quad (1.5)$$

where  $\partial\Omega$  denotes the boundary of  $\Omega$ .

*Remark 1.1.* If  $d \geq 2$  and the flow is unidirectional then  $\mathbf{u}(\mathbf{x}) = (u(\mathbf{x}), 0, \dots, 0)$ , where  $u(\mathbf{x})$  is a scalar valued function and  $\mathbf{f} = (f_1(\mathbf{x}), 0, \dots, 0)$ . For  $\mathbf{u}$  to satisfy the continuity equation (1.2), we have  $\frac{\partial u}{\partial x_1} = 0$  which implies that  $u(\mathbf{x}) = u(x_2, \dots, x_d)$ . Substituting  $\mathbf{u}(\mathbf{x})$  into equation (1.5) and writing it as a system shows that  $(\frac{\partial p}{\partial x_2}, \dots, \frac{\partial p}{\partial x_n}) = \mathbf{0}$  and  $\tilde{f} \equiv f_1(\mathbf{x}) - \frac{\partial p}{\partial x_1}$  is independent of  $x_1$ . Thus, we are left with a scalar quasilinear elliptic Dirichlet problem in the  $\mathbb{R}^{d-1}$  domain  $\tilde{\Omega} = \Omega \cap \{x_1 = 0\}$

$$-\tilde{k}\Delta_r u = \tilde{f} \quad \text{in } \tilde{\Omega}, \quad u = 0 \quad \text{on } \partial\tilde{\Omega}$$

where  $\Delta_r u = \nabla \cdot (|\nabla u|^{r-2}\nabla u)$  is the quasilinear generalization of the Laplacian known as the  $r$ -Laplacian, and  $\nabla u = (\frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_d})$ . (We note that  $\Delta_p$  is frequently called the  $p$ -Laplacian in the literature, but we are using  $p$  to denote fluid pressure here.) There has been a great deal of analytical (e.g., [10], [17], [35]) and numerical work (e.g., [9], [26], [34], [44]) devoted to problems involving  $\Delta_r u$ .

*Remark 1.2.* In our notation, the power-law index is the value  $n = r - 1$ . We recall that a fluid is considered to be Newtonian if it has power-law index  $n = 1$ , and non-Newtonian if  $n \neq 1$ . The power-law equation (1.3) is also called the Ostwald-deWaele equation. When  $0 < n < 1$ , which corresponds to our parameter  $1 < r < 2$ , power-law fluids exhibit a decrease in viscosity with increasing shear stress and they are known as pseudoplastic or shear-thinning fluids. When  $1 < n < \infty$ , corresponding to  $2 < r < \infty$ , power-law fluids exhibit an increase in viscosity with increasing shear stress and they are known as dilatant or shear-thickening fluids

[45]. For a specific example, we cite [30] where the value of  $n$  for a certain tomato paste is given as  $n = 0.257$ .

One way of studying a stationary power-law Stokes flow is to consider the velocity field  $\mathbf{u}$  as the minimizer of an appropriate energy functional. In order to enforce the constraint of a divergence free flow, we seek a minimizer  $\mathbf{u}$  in the space of divergence free vector fields, that is, we solve the problem:

$$\begin{aligned} \text{Find the minimum of } J(\mathbf{u}) &= \frac{k}{r} \int_{\Omega} |\mathbf{D}(\mathbf{u})|^r \, d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} \, d\mathbf{x}, \\ \text{where } \mathbf{u} \in \mathbf{X} &= \left\{ \mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega) : \nabla \cdot \mathbf{u} = 0 \right\}. \end{aligned} \quad (1.6)$$

This is a minimization problem with a constraint and we refer to it as the *variational formulation* of power-law flow. The Euler equation corresponding to this minimization problem describes a solution for the velocity field. Another common way of studying power-law flow is to simultaneously find a pair  $(\mathbf{u}, p)$  that satisfies (1.5). This is called the *mixed weak formulation* and it is written down precisely in (4.1) and (4.2). The connection between these two formulations comes from a technical inf-sup condition (see Theorem 5.1-5.2 below) which is frequently called the *LBB condition* named after Ladyzhenskaya, Brezzi, and Babuška. This condition can be stated as the following:  $\exists \beta > 0$  such that

$$\inf_{q \in L_0^{r'}(\Omega)} \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle \nabla \cdot \mathbf{v}, q \rangle}{\|q\|_{0,r'} \|\mathbf{v}\|_{0,r}} \geq \beta. \quad (1.7)$$

When this condition holds, the variational formulation and the mixed weak formulation are equivalent in the sense that  $\mathbf{u}$  is a solution of the variational formulation if and only if  $(\mathbf{u}, p)$  is a solution of the mixed weak formulation with  $p$  being solved in terms of  $\mathbf{u}$  using the inverse of the gradient operator. The pressure can then be computed after the velocity is known provided that the *LBB* condition holds.

The *LBB* condition is well known to hold for the linear problem ( $r = 2$ ) on Lipschitz domains in any dimension [4], [13], [31]. Since any bounded convex domain has a Lipschitz continuous boundary [27, Corollary 1.2.-2.3] we conclude that the *LBB* condition holds for our  $\Omega$  when  $r = 2$ . For the nonlinear problem  $1 < r \leq 2$ , the *LBB* has been shown ([2], [5]) for smooth domains in dimension  $d = 2$ . This is generalized in [3] where it is shown that the *LBB* condition holds for all dimensions  $d > 1$  and for the full range  $1 < r < \infty$  in Lipschitz domains. Thus, the mixed weak formulation makes sense and it is equivalent to the variational formulation in our setting in this work.

