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Multiple solutions to some singular nonlinear Schrödinger equations *

Monica Lazzo

Abstract

We consider the equation $-h^2\Delta u + V_\varepsilon(x)u = |u|^{p-2}u$ which arises in the study of standing waves of a nonlinear Schrödinger equation. We allow the potential V_ε to be unbounded below and prove existence and multiplicity results for positive solutions.

1 Introduction

In recent years, much interest has been paid to the nonlinear Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = -h^2\Delta\psi + U(x)\psi - |\psi|^{p-2}\psi, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where h is a positive constant, U is a continuous potential and p is greater than 2 and less than 2^*, the critical Sobolev exponent.

When looking for standing waves of (1.1), namely solutions of the form

$$\psi(t,x) = \exp(-i\lambda h^{-1}t)\,u(x)$$

with $\lambda \in \mathbb{R}$ and u a real function, one is led to solve the following elliptic problem in $\mathbb{R}^N$:

$$\begin{gathered} -h^2\Delta u + V(x)u = |u|^{p-2}u\,, \quad x \in \mathbb{R}^N \\ \lim_{|x|\to\infty} u(x) = 0 \end{gathered} \tag{1.2}$$

where $V(x) = U(x) + \lambda$.

The existence of solutions to (1.2) corresponding to small values of the parameter h and their behaviour as h tends to zero are of particular concern in the so-called semiclassical analysis. The first result on semiclassical solutions for (1.2) is ascribed to Floer and Weinstein [11], who consider a cubic nonlinearity in the one dimensional case. They assume that the potential V is

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bounded and it has a nondegenerate critical point x_0; via a Lyapunov-Schmidt finite dimensional reduction, they find a solution to (1.2), for small h. Furthermore, they prove that a concentration phenomenon occurs: as h tends to zero, their solutions tend in a suitable sense to the solution of the limit equation $-u'' + V(x_0)u = u^3$, rescaled around x_0.

Floer and Weinstein's results were generalized to higher dimensions and arbitrary subcritical exponents by Oh [17]. Afterwards, many authors contributed to solving (1.2) by using various methods which in turn required various assumptions on the potential V; see, for instance, [1], [2], [10], [18] and references therein for a partial account on the topic.

Our results are mainly inspired by a paper by Rabinowitz [18]. If

$$0 < V_0 \equiv \inf_{\mathbb{R}^N} V < \liminf_{|x|\to\infty} V(x)\,, \tag{1.3}$$

then (1.2) has a solution u_h, for h sufficiently small (see Theorem 4.33 in [18]). The approach in [18] is a variational one: solutions to (1.2) are found as critical points of the energy functional

$$I_h(u) = \frac{h^2}{2}\int |\nabla u|^2 + \frac{1}{2}\int V(x)|u|^2 - \frac{1}{p}\int |u|^p$$

in a suitable Hilbert space. The functional I_h exhibits a mountain pass–type geometry; the lack of compactness, due to the unboundedness of the domain, is overcome by means of (1.3), and u_h is obtained via a mountain pass–type argument.

Our goal in this paper is to show that a result in the spirit of [18] holds if the potential in (1.2) is perturbed by adding a negative potential which may be singular, so that the resulting potential may be unbounded below.

Precisely, we consider the potential

$$V_\varepsilon(x) = V(x) - \varepsilon(h)W(x)\,,$$

where $\varepsilon : [0,+\infty) \longrightarrow [0,+\infty)$ and $W : \mathbb{R}^N \longrightarrow [0,+\infty)$ is a measurable function such that, for some $\alpha_1 > 0$ and $\alpha_2 \geq 0$, the inequality

$$\int W(x)|u|^2 \leq \alpha_1\|\nabla u\|_2^2 + \alpha_2\|u\|_2^2 \tag{1.4}$$

holds for any $u \in H^1(\mathbb{R}^N)$.

We are interested in existence and multiplicity of solutions for the problem

$$\begin{gathered} -h^2\Delta u + V_\varepsilon(x)\,u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N \\ \lim\nolimits_{|x|\to\infty} u(x) = 0\,. \end{gathered} \tag{1.5}$$

Our first result is the following

Theorem 1.1 *Assume* (1.3) *and* (1.4)*. There exists $\varepsilon^* > 0$ such that, if*

$$\limsup_{h\to 0} \frac{\varepsilon(h)}{h^2} < \varepsilon^*\,, \tag{1.6}$$

then (1.5) *has a positive solution, for h sufficiently small.*

Let us remark that in [18], as well as in the other papers quoted above, the potential that defines the Schrödinger operator is bounded below. On the other hand, the potential V_ε we consider may be unbounded below, since (1.4) may well be satisfied by potentials W which are unbounded above. From this standpoint Theorem 1.1, if very natural and quite simple to prove, seems not to be known.

In the paper we also obtain a multiplicity result, by relating the number of solutions of (1.5) with the topology of the set of global minima of V. In order to state our result we need the following standard notation: if Y is a closed subset of a topological space X, $\mathrm{cat}_X(Y)$ is the Ljusternik–Schnirelman category of Y in X, namely the least number of closed and contractible sets in X which cover Y. If $X = Y$, we set $\mathrm{cat}_X(X) = \mathrm{cat}(Y)$.