We note that, in the variational formulation,  $\mathbf{u}$  is defined as the minimum of a convex functional on a separable Banach space, thus (1.6) always has a unique solution  $\mathbf{u}$  for any  $1 < r < \infty$ , whether or not the *LBB* condition holds. However, the pressure  $p$  is not necessarily well defined, so that results involving the pressure function  $p$  typically require the additional assumption of an *LBB* condition and it is not known if this condition holds in a nonconvex domain.

The two formulations discussed above for the power-law Stokes problem are both useful for the numerical analysis of the problem. A finite element analysis of the power-law Stokes problem using the mixed weak formulation has been studied by several authors, for example, in [5], [6], [7], [8]. Finite element analysis using the direct variational formulation (1.6) requires one to solve a constrained minimization problem and construct finite element spaces with approximately divergence

free interpolation functions. Thus, the variational formulation and its associated constrained minimization problem is more difficult for both analysis and numerical approximation. A natural way to overcome this difficulty is to introduce a penalty functional that eliminates the constraint. The first use of the penalty function method in conjunction with the finite element method is due to Babuška [4]. The method was quickly adopted as a standard tool for the finite element analysis of viscous, incompressible fluid flows [47]. Extensive studies of the penalty method applied to fluid flow problems, both experimentally and mathematically, have appeared from the late seventies to the present day. Here we cite only several important articles among them [21], [22], [23], [28], [29], [36], [37], [38], [39], [40], and [47]. A very general mathematical analysis of the penalty method applied to nonlinear problems including a class of non-Newtonian fluid flow problems was presented by Oden [38]. His work provides some important convergence results. It appears that when the penalty method is applied to the Newtonian Stokesian flow problems, the resultant matrices are ill-conditioned. However, this deficiency can be overcome by the use of reduced integration techniques.

For a given penalty parameter  $\epsilon > 0$ , the penalty formulation requires the unconstrained minimization of the nonlinear convex functional

$$J_\epsilon(\mathbf{u}) = J(\mathbf{u}) + \frac{1}{r\epsilon} \int_\Omega |\nabla \cdot \mathbf{u}|^r dx$$

over the Sobolev space  $\mathbf{W}_0^{1,r}(\Omega)$ . The corresponding pressure  $p^\epsilon$  is defined in terms of the minimizer  $\mathbf{u}^\epsilon$ . We prove that the penalty approximation  $\mathbf{u}^\epsilon$  of the unconstrained minimization problem  $\min\{J_\epsilon(\mathbf{u}) : \mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega)\}$  converges to the true solution  $\mathbf{u}$  of  $\min\{J(\mathbf{u}) : \mathbf{u} \in X\}$  as  $\epsilon \rightarrow 0$  for any  $1 < r < \infty$  without assuming that the domain  $\Omega$  is convex. This convergence result is only for the velocity field since the pressure may be undefined. However, because of the more general variational setting, this result establishes the validity of a penalty approximation even when the *LBB* condition fails to hold. This is a convergence result, not an error estimate, but it doesn't require the *LBB* condition (1.7). Here we are writing  $(\mathbf{u}, p)$  for the unique solution of the mixed formulation (4.2), and  $\mathbf{u}^\epsilon$  the penalty solution. When the *LBB* condition holds, we obtain error estimates for the velocity field  $\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r} = O(\epsilon^{g_1(r)})$ , where  $g_1(r) = \frac{1}{(r-1)(3-r)}$  for  $1 < r \leq 2$ , and  $g_1(r) = \frac{1}{(r-1)^2}$  for  $2 \leq r < \infty$ . Let  $\phi(z) = |z|^{r-2}z$ ,  $z \in \mathbb{R}$  and let  $p^\epsilon = c - \phi(\nabla \cdot \mathbf{u}^\epsilon)$ , where  $c = \int_\Omega \phi(\nabla \cdot \mathbf{u}^\epsilon) dx$ . We also show error estimates for the pressure  $\|p - p^\epsilon\|_{0,r'} = O(\epsilon^{g_2(r)})$  where  $g_2(r) = \frac{1}{(3-r)}$  for  $1 < r \leq 2$  and  $g_2(r) = \frac{1}{(r-1)^2}$  for  $2 \leq r < \infty$ . These rates of convergence reduce to known results  $\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,2} + \|p - p^\epsilon\|_{0,2} = O(\epsilon)$  for the Newtonian case  $r = 2$  as discussed in [25], [36], [39], and [43].

This is an important feature of the unconstrained penalty minimization formulation which makes it convenient for error analysis and numerical implementation. In contrast, the mixed weak formulation requires the solution of a system of nonlinear equations and a discrete *LBB* condition. Thus, we provide a mathematical analysis of the penalty method applied to the power-law Stokes problem in the variational formulation. To our knowledge, this is a generalization of the analysis available in the literature for Newtonian flows. Numerical experiments have been performed on power-law flow problems using the penalty method in the engineering literature [32], [33], [39], [41], and [47]. Since pressure must then be calculated from

the computed velocity field, the accuracy of the pressure is lower than that of the velocity as shown in our error estimates. It is interesting to note that the penalty term  $\frac{1}{2\epsilon} \int_{\Omega} |\nabla \cdot \mathbf{u}|^2 d\mathbf{x}$  was used in [41] to approximate power-law flows instead of  $\frac{1}{r\epsilon} \int_{\Omega} |\nabla \cdot \mathbf{u}|^r d\mathbf{x}$ , which is used in this work and reasonable numerical results were obtained without a mathematical analysis.

## 2. PRELIMINARIES

We begin by establishing some notation. Let  $L^r(\Omega)$  for  $1 < r < \infty$  be the space of real scalar functions defined on  $\Omega$  whose  $r^{\text{th}}$  power is absolutely integrable with respect to Lebesgue measure  $d\mathbf{x} = dx_1 \dots dx_d$ . This is a Banach space with the norm  $\|u\|_{0,r} = (\int_{\Omega} |u(\mathbf{x})|^r d\mathbf{x})^{1/r}$ . The Sobolev space  $W^{k,r}(\Omega)$  is the space of functions in  $L^r(\Omega)$  with distributional derivatives up to order  $k$  also in  $L^r(\Omega)$ . The norm for this space is  $\|u\|_{k,r} = (\int_{\Omega} \sum_{|j| \leq k} |D^j u(\mathbf{x})|^r d\mathbf{x})^{1/r}$ , where we use the standard multi-index notation. That is, for  $j = (j_1 \dots j_d) \in \mathbb{N}^d$ , where  $\mathbb{N}$  is the set of natural numbers, define  $|j| = \sum_{i=1}^d j_i$  and write the partial derivative  $D^j u(\mathbf{x}) = \frac{\partial^{|j|} u}{\partial^{j_1} x_1 \dots \partial^{j_d} x_d}$ . The closure of  $C_0^\infty(\Omega)$  in  $W^{k,r}(\Omega)$  is denoted by  $W_0^{k,r}(\Omega)$ . For systems of equations, we need the product spaces defined by  $\mathbf{L}^r(\Omega) = [L^r(\Omega)]^d$ ,  $\mathbf{W}^{k,r}(\Omega) = [W^{k,r}(\Omega)]^d$ , and  $\mathbf{W}_0^{k,r}(\Omega) = [W_0^{k,r}(\Omega)]^d$ . The norm for  $\mathbf{v} = (v_1, \dots, v_d) \in \mathbf{L}^r(\Omega)$  is  $\|\mathbf{v}\|_{0,r} = (\int_{\Omega} \sum_{i=1}^d |v_i|^r d\mathbf{x})^{1/r}$ . For  $\mathbf{v} \in \mathbf{W}^{k,r}(\Omega)$  we have norm  $\|\mathbf{v}\|_{k,r} = (\int_{\Omega} \sum_{i=1}^d \sum_{|j| \leq k} |D^j v_i|^r d\mathbf{x})^{1/r}$ . It is well known [1], that the seminorm  $|\mathbf{v}|_{k,r} = (\int_{\Omega} (\sum_{i=1}^d \sum_{|j|=k} |D^j v_i|^r d\mathbf{x})^{1/r}$  is equivalent to  $\|\mathbf{v}\|_{k,r}$  for  $\mathbf{v} \in \mathbf{W}_0^{k,r}(\Omega)$ . In addition, by Korn's inequality, see [1], the norm  $\|\mathbf{D}(\mathbf{u})\|_{0,r} = (\int_{\Omega} \sum_{i,j=1}^d |D_{ij}(\mathbf{u})|^r d\mathbf{x})^{1/r}$  is equivalent to  $\|\mathbf{u}\|_{1,r}$  in  $\mathbf{W}_0^{1,r}(\Omega)$ . For  $1 < r < \infty$  let  $r'$  satisfy  $\frac{1}{r} + \frac{1}{r'} = 1$ , which is equivalent to  $r' = \frac{r}{r-1}$ . Let  $\|\cdot\|_{-1,r'}$  denote the norm on  $\mathbf{W}^{-1,r'}(\Omega)$  which is the dual space of  $\mathbf{W}_0^{1,r}(\Omega)$ . Let  $\phi(\mathbf{x}) = |\mathbf{x}|^{r-2}\mathbf{x}$ , where  $\mathbf{x} \in \mathbb{R}^d$ . Note that  $\phi^{-1}(\mathbf{x}) = |\mathbf{x}|^{r'-2}\mathbf{x}$ . Finally, let  $L_0^r(\Omega) = \{q \in L^r(\Omega) : \int_{\Omega} q d\mathbf{x} = 0\}$ .

The following inequalities hold for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ ; the constant  $C > 0$  is independent of  $\mathbf{x}$  and  $\mathbf{y}$ .