Let M be the set of global minima of V and, for any positive δ, let $M_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \le \delta\}$.

Theorem 1.2 *Assume* (1.3) *and* (1.4). *For any $\delta > 0$ there exists $\varepsilon^{**}(\delta) > 0$ such that, if*

$$\limsup_{h\to 0} \frac{\varepsilon(h)}{h^2} < \varepsilon^{**}(\delta)\,, \tag{1.7}$$

then (1.5) *has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions, for h sufficiently small.*

In several situations, $\mathrm{cat}_{M_\delta}(M)$ and $\mathrm{cat}(M)$ agree, at least for small δ. This is the case, for instance, if M is the closure of a bounded open set with smooth boundary, a smooth and compact submanifold of $\mathbb{R}^N$ or a finite set; in the last case, the category of M is nothing but the cardinality of M.

As a motivation for Theorem 1.2, we recall that, as proved by Wang [19], the family of solutions u_h found in [18] concentrates near global minima of V, as h tends to 0. Therefore, a rather natural question is: is it possible to relate the multiplicity of solutions for (1.5) with the topological richness of the set of minimum points of V? In [6] an affirmative answer was given for the unperturbed problem (1.2), that is, for $\varepsilon(h)$ identically zero. Theorem 1.2 is a natural generalization of the result in [6]: the number of solutions to (1.5) can still be related with the topology of the global minima set of the unperturbed potential, provided the perturbation is small with respect to the coefficient of the differential term, in the sense of (1.7).

Let us end this section by giving some examples of potentials satisfying (1.4). Let W be in the so-called Kato–Rellich class, namely $W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, with $q = 2$ if $N \le 3$, $q > 2$ if $N = 4$ and $q \ge N/2$ if $N \ge 5$. Then the following property, that obviously implies (1.4), holds: for any $\xi > 0$ there exists $\alpha_\xi > 0$ such that $\int W(x)|u|^2 \le \xi \|\nabla u\|_2^2 + \alpha_\xi \|u\|_2^2$ for any $u \in H^1(\mathbb{R}^N)$ (for the proof, see for instance [21]). Notice that, for example, the Coulomb potential $|x|^{-1}$ is in the Kato–Rellich class, for $N \ge 3$. Next, let $W \in L^{N/2}(\mathbb{R}^N) \cap L^\beta(\mathbb{R}^N)$, for some $\beta > N/2$, then the eigenvalue problem $-\Delta u = \lambda W(x)u$, $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, has the same properties as an eigenvalue problem for $-\Delta$ in a bounded domain (see [9]). In particular, the first eigenvalue $\lambda_1(-\Delta, \mathbb{R}^N, W)$ is strictly positive and, as a consequence, (1.4) is fulfilled with $\alpha_1 = \lambda_1(-\Delta, \mathbb{R}^N, W)$ and $\alpha_2 = 0$. Finally, let $W(x) = |x|^{-2}$ (such a potential is in none of the previous classes). In this case, Hardy inequality gives (1.4), with $\alpha_1 = 4/(N-2)^2$ and $\alpha_2 = 0$.

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2 Preliminaries

Let $H^1(\mathbb{R}^N)$ be the standard Sobolev space endowed with the standard norm $\|\cdot\|_{H^1}$ and $\mathcal{H} = \left\{u \in H^1(\mathbb{R}^N) : \int V(x)|u|^2 < +\infty\right\}$; unless otherwise stated, the integration set $\mathbb{R}^N$ will be understood.

In $\mathcal{H}$ we define the functionals

$$J_{h,\varepsilon}(u) = \int h^2|\nabla u|^2 + V_\varepsilon(x)|u|^2\,, \qquad J_{h,0}(u) = \int h^2|\nabla u|^2 + V(x)|u|^2\,.$$

Clearly, $J_{h,\varepsilon}(u) \le J_{h,0}(u)$ for any u . Conversely, if (1.4) holds and $0 < h^2 \le V_0\,\alpha_1\,\alpha_2^{-1}$ (no restrictions on h if $\alpha_2 = 0$), then for any $u \in \mathcal{H}$ we have

$$\left(1 - \alpha_1\,\frac{\varepsilon(h)}{h^2}\right) J_{h,0}(u) \le J_{h,\varepsilon}(u)\,. \tag{2.1}$$

Indeed,

$$\int W(x)|u|^2 \le \alpha_1 \int |\nabla u|^2 + \frac{\alpha_2}{V_0}\int V(x)|u|^2 \le \frac{\alpha_1}{h^2}\,J_{h,0}(u)\,.$$

As a consequence,

$$J_{h,0}(u) = J_{h,\varepsilon}(u) + \varepsilon(h)\int W(x)|u|^2 \le J_{h,\varepsilon}(u) + \alpha_1\,\frac{\varepsilon(h)}{h^2} J_{h,0}(u)$$

whence (2.1) follows. From (2.1), if $\limsup_{h\to 0}\varepsilon(h)h^{-2} < \alpha_1^{-1}$ there exist $\alpha_0, h_0^* > 0$ such that

$$J_{h,\varepsilon}(u) \ge \min\left\{h^2, V_0\right\}\alpha_0\,\|u\|_{H^1}^2 \tag{2.2}$$

for any $u \in \mathcal{H}$, for any $0 < h < h_0^*$. As a result, the set $\mathcal{H}$, endowed with the norm $\|u\|_h^2 = J_{h,\varepsilon}(u)$, is a Hilbert space and it is continuously embedded in $H^1(\mathbb{R}^N)$.