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &\leq C(\phi(\mathbf{x}) - \phi(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y})(|\mathbf{x}| + |\mathbf{y}|)^{2-r}, \\ |\phi(\mathbf{x}) - \phi(\mathbf{y})| &\leq C|\mathbf{x} - \mathbf{y}|^{r-1}, \text{ for } 1 < r < 2; \end{aligned} \tag{2.1}$$

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^r &\leq C(\phi(\mathbf{x}) - \phi(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}), \\ |\phi(\mathbf{x}) - \phi(\mathbf{y})| &\leq C|\mathbf{x} - \mathbf{y}|(|\mathbf{x}| + |\mathbf{y}|)^{r-2}, \text{ for } 2 \leq r < \infty. \end{aligned} \tag{2.2}$$

They were proved for the case  $d = 2$  in Glowinski and Marroco [26] and were generalized by Barrett and Liu [8], see also [15]. A simple proof for general  $d$  is shown in DiBenedetto [16]. Finally, for completeness, we quote some important results from convex analysis and functional analysis which will apply to our problem. See [14] or [24] for further details. Let  $X$  be a reflexive Banach space with dual space  $X^*$ . Suppose the operator  $T : X \rightarrow X^*$ . Let  $\langle u^*, u \rangle$  denote the duality pairing between  $u \in X$  and  $u^* \in X^*$ . We say  $T$  is *bounded* if  $\exists C > 0$  such that  $\|Tu\|_{X^*} \leq C\|u\|_X$  for all  $u \in X$ . The operator  $T$  is *monotone* if  $\langle Tu - Tv, u - v \rangle \geq 0$  for all  $u, v \in X$ .  $T$  is *strictly monotone* if the inequality is strict for all  $u, v \in X$

with  $u \neq v$ . A *coercive* operator  $T$  satisfies  $\lim_{\|u\| \rightarrow \infty} \frac{\langle Tu, u \rangle}{\|u\|} = \infty$ . Finally, we say  $T$  is *potential* if  $\exists$  a functional  $J : X \rightarrow \mathbb{R}$  such that  $J'(u) = Tu$  for all  $u \in X$  (i.e.  $\langle J'(u), v \rangle = \langle T(u), v \rangle$  for all  $u, v \in X$ ).

**Theorem 2.1.** *Let  $T : X \rightarrow X^*$  be a bounded, monotone, coercive, potential operator. Then  $TX = X^*$ . Thus,  $Tu = f$  has a solution for every  $f \in X^*$ . Moreover, if  $T$  is strictly monotone, then  $Tu = f$  has a unique solution.*

**Theorem 2.2.** *Let  $J$  be a functional defined on  $X$  such that  $\lim_{\|u\| \rightarrow \infty} J(u) = \infty$ . If  $J$  is either (i) continuous and convex on  $X$  or (ii) weakly lower semicontinuous on  $X$  then  $\inf_{u \in X} J(u) > -\infty$  and there exists at least one  $u_0 \in X$  such that  $J(u_0) = \inf_{u \in X} J(u)$ . Moreover, if  $J$  is continuous and strictly convex on  $X$  then there is precisely one such  $u_0$ .*

**Theorem 2.3.** *Let  $J : X \rightarrow \mathbb{R}$  be a functional with a local extremum at  $u_0 \in X$ . If  $\langle J'(u_0), v \rangle$  exists for some  $v \in X$  then  $\langle J'(u_0), v \rangle = 0$ .*

### 3. THE VARIATIONAL FORMULATION (VF) OF THE STOKES PROBLEM

In order to apply the penalty method, we first consider the variational formulation of (1.5) given in (1.6). The Euler-Lagrange equation associated to problem (1.6) is

$$\langle A(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X} \quad (3.1)$$

where  $A : \mathbf{W}_0^{1,r}(\Omega) \rightarrow \mathbf{W}^{-1,r'}(\Omega)$  is defined by

$$\langle A(\mathbf{u}), \mathbf{v} \rangle = k \int_{\Omega} |\mathbf{D}(\mathbf{u})|^{r-2} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) dx \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega).$$

We use  $\langle \cdot, \cdot \rangle$  to denote the duality pairing between  $\mathbf{W}_0^{1,r}(\Omega)$  and  $\mathbf{W}^{-1,r'}(\Omega)$  as well as between  $\mathbf{L}^r(\Omega)$  and  $\mathbf{L}^{r'}(\Omega)$ . In particular, we only need to assume  $\mathbf{f} \in \mathbf{W}^{-1,r'}(\Omega)$  for this formulation.

By using (2.1) and (2.2), we obtain the following where  $C > 0$  denotes a generic constant independent of  $\mathbf{u}$  and  $\mathbf{v}$ .

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|_{1,r}^2 &\leq C \langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle (\|\mathbf{u}\|_{1,r} + \|\mathbf{v}\|_{1,r})^{2-r}, \\ \|A(\mathbf{u}) - A(\mathbf{v})\|_{-1,r'} &\leq C \|\mathbf{u} - \mathbf{v}\|_{1,r}^{r-1}, \text{ for } 1 < r \leq 2; \end{aligned} \quad (3.2)$$

$$\begin{aligned} \|\mathbf{u} - \mathbf{v}\|_{1,r}^r &\leq C \langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle, \\ \|A(\mathbf{u}) - A(\mathbf{v})\|_{-1,r'} &\leq C \|\mathbf{u} - \mathbf{v}\|_{1,r} (\|\mathbf{u}\|_{1,r} + \|\mathbf{v}\|_{1,r})^{r-2}, \text{ for } 2 \leq r < \infty. \end{aligned} \quad (3.3)$$

From (3.2) and (3.3),  $A$  can be shown to be a bounded, monotone, coercive, potential operator on  $\mathbf{X} = \{\mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega) : \nabla \cdot \mathbf{u} = 0\}$ . We conclude using Theorem 2.1 that (3.1) has a unique solution  $\mathbf{u}$ . It follows that  $J : \mathbf{X} \rightarrow \mathbb{R}$  is a continuous, strictly convex functional on  $\mathbf{X}$  and

$$\lim_{\|\mathbf{u}\|_{1,r} \rightarrow \infty} J(\mathbf{u}) = \infty.$$

Thus, by Theorem 2.2 problem (1.6) has one and only one solution  $\mathbf{u}$  and hence (3.1) and (1.6) are equivalent. Note that in this formulation, the pressure function  $p$  does not appear.

## 4. THE MIXED WEAK FORMULATION (MWF) OF THE STOKES PROBLEM

For the mixed weak formulation of the Stokes problem (1.5), we suppose  $\mathbf{f} \in \mathbf{W}^{-1,r'}(\Omega)$ . The problem is then to simultaneously find  $\mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega)$  and  $p \in L_0^{r'}(\Omega)$ , such that

$$\begin{aligned} k \int_{\Omega} |\mathbf{D}(\mathbf{u})|^{r-2} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) \, d\mathbf{x} - \int_{\Omega} p \nabla \cdot \mathbf{v} \, d\mathbf{x} &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega) \\ \int_{\Omega} q \nabla \cdot \mathbf{u} \, d\mathbf{x} &= 0 \quad \forall q \in L_0^{r'}(\Omega). \end{aligned} \quad (4.1)$$

If we let  $b(p, \mathbf{v})$  be the bilinear form defined on  $L_0^{r'}(\Omega) \times \mathbf{W}_0^{1,r}(\Omega)$  by  $b(p, \mathbf{v}) = \int_{\Omega} p \nabla \cdot \mathbf{v} \, d\mathbf{x}$ , then the weak formulation (4.1) can be rewritten as the problem of finding  $(\mathbf{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times L_0^{r'}(\Omega)$  such that

$$\begin{aligned} \langle A(\mathbf{u}), \mathbf{v} \rangle - b(p, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega), \\ b(q, \mathbf{u}) &= 0 \quad \forall q \in L_0^{r'}(\Omega). \end{aligned} \quad (4.2)$$

The existence and uniqueness of solutions of (4.1) and (4.2) was studied by J. Baranger and Najib [5] and J. W. Barrett and W. B. Liu [8].

This mixed weak formulation requires the pressure  $p \in L_0^{r'}(\Omega)$  and the *LBB* condition is a sufficient condition to guarantee  $p \in L_0^{r'}(\Omega)$ . It is possible that the velocity field is well defined within  $\mathbf{W}_0^{1,r}(\Omega)$ , but the mixed weak formulation is not well-posed when, e.g., when the domain is nonconvex and the *LBB* condition fails to hold. In this case, good approximations of the pressure from the velocity field are not expected from the penalty method.

5. THE *LBB* CONDITION AND EQUIVALENCE OF (VF) AND (MWF)

Baranger and Najib prove in [5] that (4.2) is equivalent to (3.1) (and hence (1.6)) for any  $1 < r < \infty$  provided  $\Omega$  is 2-dimensional and  $\partial\Omega$  is smooth. They actually prove the following.

**Theorem 5.1.** *Suppose  $\Omega \subset \mathbb{R}^2$  has a smooth boundary. Let  $1 < r < \infty$ . Problem (4.2) has a unique solution if and only if (3.1) has a solution and the divergence operator  $B = \nabla \cdot$  is surjective and satisfies the following condition*

$$0 < \alpha \leq \inf_{q \in L_0^{r'}(\Omega)} \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle B\mathbf{v}, q \rangle}{\|q\|_{0,r'} \|\mathbf{v}\|_{1,r}}. \quad (5.1)$$

The same result is also stated in [8]. The inequality in Theorem 5.1 is often referred to as the *LBB condition* for the continuous model. Amrouche and Girault stated [3] the following generalization.

**Theorem 5.2.** *Let  $\Omega$  be a bounded, connected, Lipschitz-continuous domain in  $\mathbb{R}^d$  and let  $r$  be any real number with  $1 < r < \infty$ , and  $r'$  its conjugate. There exists a constant  $\beta > 0$  such that*

$$0 < \beta \leq \inf_{q \in L_0^{r'}(\Omega)} \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle B\mathbf{v}, q \rangle}{\|q\|_{0,r'} \|\mathbf{v}\|_{1,r}}. \quad (5.2)$$

This allows us to conclude the equivalence of (3.1) and (4.2) in our more general setting.

6. AN *A Priori* BOUND

Using Theorems 5.1 and 5.2, we conclude, (4.2) has an unique solution  $(\mathbf{u}, p)$  in which  $\mathbf{u}$  is the unique solution of (1.6). See, e.g, [5] and [25].

**Lemma 6.1.** *Let  $\mathbf{u}$  be the solution of (1.6), then  $\|\mathbf{u}\|_{1,r} \leq C$ . Suppose further that (5.2) holds, and let  $(\mathbf{u}, p)$  be the solution of (4.2). Then  $\|\mathbf{u}\|_{1,r} \leq C$  and  $\|p\|_{0,r'} \leq C$ . These constants  $C > 0$  depend only on  $r$ ,  $\Omega$  and  $\mathbf{f}$ .*