Let us define the manifold $\Sigma = \left\{u \in \mathcal{H}\ : \int |u|^p = 1\right\}$. Plainly, $J_{h,\varepsilon}$ is well defined and smooth on Σ; moreover, for any critical point u of $J_{h,\varepsilon}$ on Σ, $\left(J_{h,\varepsilon}(u)\right)^{\frac{1}{p-2}} u$ is a weak solution for (1.5).

We are interested in positive solutions for (1.5). As it is easily guessed, low energy solutions do not change sign; this is the content of the next proposition. First we need some notations.

We recall that, for any positive h and λ, the equation with constant coefficients

$$-h^2\Delta u + \lambda\, u = |u|^{p-2}u \qquad \text{in } \mathbb{R}^N$$

has a unique positive solution $\tilde{\omega}(h;\lambda) \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$, which is radially symmetric around the origin and decays exponentially at infinity (see for instance [4], [8], [12], [13]). The infimum

$$m(h;\lambda) = \inf\Big\{ \frac{h^2\|\nabla u\|_2^2 + \lambda\|u\|_2^2}{\|u\|_p^2} \,:\, u \in H^1(\mathbb{R}^N)\,,\ u \neq 0 \Big\}$$

is achieved in $\omega(h;\lambda) = \tilde{\omega}(h;\lambda)/\|\tilde{\omega}(h;\lambda)\|_p$. A straighforward computation gives $m(h;\lambda) = h^\theta\, m(1;\lambda)$ with $\theta = \frac{N(p-2)}{p}$. For convenience, we set

$$m_0 = m(1;V_0)\,.$$

We are ready to state our result on the sign of solutions to (1.5).

Proposition 2.1 *Assume* (1.3), (1.4) *and*

$$\limsup_{h\to 0} \frac{\varepsilon(h)}{h^2} < \frac{1}{\alpha_1}\,\Big(1 - 2^{\frac{2-p}{p}}\Big)\,. \tag{2.3}$$

Then there exist $k_1^*, h_1^* > 0$ *such that, for any* $0 < h < h_1^*$*, every critical point* u *of* $J_{h,\varepsilon}$ *on* Σ *satisfying*

$$J_{h,\varepsilon}(u) \leq (m_0 + k_1^*)\,h^\theta \tag{2.4}$$

does not change sign.

Proof. Let ε_0 be the left-hand side in (2.3). Fix $\eta_0 > 0$ such that $0 < \alpha_1(\varepsilon_0 + \eta_0) < 1 - 2^{\frac{2-p}{p}}$ and let $h_1^* \in \big(0, h_0^*\big)$ be such that $\varepsilon(h) < (\varepsilon_0 + \eta_0)h^2$ for any $0 < h < h_1^*$. Finally, choose

$$0 < k_1^* < \Big(2^{\frac{p-2}{p}}\big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big) - 1\Big)m_0\,. \tag{2.5}$$

Now, let $0 < h < h_1^*$ and let $u = u^+ + u^-$ be a critical point of $J_{h,\varepsilon}$ on Σ such that $u^+, u^- \not\equiv 0$. If we multiply

$$-h^2\Delta u + V_\varepsilon(x)u = J_{h,\varepsilon}(u)|u|^{p-2}u$$

by u^+ and integrate on $\mathbb{R}^N$, we get $J_{h,\varepsilon}(u)\|u^+\|_p^p = J_{h,\varepsilon}(u^+) \geq c_h\|u^+\|_p^2$, thus $\|u^+\|_p^p \geq (c_h/J_{h,\varepsilon}(u))^{\frac{p}{p-2}}$. Obviously the same inequality holds for u^-, thus $1 = \|u^+\|_p^p + \|u^-\|_p^p \geq 2\,(c_h/J_{h,\varepsilon}(u))^{\frac{p}{p-2}}$, whence $J_{h,\varepsilon}(u) \geq 2^{\frac{p-2}{p}}c_h$. Then (2.4), (2.1) and the definition of m_0 give

$$\big(m_0 + k_1^*\big)h^\theta \geq J_{h,\varepsilon}(u) \geq 2^{\frac{p-2}{p}}\Big(1 - \alpha_1\,(\varepsilon_0 + \eta_0)\Big)m_0\,h^\theta\,;$$

if we divide by h^θ we contradict (2.5). $\diamondsuit$

3 Palais–Smale condition

Before looking for critical points of $J_{h,\varepsilon}$ on Σ, we deal with the compactness issue. It is well known that (1.5) is affected by a lack of compactness, due to the noncompact Sobolev embedding $H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$. As a result, $J_{h,\varepsilon}$ may not satisfy Palais–Smale condition globally on Σ; nevertheless, we can show that Palais–Smale condition holds below some level, related to $\liminf_{|x|\to\infty} V(x)$. In order to state this result, we need some more notations.