*Proof.* In (3.1), let  $\mathbf{v} = \mathbf{u}$ . Then  $\langle A(\mathbf{u}), \mathbf{u} \rangle = \langle \mathbf{f}, \mathbf{u} \rangle$ . We have, by (3.2),  $\|\mathbf{u}\|_{1,r}^r \leq C \langle A(\mathbf{u}), \mathbf{u} \rangle = C \langle \mathbf{f}, \mathbf{u} \rangle \leq C \|\mathbf{f}\|_{-1,r'} \|\mathbf{u}\|_{1,r}$  which implies

$$\|\mathbf{u}\|_{1,r} \leq C \|\mathbf{f}\|_{-1,r'}^{\frac{1}{r-1}}, \quad (6.1)$$

for  $2 \leq r \leq \infty$ . Similarly, by (3.3), we have  $\|\mathbf{u}\|_{1,r}^2 \leq C \langle A(\mathbf{u}), \mathbf{u} \rangle \|\mathbf{u}\|_{1,r}^{2-r} = C \langle \mathbf{f}, \mathbf{u} \rangle \|\mathbf{u}\|_{1,r}^{2-r}$  which gives (6.1) for  $1 < r \leq 2$ . By (5.2) and (4.2)

$$\begin{aligned} \|p\|_{0,r'} &\leq C \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle \nabla \cdot \mathbf{v}, p \rangle}{\|\mathbf{v}\|_{1,r}} \\ &= C \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle A(\mathbf{u}), \mathbf{v} \rangle - \langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{1,r}} \\ &\leq C (\|A(\mathbf{u})\|_{-1,r'} + \|\mathbf{f}\|_{-1,r'}). \end{aligned} \quad (6.2)$$

Upon applying (3.2) and (3.3) to the right hand side of (6.2) and using (6.1) we conclude that  $\|p\|_{0,r'} \leq C$  for  $1 < r < \infty$ .  $\square$

## 7. THE PENALTY FORMULATION FOR THE STOKES PROBLEM

Let  $\epsilon$  be a positive number and consider the following functional

$$J_\epsilon(\mathbf{u}) = J(\mathbf{u}) + \frac{1}{r\epsilon} \int_\Omega |\nabla \cdot \mathbf{u}|^r dx,$$

where  $J(\mathbf{u})$  is defined in (1.6). The minimizer  $\mathbf{u}$  of  $J_\epsilon(\mathbf{u})$  over  $\mathbf{W}_0^{1,r}(\Omega)$  satisfies the Euler-Lagrange equation

$$\langle A(\mathbf{u}), \mathbf{v} \rangle + \frac{1}{\epsilon} \langle \phi(\nabla \cdot \mathbf{u}), \nabla \cdot \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega). \quad (7.1)$$

Note that  $J_\epsilon(\mathbf{u})$  is strictly convex and the operator  $T_\epsilon : \mathbf{W}_0^{1,r}(\Omega) \rightarrow \mathbf{W}^{-1,r'}(\Omega)$  defined by  $T_\epsilon(\mathbf{u}) = J'_\epsilon(\mathbf{u})$  is a bounded, coercive, and strictly monotone operator on  $\mathbf{W}_0^{1,r}(\Omega)$ , since  $\phi$  satisfies (2.1) and (2.2). Therefore, by Theorem 2.1, (7.1) has a solution  $\mathbf{u}^\epsilon$  which is the unique solution of

$$\min_{\mathbf{u} \in \mathbf{W}_0^{1,r}(\Omega)} J_\epsilon(\mathbf{u}). \quad (7.2)$$

We now prove two main results related to the penalty approximations  $\mathbf{u}^\epsilon$  of solutions  $\mathbf{u}$  of (1.6). The first is a general convergence result, and the second is a more precise error estimate which holds provided the *LBB* condition also holds.

**Theorem 7.1.** *Let  $\epsilon > 0$  be given and suppose that  $\mathbf{u}$  is the solution of (1.6) and  $\mathbf{u}^\epsilon$  is the solution of (7.2). Then  $\mathbf{u}^\epsilon$  converges strongly to  $\mathbf{u}$  in  $\mathbf{W}_0^{1,r}(\Omega)$  as  $\epsilon \rightarrow 0$ . Furthermore,  $\exists C$  independent of  $\epsilon$  such that  $\|\mathbf{u}^\epsilon\|_{1,r} \leq C$  and  $\|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r} \leq C\epsilon^{\frac{1}{r}}$ . Therefore, it follows that  $\nabla \cdot \mathbf{u}^\epsilon \rightarrow 0$  in  $L^r(\Omega)$  as  $\epsilon \rightarrow 0$ .*



*Proof.* The proof that  $\mathbf{u}^\epsilon$  converges strongly to  $\mathbf{u}$  in  $\mathbf{W}_0^{1,r}(\Omega)$  as  $\epsilon \rightarrow 0$  follows along the lines of [46, Theorem 46.D. and Corollary 46.7]. We only need to check that the functional  $G(\mathbf{v}) = \|\nabla \cdot \mathbf{v}\|_{0,r}$  is weakly sequentially continuous in  $\mathbf{W}_0^{1,r}(\Omega)$ . To this end, let  $\mathbf{v}_n \rightharpoonup \mathbf{v}$  in  $\mathbf{W}_0^{1,r}(\Omega)$ . Then  $\langle \nabla \cdot \mathbf{v}_n, \eta \rangle \rightarrow \langle \nabla \cdot \mathbf{v}, \eta \rangle$  as  $n \rightarrow \infty$  for any  $\eta \in L^{r'}(\Omega)$ . Indeed,  $C^\infty(\Omega)$  is dense in  $L^{r'}(\Omega)$  and

$$\langle \nabla \cdot \mathbf{v}_n, \eta \rangle = -\langle \mathbf{v}_n, \nabla \eta \rangle \rightarrow -\langle \mathbf{v}, \nabla \eta \rangle = \langle \nabla \cdot \mathbf{v}, \eta \rangle, \forall \eta \in C^\infty(\Omega).$$

Therefore,  $\nabla \cdot \mathbf{v}_n \rightharpoonup \nabla \cdot \mathbf{v}$  in  $L^r(\Omega)$ . It is well known that the norm  $\|\cdot\|_{0,r}$  is weakly sequentially continuous in  $L^r(\Omega)$ . We conclude that  $\lim_{n \rightarrow \infty} \|\nabla \cdot \mathbf{v}_n\|_{0,r} = \|\nabla \cdot \mathbf{v}\|_{0,r}$  and hence  $\mathbf{u}^\epsilon \rightarrow \mathbf{u}$  in  $\mathbf{W}_0^{1,r}(\Omega)$  because of [46, Theorem 46.D. and Corollary 46.7]. To complete the proof, let  $\mathbf{u} = \mathbf{v} = \mathbf{u}^\epsilon$  in (7.1). We have

$$\|\mathbf{u}^\epsilon\|_{1,r}^r + \frac{1}{\epsilon} \|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r}^r \leq C \|\mathbf{f}\|_{-1,r'} \|\mathbf{u}^\epsilon\|_{1,r}$$

which implies  $\|\mathbf{u}^\epsilon\|_{1,r}^{r-1} \leq C \|\mathbf{f}\|_{-1,r'}$  and  $\|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r}^r \leq \epsilon C \|\mathbf{f}\|_{-1,r'}^r$ . Therefore  $\|\mathbf{u}^\epsilon\|_{1,r} \leq C$  and  $\nabla \cdot \mathbf{u}^\epsilon \rightarrow 0$  in  $L^r(\Omega)$  as  $\epsilon \rightarrow 0$ .  $\square$

Note that when  $r = 2$ , Theorem 7.1 is a well-known result [43]. A generalized version of it for convergence in higher order derivative norms can be found in [3].

**Lemma 7.2.** *For each  $\epsilon > 0$ , let  $\mathbf{u}^\epsilon$  denote the unique minimizer of (7.2) and let  $p^\epsilon = c - \frac{1}{\epsilon} \phi(\nabla \cdot \mathbf{u}^\epsilon)$ , where  $c = \frac{1}{\epsilon|\Omega|} \int_\Omega \phi(\nabla \cdot \mathbf{u}^\epsilon) dx$ . If (5.2) holds (which is the LBB condition), then there exists  $C > 0$  which depends only on  $r, \Omega$  and  $\mathbf{f}$  such that  $\|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r} \leq C\epsilon^{\frac{1}{r-1}}$  and  $\|p^\epsilon\|_{0,r'} \leq C$ .*

*Proof.* The pair  $(\mathbf{u}^\epsilon, p^\epsilon)$  satisfies

$$\langle A(\mathbf{u}^\epsilon), \mathbf{v} \rangle - b(p^\epsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega), \tag{7.3}$$

since  $\mathbf{u}^\epsilon$  satisfies (7.1) and  $\int_\Omega c \nabla \cdot \mathbf{v} dx = c \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} ds = 0$  by Gauss' Theorem. Moreover,  $p^\epsilon \in L_0^{r'}(\Omega)$ , and by (5.2) and (7.3)

$$\begin{aligned} \|p^\epsilon\|_{0,r'} &\leq C \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle \nabla \cdot \mathbf{v}, p^\epsilon \rangle}{\|\mathbf{v}\|_{1,r}} = C \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{b(p^\epsilon, \mathbf{v})}{\|\mathbf{v}\|_{1,r}} \\ &= C \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\langle A(\mathbf{u}^\epsilon), \mathbf{v} \rangle - \langle \mathbf{f}, \mathbf{v} \rangle}{\|\mathbf{v}\|_{1,r}} \\ &\leq C(\|A(\mathbf{u}^\epsilon)\|_{-1,r'} + \|\mathbf{f}\|_{-1,r'}). \end{aligned}$$

Using (3.2) and (3.3) we conclude  $\|p^\epsilon\|_{0,r'} \leq C(\|\mathbf{u}^\epsilon\|_{1,r}^{r-1} + \|\mathbf{f}\|_{-1,r'})$ , therefore by Theorem 7.1

$$\|p^\epsilon\|_{0,r'} \leq C \|\mathbf{f}\|_{-1,r'}.$$

Let  $\mathbf{v} = \mathbf{u} - \mathbf{u}^\epsilon$  in (4.2) and (7.3) and then subtract (7.3) from (4.2) to get

$$\langle A(\mathbf{u}) - A(\mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon \rangle = b(p - p^\epsilon, \mathbf{u} - \mathbf{u}^\epsilon). \tag{7.4}$$

Since  $\nabla \cdot \mathbf{u} = 0$  and  $\int_\Omega \nabla \cdot \mathbf{u}^\epsilon dx = 0$ , the above equation gives

$$\langle A(\mathbf{u}) - A(\mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon \rangle + \frac{1}{\epsilon} \langle \phi(\nabla \cdot \mathbf{u}^\epsilon), \nabla \cdot \mathbf{u}^\epsilon \rangle = -b(p, \mathbf{u}^\epsilon).$$