By (1.3), we can choose $V_\infty \in \mathbb{R}$ such that

$$V_0 < V_\infty \le \liminf_{|x|\to\infty} V(x)\,. \tag{3.1}$$

Let us denote

$$m_\infty = m(1; V_\infty)\,;$$

the map $\lambda \mapsto m(1;\lambda)$ being strictly increasing, (3.1) implies

$$m_0 < m_\infty\,. \tag{3.2}$$

Proposition 3.1 *Assume* (1.3), (1.4) *and*

$$\limsup_{h\to 0} \frac{\varepsilon(h)}{h^2} < \frac{1}{\alpha_1}\left(1 - \frac{m_0}{m_\infty}\right). \tag{3.3}$$

Then there exist $k_2^* \in (0, m_\infty - m_0)$ and $h_2^* > 0$ such that $J_{h,\varepsilon}$ satisfies Palais–Smale condition in the sublevel $\{u \in \Sigma \,:\, J_{h,\varepsilon}(u) < (m_0 + k_2^*)\,h^\theta\}$, for any $0 < h < h_2^*$.

Proof. Let ε_0 be the left-hand side in (3.3), let $\widetilde{C} \in \big(m_0, (1-\alpha_1\varepsilon_0)\,m_\infty\big)$ and fix $\eta_0 > 0$ such that

$$\widetilde{C} + \alpha_1\,\eta_0\,m_\infty < (1-\alpha_1\varepsilon_0)\,m_\infty\,; \tag{3.4}$$

obviously, for h small we have

$$\varepsilon(h)\,h^{-2} \le \varepsilon_0 + \eta_0\,. \tag{3.5}$$

Next, let $C < \widetilde{C}$ and let $\{u_n\} \subset \Sigma$ be a Palais–Smale sequence for $J_{h,\varepsilon}$ on Σ at the level $C_h \equiv C\,h^\theta$, namely

$$J_{h,\varepsilon}(u_n) = C_h + o(1) \tag{3.6}$$

$$-h^2\Delta u_n + V_\varepsilon(x)u_n - \lambda_n|u_n|^{p-2}u_n = o(1) \quad \text{in } \mathcal{H}^{-1} \tag{3.7}$$

as $n\to\infty$; it is easily seen that $\lambda_n = C_h + o(1)$. Trivially $\{u_n\}$ is bounded in $\mathcal{H}$, therefore it has a weak limit $u \in \mathcal{H}$. In order to prove that $\{u_n\}$ converges to u strongly in $\mathcal{H}$ we apply Lions' Concentration–Compactness Lemma (see [15], [16]) to the sequence of measures $\rho_n = h^2|\nabla u_n|^2 + V_\varepsilon(x)|u_n|^2$. By definition, $\int \rho_n \to C_h$ as $n\to\infty$, and $C_h > 0$ because of (2.2). Vanishing is easily ruled out since $u_n \in \Sigma$. If dichotomy occurs, there exist $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 = C_h$ such that for any $\xi > 0$ there are $y_n \in \mathbb{R}^N$, $R > 0$, $R_n \to \infty$ such that

$$\int_{|x-y_n|<R} \rho_n \geq \delta_1 - \xi\,, \quad \int_{|x-y_n|>2R_n} \rho_n \geq \delta_2 - \xi\,. \tag{3.8}$$

As a consequence,

$$\int_{2R<|x-y_n|<R_n} \rho_n \leq 2\xi\,. \tag{3.9}$$

Let $\zeta : [0, +\infty) \to [0, 1]$ be a smooth, non increasing function, such that $\zeta(t) = 1$ if $0 \leq t \leq 1$, $\zeta(t) = 0$ if $t \geq 2$. If we define

$$u_n^1(x) = u_n(x)\,\zeta\Big(\frac{x-y_n}{R}\Big), \quad u_n^2(x) = u_n(x) - u_n(x)\,\zeta\Big(\frac{x-y_n}{R_n}\Big), \tag{3.10}$$

then (3.8) yields

$$\int h^2|\nabla u_n^i|^2 + V_\varepsilon(x)|u_n^i|^2 \geq \delta_i - \xi\,, \quad i = 1, 2\,.$$

From the definition of u_n^i and (3.9) we get

$$\int \nabla u_n \cdot \nabla u_n^i = \int |\nabla u_n^i|^2 + O(\xi)\,, \quad \int V_\varepsilon(x) u_n u_n^i = \int V_\varepsilon(x)|u_n^i|^2 + O(\xi)\,,$$
$$\int |u_n|^{p-2} u_n u_n^i = \int |u_n^i|^p + O(\xi)$$

whence, by taking (3.7) into account,

$$J_{h,\varepsilon}(u_n^i) = \int h^2|\nabla u_n^i|^2 + V_\varepsilon(x)|u_n^i|^2 = C_h \int |u_n^i|^p + o(1) + O(\xi)\,. \tag{3.11}$$