Since  $\langle A(\mathbf{u}) - A(\mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon \rangle \geq 0$  and  $|b(p, \mathbf{u}^\epsilon)| \leq \|p\|_{0,r'} \|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r}$  we have

$$\frac{1}{\epsilon} \|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r}^r = \frac{1}{\epsilon} \langle \phi(\nabla \cdot \mathbf{u}^\epsilon), \nabla \cdot \mathbf{u}^\epsilon \rangle \leq \|p\|_{0,r'} \|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r},$$

which gives  $\|\nabla \cdot \mathbf{u}^\epsilon\|_{0,r} \leq C\epsilon^{\frac{1}{r-1}}$  by using Lemma 6.1 to bound  $\|p\|_{0,r'}$ .  $\square$

**Theorem 7.3.** *Suppose (5.2) (the LBB condition) holds. Let  $(\mathbf{u}, p)$  be the unique solution of (4.2) and  $\mathbf{u}^\epsilon$  be a solution of (7.2). Let  $p^\epsilon = c - \frac{1}{\epsilon}\phi(\nabla \cdot \mathbf{u}^\epsilon)$ , where  $c = \frac{1}{\epsilon|\Omega|} \int_\Omega \phi(\nabla \cdot \mathbf{u}^\epsilon) dx$ . Then there exists  $C > 0$  which depends only on  $r$ ,  $\Omega$ , and  $\mathbf{f}$  such that*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r} &\leq C\epsilon^{\frac{1}{(r-1)(3-r)}} \quad \text{for } 1 < r \leq 2 \\ \text{and } \|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r} &\leq C\epsilon^{\frac{1}{(r-1)^2}} \quad \text{for } 2 \leq r < \infty. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|p - p^\epsilon\|_{0,r'} &\leq C\epsilon^{\frac{1}{(3-r)}} \quad \text{for } 1 < r \leq 2 \\ \text{and } \|p - p^\epsilon\|_{0,r'} &\leq C\epsilon^{\frac{1}{(r-1)^2}} \quad \text{for } 2 \leq r < \infty. \end{aligned}$$

*Proof.* By (5.2), we have

$$\|p - p^\epsilon\|_{0,r'} \leq C \sup_{\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)} \frac{b(p - p^\epsilon, \mathbf{v})}{\|\mathbf{v}\|_{1,r}}. \quad (7.5)$$

Subtracting (7.3) from (4.2) gives

$$b(p - p^\epsilon, \mathbf{v}) = \langle A(\mathbf{u}), \mathbf{v} \rangle - \langle A(\mathbf{u}^\epsilon), \mathbf{v} \rangle. \quad (7.6)$$

Using (7.5) and (7.6) we get

$$\|p - p^\epsilon\|_{0,r'} \leq C \|A(\mathbf{u}) - A(\mathbf{u}^\epsilon)\|_{-1,r'}. \quad (7.7)$$

By (7.7), (3.2), Lemma 6.1 and Theorem 7.1, we have

$$\|p - p^\epsilon\|_{0,r'} \leq C \|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r} (\|\mathbf{u}\|_{1,r} + \|\mathbf{u}^\epsilon\|_{1,r})^{r-2} \leq C \|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r} \quad (7.8)$$

for  $2 \leq r < \infty$ . Similarly, using (7.7) and (3.3) we get for  $1 < r \leq 2$

$$\|p - p^\epsilon\|_{0,r'} \leq C \|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r}^{r-1}. \quad (7.9)$$

Since  $(\mathbf{u}, p)$  solves (4.2) we have  $\nabla \cdot \mathbf{u} = 0$  and hence by (7.4)

$$\langle A(\mathbf{u}) - A(\mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon \rangle = -b(p - p^\epsilon, \mathbf{u}^\epsilon).$$

Therefore, for  $1 < r \leq 2$ , by (3.3), (7.4), and the uniform bounds on  $\|\mathbf{u}\|_{1,r}$  and  $\|\mathbf{u}^\epsilon\|_{1,r}$  in Lemma 6.1 and Theorem 7.1 we have

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r}^2 \leq C \langle A(\mathbf{u}) - A(\mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon \rangle (\|\mathbf{u}\|_{1,r} + \|\mathbf{u}^\epsilon\|_{1,r})^{2-r} \leq C |b(p - p^\epsilon, \mathbf{u}^\epsilon)|.$$

Similarly, for  $2 \leq r < \infty$

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r}^r \leq C \langle A(\mathbf{u}) - A(\mathbf{u}^\epsilon), \mathbf{u} - \mathbf{u}^\epsilon \rangle \leq C |b(p - p^\epsilon, \mathbf{u}^\epsilon)|.$$

Applying Lemma 7.2 and the bounds in (7.8), (7.9) we have for  $2 \leq r < \infty$

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r}^{r-1} \leq C\epsilon^{\frac{1}{r-1}}, \quad (7.10)$$

and for  $1 < r \leq 2$

$$\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r}^{3-r} \leq C\epsilon^{\frac{1}{r-1}}. \quad (7.11)$$

This gives the desired estimates for  $\|\mathbf{u} - \mathbf{u}^\epsilon\|_{1,r}$ . Using (7.8), (7.9) and (7.10), (7.11) we have the error estimates for  $\|p - p^\epsilon\|_{0,r'}$ .  $\square$

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