Now, if the sequence $\{y_n\}$ is unbounded in $\mathbb{R}^N$, for large n we have $V(x) \geq V_\infty - \xi$ for any $x \in B_R(y_n)$. Thus from (2.1), (3.5), the definition of $m(h; V_\infty)$ and (3.11) we get

$$\begin{aligned} J_{h,\varepsilon}(u_n^1) &\geq \Big(1 - \alpha_1\,\frac{\varepsilon(h)}{h^2}\Big) \int h^2|\nabla u_n^1|^2 + V(x)|u_n^1|^2 \\ &\geq O(\xi) + \big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big) \int h^2|\nabla u_n^1|^2 + V_\infty|u_n^1|^2 \\ &\geq O(\xi) + \big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big)\, m(h; V_\infty)\|u_n^1\|_p^2 \\ &= O(\xi) + o(1) + \big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big)\, m(h; V_\infty) \left(\frac{J_{h,\varepsilon}(u_n^1)}{C_h}\right)^{2/p} \end{aligned}$$

whence

$$J_{h,\varepsilon}(u_n^1) \geq O(\xi) + o(1) + \big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big)^{\frac{p}{p-2}}\, m(h; V_\infty)^{\frac{p}{p-2}}\, C_h^{\frac{2}{2-p}}\,. \tag{3.12}$$

From (3.6) and (3.12)

$$\begin{aligned} C_h + o(1) &\geq J_{h,\varepsilon}(u_n^1) + O(\xi) \\ &\geq O(\xi) + o(1) + \big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big)^{\frac{p}{p-2}} m(h; V_\infty)^{\frac{p}{p-2}} C_h^{\frac{2}{2-p}} ; \end{aligned}$$

letting $\xi \to 0$, $n \to \infty$ and dividing by h^θ yields

$$C \geq \big(1 - \alpha_1(\varepsilon_0 + \eta_0)\big)\, m_\infty$$

and, from (3.4), $C > \widetilde{C}$, a contradiction. If the sequence $\{y_n\}$ is bounded in $\mathbb{R}^N$, for large n we have $V(x) \geq V_\infty - \xi$ for any x such that $|x - y_n| > R_n$, and again we get a contradiction by taking u_n^2 into account. Dicotomy is therefore ruled out in any case. As a result, the sequence $\{\rho_n\}$ is tight: there exists $\{y_n\} \subset \mathbb{R}^N$ such that for any $\xi > 0$

$$\int_{|x-y_n|<R} h^2|\nabla u_n|^2 + V_\varepsilon(x)|u_n|^2 \geq C_h - \xi$$

for a suitable $R > 0$. If the sequence $\{y_n\}$ were unbounded in $\mathbb{R}^N$, we could define u_n^1 as in (3.10) and, noticing that

$$\int h^2|\nabla u_n^1|^2 + V_\varepsilon(x)|u_n^1|^2 \geq C_h - \xi\,,$$

we could get a contradiction exactly as before. So $\{y_n\}$ is bounded in $\mathbb{R}^N$, and for some $\overline{R} > 0$ we have

$$\int_{|x|>\overline{R}} h^2|\nabla u_n|^2 + V_\varepsilon(x)|u_n|^2 < \xi + o(1)\,.$$

At this point, the compactness of the embedding $H^1 \subset L^p$ on bounded domains implies that $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$, so that

$$\begin{aligned} \int h^2|\nabla u_n|^2 + V_\varepsilon(x)|u_n|^2 &= C_h \int |u_n|^p + o(1) = C_h \int |u|^p + o(1) \\ &= \int h^2|\nabla u|^2 + V_\varepsilon(x)|u|^2 + o(1) + O(\xi)\,. \end{aligned}$$

In other words, $\|u_n\|_h^2 \to \|u\|_h^2$, thus $u_n \to u$ strongly in $\mathcal{H}$. $\diamondsuit$

Remark 3.2 *Proposition 3.1 and the choice of V_∞ imply that, when V is coercive, $J_{h,\varepsilon}$ satisfies Palais–Smale condition on Σ at any level. With no loss of generality, we shall henceforth assume $V_\infty = \liminf_{|x|\to\infty} V(x) < +\infty$.*

4 Proof of Theorem 1.1

In order to find a solution to (1.5), it suffices to prove that the minimization problem

$$c_h = \inf_{u\in\Sigma} J_{h,\varepsilon}(u)$$

is solvable. As it is well known, c_h is attained if, for instance, $J_{h,\varepsilon}$ satisfies Palais–Smale condition below $c_h + \alpha$, for some positive α. Thus, in view of Proposition 3.1, it is enough to prove that c_h is less than $\big(m_0 + k_2^*\big)\, h^\theta$.

Let us remark that, in the spirit of Lieb's Lemma (see [5], Lemma 1.2), we can prove that c_h is attained provided it less than $(m_0+K_2)\, h^\theta$, without referring to Palais–Smale condition. Although less straightforward, we have chosen this approach because it is useful in Section 5, where we need more compactness in order to get a multiplicity result.

Proposition 4.1 *Under the same assumptions as in Proposition 3.1, there exists $h^* > 0$ such that c_h is attained for any $0 < h < h^*$.*

Proof. Due to our previous remarks, we only have to prove that $c_h < (m_0 + k_2^*)\, h^\theta$ for small h. To this aim, it is enough to find a test function whose energy is less than $(m_0 + k_2^*)\, h^\theta$.

Let $\delta > 0$ be fixed and let $\eta : [0, +\infty) \to [0, 1]$ be a smooth, non increasing function, such that $\eta(t) = 1$ if $0 \le t \le \delta/2$ and $\eta(t) = 0$ if $t \ge \delta$.

Let $\omega = \omega(1; V_0)$ (cf. Section 2, where the functions $\omega(h; \lambda)$ were defined), fix any x_0 such that $V(x_0) = V_0$ and set

$$\varphi_{h,x_0}(x) = \nu_h\, \omega\Big(\frac{x - x_0}{h}\Big)\eta(|x - x_0|)\,; \tag{4.1}$$

the constant ν_h is chosen in such a way that $\|\varphi_{h,x_0}\|_p = 1$. Then, by its very definition, $\varphi_{h,x_0} \in \Sigma$. It is easy to see that

$$\begin{aligned}
J_{h,\varepsilon}(\varphi_{h,x_0}) \;&\le\; J_{h,0}(\varphi_{h,x_0}) = \int h^2|\nabla\varphi_{h,x_0}|^2 + V(x)|\varphi_{h,x_0}|^2 \\
&=\; \frac{h^N \int |\nabla(\omega(x)\eta(h|x|))|^2 + V(hx + x_0)|\omega(x)\eta(h|x|)|^2}{\big(h^N \int |\omega(x)\eta(h|x|)|^p\big)^{2/p}} \\
&=\; \frac{\int |\nabla\omega(x)|^2 + V(x_0)|\omega(x)|^2 + o(1)}{\big(\int |\omega(x)|^p + o(1)\big)^{2/p}}\, h^\theta \\
&=\; \big(m_0 + o(1)\big)\, h^\theta.
\end{aligned} \tag{4.2}$$

Clearly (4.2) yields $c_h < (m_0 + k_2^*)\, h^\theta$, provided h is small enough. $\diamondsuit$

Proof of Theorem 1.1 Let

$$\varepsilon^* = \min\left\{\frac{1}{\alpha_1}\left(1 - 2^{\frac{2-p}{p}}\right), \frac{1}{\alpha_1}\left(1 - \frac{m_0}{m_\infty}\right)\right\}$$

and assume (1.6). If $0 < h < h^*$, Proposition 4.1 implies that there exists $u \in \Sigma$ such that $J_{h,\varepsilon}(u) = c_h$. From (4.2) we deduce $J_{h,\varepsilon}(u) \le \big(m_0 + o(1)\big)\, h^\theta$ so that, for h small, (2.4) holds, Proposition 2.1 applies and u does not change sign. We can therefore assume u to be positive and, as a result, $\big(J_{h,\varepsilon}(u)\big)^{\frac{1}{p-2}} u$ is a positive solution to (1.5). $\diamondsuit$

5 Proof of Theorem 1.2

Let us roughly describe the argument we use in proving Theorem 1.2. We know that $J_{h,\varepsilon}$ is bounded below on Σ; moreover, if $\limsup_{h\to 0} \varepsilon(h)\, h^{-2}$ is small enough, then $J_{h,\varepsilon}$ satisfies Palais–Smale condition in the sublevel $J^a_{h,\varepsilon} = \{u \in \Sigma : J_{h,\varepsilon}(u) \leq a\}$ for any $a < (m_0 + k_2^*)\, h^\theta$ (cf. Prop. 3.1). A classical result in Ljusternik–Schnirelman Theory implies that the number of critical points of $J_{h,\varepsilon}$ on Σ is bounded below by $\operatorname{cat}(J^a_{h,\varepsilon})$. Thus, in order to relate the number of solutions of (1.5) with the topology of M, it is enough to find a suitable level a such that the category of the corresponding sublevel is bounded below by the category of M. To this aim, the following proposition is very useful. For the proof, based on the very definition of category and homotopical equivalence, we refer for instance to [3].

Proposition 5.1 *Let $a > 0$ and let J^* be a closed subset of $J^a_{h,\varepsilon}$. Let $\Phi_h : M \longrightarrow J^*$ and $\beta : J^a_{h,\varepsilon} \longrightarrow M_\delta$ be continuous maps such that $\beta\circ\Phi_h$ is homotopically equivalent to the embedding $j : M \longrightarrow M_\delta$. Then $\operatorname{cat}_{J^a_{h,\varepsilon}}(J^*) \geq \operatorname{cat}_{M_\delta}(M)$.*

In our setting, the construction of the map Φ_h is very simple, and we already have all the ingredients we need. Indeed, for any $x_0 \in M$ and for any h we define $\Phi_h(x_0) = \varphi_{h,x_0}$ (cf. (4.1), where φ_{h,x_0} was first defined).

Next we define a barycenter map $\beta : \Sigma \to \mathbb{R}^N$ by $\beta(u) = \int \chi(x)|u(x)|^p$; here $\chi(x) = x$ if $|x| \leq \rho$, $\chi(x) = \rho\, x/|x|$ if $|x| \geq \rho$ and $\rho > 0$ is such that $M_\delta \subset \big\{x \in \mathbb{R}^N : |x| \leq \rho\big\}$. A simple computation gives

$$\beta\big(\Phi_h(x_0)\big) \longrightarrow x_0 \tag{5.1}$$

as $h \to 0$, uniformly for $x_0 \in M$.

The content of the next proposition is that barycenters of low energy functions are close to M.

Proposition 5.2 *Assume* (1.3) *and* (1.4)*. For any $\delta > 0$ there exists $\varepsilon_1^{**}(\delta) > 0$ such that, if*

$$\limsup_{h\to 0} \frac{\varepsilon(h)}{h^2} < \varepsilon_1^{**}(\delta)\,, \tag{5.2}$$

then there exist $k_3^*, h_3^* > 0$ such that $0 < h < h_3^*$, $u \in \Sigma$ and $J_{h,\varepsilon}(u) \leq (m_0 + k_3^*)\, h^\theta$ imply $\beta(u) \in M_\delta$.

Proof. By contradiction, let us assume that for some $\delta > 0$ we can find $\varepsilon_m \geq 0$ such that $\varepsilon_m \to 0$ as $m \to \infty$, $\limsup_{h\to 0} \varepsilon(h)\, h^{-2} = \varepsilon_m$ and the claim in Proposition 5.2 does not hold.
For h small we have $\varepsilon(h)h^{-2} < \varepsilon_m + \frac{1}{m}$ and, by (2.1),

$$\left(1 - \alpha_1\left(\varepsilon_m + \frac{1}{m}\right)\right) J_{h,0}(u) \leq J_{h,\varepsilon}(u)\,. \tag{5.3}$$

Let $h_n, k_n \to 0^+$ as $n \to \infty$ and $u_n \in \Sigma$ be such that $J_{h_n,\varepsilon}(u_n) \le (m_0 + k_n)\, h_n^\theta$ and $\beta(u_n) \notin M_\delta$. Let $v_n(x) = h_n^{N/p} u_n(h_n x)$; from (5.3) we get

$$\int |\nabla v_n|^2 + V(h_n x)|v_n|^2 \le \frac{m_0 + k_n}{1 - \alpha_1 \left(\varepsilon_m + \frac{1}{m}\right)}\,. \tag{5.4}$$

We apply Lions' Lemma to the sequence of probability measures $\sigma_n = |v_n|^p$. Vanishing is easily ruled out. If dichotomy occurs, there exist $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 = 1$ such that for any $\xi > 0$ there are $y_n \in \mathbb{R}^N$, $R > 0$, $R_n \to \infty$ such that

$$\int_{|x-y_n|<R} \sigma_n \ge \delta_1 - \xi\,, \quad \int_{|x-y_n|>2R_n} \sigma_n \ge \delta_2 - \xi\,. \tag{5.5}$$

Let us consider ζ as in the proof of Proposition 3.1 and define v_n^1, v_n^2 accordingly as in (3.10). Inequalities (5.5) give

$$\int |v_n^i|^p \ge \delta_i - \xi\,, \quad i = 1, 2\,. \tag{5.6}$$

From (5.4) and (5.6) we get

$$\begin{aligned} \frac{m_0 + k_n}{1 - \alpha_1 \left(\varepsilon_m + \frac{1}{m}\right)} &\ge \int |\nabla v_n^1|^2 + V_0 |v_n^1|^2 + \int |\nabla v_n^2|^2 + V_0 |v_n^2|^2 + O(\xi) \\ &\ge m_0 \Big(|v_n^1|_p^2 + |v_n^2|_p^2 \Big) + O(\xi) \\ &\ge m_0 \big((\delta_1 - \xi)^{\frac{2}{p}} + (\delta_2 - \xi)^{\frac{2}{p}} \big)\,. \end{aligned}$$

As $n, m \to \infty$ and $\xi \to 0$ we deduce $1 \ge \delta_1^{2/p} + \delta_2^{2/p}$, a contradiction. Thus $\{\sigma_n\}$ is tight: there exists $\{y_n\} \subset \mathbb{R}^N$ such that for any $\xi > 0$

$$\int_{|x-y_n|<R} |v_n(x)|^p \ge 1 - \xi \tag{5.7}$$

for a suitable $R > 0$. The sequence $\widehat{v}_n = v_n(\cdot + y_n)$ converges to some $\widehat{v}$ weakly in $H^1(\mathbb{R}^N)$ and, due to the compactness property (5.7), strongly in $L^p(\mathbb{R}^N)$. If the sequence $x_n \equiv h_n y_n$ goes to infinity, then (5.4) gives

$$m_0 \ge \int |\nabla \widehat{v}|^2 + \liminf_{n\to\infty} \int V(h_n x + x_n)|\widehat{v}_n|^2 \ge \int |\nabla \widehat{v}|^2 + \int V_\infty |\widehat{v}|^2 \ge m_\infty\,,$$

which contradicts (3.2). Thus we can assume $x_n \to \widehat{x}$, and arguing as before we obtain

$$m_0 \ge \int |\nabla \widehat{v}|^2 + V_\varepsilon(\widehat{x})|\widehat{v}|^2 \ge m(1; V(\widehat{x})) \ge m_0\,.$$

From this we get $V(\widehat{x}) = V_0$ and $\int |\nabla \widehat{v}|^2 + V_0 |\widehat{v}|^2 = m_0$, whence $\widehat{v} = \omega$ (ω was introduced in the proof of Proposition 4.1). Furthermore, since $\int |\nabla \widehat{v}_n|^2 + V_0 |\widehat{v}_n|^2 \ge m_0$, from (5.4) we get $\int |\nabla \widehat{v}_n|^2 + V_0 |\widehat{v}_n|^2 \to m_0 = \int |\nabla \omega|^2 + V_0 |\omega|^2$

as $n \to \infty$, so that $\widehat{v}_n$ converges to ω strongly in $H^1(\mathbb{R}^N)$. Finally, a simple computation gives

$$\left|\beta(u_n) - \beta\big(\Phi_{h_n}(x_n)\big)\right| \leq \rho \int \left||\widehat{v}_n(x)|^p - |\omega(x)|^p\right| = o(1)\,;$$

(5.1) thus implies $|\beta(u_n) - x_n| = o(1)$, which contradicts $\beta(u_n) \notin M_\delta$. This concludes the proof. $\diamondsuit$

Proof of Theorem 1.2 Let $\delta > 0$ be fixed and $\varepsilon_1^*(\delta)$ be as in Proposition 5.2. Let

$$\varepsilon^{**}(\delta) = \min\left\{\frac{1}{\alpha_1}\left(1 - 2^{\frac{2-p}{p}}\right), \frac{1}{\alpha_1}\left(1 - \frac{m_0}{m_\infty}\right), \varepsilon_1^*(\delta)\right\}$$

and assume (1.7). Let $0 < h^* \leq \min\{h_i^* : i = 1,2,3\}$ and $k^* = \min\{k_i^* : i = 1,2,3\}$, the constants h_i^*, k_i^* being defined in Propositions 2.1, 3.1 and 5.2. Let $0 < h < h^*$; we can assume that $a(h) \equiv (m_0 + k^*)\, h^\theta$ is not a critical value for $J_{h,\varepsilon}$ on Σ. For convenience, we set $\Sigma_h = \{u \in \Sigma : J_{h,\varepsilon}(u) \leq a(h)\}$, $\Sigma_h^+ = \{u \in \Sigma_h : u \geq 0\}$ and $\Sigma_h^- = \{u \in \Sigma_h : u \leq 0\}$.
If h^* is small enough, (4.2) gives $J_{h,\varepsilon}\big(\Phi_h(x_0)\big) \leq (m_0 + k^*)\, h^\theta$ for any $x_0 \in M$. In other words, $\Phi_h(x_0) \in \Sigma_h^+$ for any $x_0 \in M$. Furthermore, Proposition 5.2 implies $\beta(u) \in M_\delta$ for any $u \in \Sigma_h$. Finally, as a consequence of (5.1) it is easy to see that $\beta \circ \Phi_h$ is homotopically equivalent to the embedding $j : M \to M_\delta$. Thus Proposition 5.1 gives $\mathrm{cat}_{\Sigma_h}(\Sigma_h^+) \geq \mathrm{cat}_{M_\delta}(M)$. If we use the map $-\Phi_h$ we also get $\mathrm{cat}_{\Sigma_h}(\Sigma_h^-) \geq \mathrm{cat}_{M_\delta}(M)$, whence $\mathrm{cat}(\Sigma_h) \geq 2\mathrm{cat}_{M_\delta}(M)$, for h small.

Proposition 3.1 guarantees that Palais–Smale condition holds in a sublevel containing Σ_h. Thus Ljusternik–Schnirelman Theory applies and we deduce that $J_{h,\varepsilon}$ has at least $2\,\mathrm{cat}_{M_\delta}(M)$ critical points on Σ, satisfying $J_{h,\varepsilon}(u) \leq a(h) < (m_0 + k_1^*)h^\theta$. Therefore, by Proposition 2.1 they do not change sign and we can assume that at least $\mathrm{cat}_{M_\delta}(M)$ critical points are positive. $\diamondsuit$

As a final comment, let us point out that, in proving Theorem 1.2, we adapted the arguments used in [6] to deal with the unperturbed problem. The same kind of approach was used in [7] to study the equation $-h^2\Delta u + V(x)u = K(x)|u|^{p-2}u + Q(x)|u|^{q-2}u$, where V, K, Q are competing potentials. In [7] the number of solutions is related with the global minima set of the so-called ground energy function (cf. [20] and also [14], where a more general subcritical nonlinearity is allowed).

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Monica Lazzo
Dipartimento di Matematica, Università di Bari
via Orabona 4, 70125 Bari, Italy
e-mail: lazzo@dm.uniba.it