Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 13, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

MULTIPLE POSITIVE SOLUTIONS FOR A NONLOCAL BOUNDARY-VALUE PROBLEM WITH RESPONSE FUNCTION QUIET AT ZERO

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Abstract. The existence of positive solutions of a nonlocal boundary value problem for a second order differential equation is investigated. By assuming that the response function is quiet at zero, in a sense introduced here, and it satisfies some easy conditions, existence results for a countable set of positive solutions are given.

1. Introduction

In a recent paper the authors gave sufficient conditions for the existence of a positive solution of the nonlocal boundary value problem

$$(p(t)x')' + q(t)f(x) = 0, \quad \text{a.a. } t \in [0,1] \tag{1.1}$$

$$x(0) = 0, \tag{1.2}$$

$$x'(1) = \int_\eta^1 x'(s)dg(s) \tag{1.3}$$

where $\eta \in (0,1)$, see [25]. Among these conditions, the monotonicity of the response function f seemed to be crucial in the proof. In this paper we weaken the monotonicity condition on f by assuming that this function is quiet at zero in the following sense: Given any pair of sequences (x_n), (y_n) with $0 \leq x_n \leq y_n$ converging to zero it holds $f(x_n) = O(f(y_n))$, (where O stands for the big-O symbol). This definition, which is introduced here, refers to functions f which do not vanish at least on $(0, +\infty)$. It is not difficult to see that if $f(0) > 0$, or, if f is increasing in a right neighborhood of zero, then f is quiet at zero.

Moreover we extend the results of [25] and show that our boundary value problem can admit a countable family of positive solutions.

Here we have to mention that boundary value problems of the form (1.1), (1.2), (1.3) are mainly motivated by the works of Bitsadze [8], Bitsadze and Samarskii [9] and Il'in and Moiseev [23] and includes as special cases multipoint boundary value problems considered in [19] and [20]. Moreover, the authors in [25-28] proved recently existence results for some relative nonlocal boundary value problems. On the other hand the problem of the existence of multiple solutions (at least two) for

Mathematics Subject Classifications: 34B18.
Key words: Multiple positive solutions, nonlocal boundary value problems, functions quiet at zero, Krasnoselskii's fixed point theorem.

Submitted January 23, 2001. Published February 19, 2001.

various types of boundary value problems is recently the subject of many papers. Among others we refer to [1-4, 6, 7, 10-12, 14-17, 21, 22, 24, 29, 32]. The technique in these papers is based on fixed point results in cones. Most of them are based on the following well known fixed point theorem due to Krasnoselskii [30].

Theorem 1.1. *Let $\mathcal{B}$ a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of E, with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$A\colon \mathbb{K} \cap (\Omega_2 \setminus \overline{\Omega_1}) \to \mathbb{K}$$

be a completely continuous operator such that either

$$\|Au\| \le \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_1, \ \ \|Au\| \ge \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_2$$

or

$$\|Au\| \ge \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_1, \ \ \|Au\| \le \|u\|, \ \ u \in \mathbb{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathbb{K} \cap (\Omega_2 \setminus \overline{\Omega_1})$.

In this norm form and in its degree form (see [18]), Theorem 1.1 is applied in [1, 2, 3, 11, 14-17, 22, 24, 29, 32]. Some interesting versions of the theorem (see, e.g., [5, 13, 31]) are also applied in [4, 6, 7, 12, 21]. Finally, we mention that by using a different fixed point theorem due to Ricceri [33] a multiplicity existence result is obtained in [10].

Here we apply Theorem 1.1 to obtain existence results for a countable set of positive solutions of the boundary value problem (1.1), (1.2), (1.3), where the main hypothesis is that the function f is a quiet at zero function. This meaning is given in the following section.

2. Quietness at zero

We introduce the following definition:

Definition. A continuous function $f : [0, +\infty) \to \mathbb{R}$, with $f(x) > 0$ when $x > 0$, is said to be quiet at zero, if for any pair of sequences (x_n), (y_n) with $0 \le x_n \le y_n$, $n = 1, 2, ...$, which converge to zero, it holds

$$f(x_n) = O(f(y_n)).$$

This means that there is a $K > 0$ such that $f(x_n) \le Kf(y_n)$ for all n. An equivalent form of this definition, which will be used in our proofs, is given by the following lemma:

Lemma 2.1. *A continuous function $f : [0, +\infty) \to \mathbb{R}$, with $f(x) > 0$ when $x > 0$ is quiet at zero, if and only if for each $T > 0$ there is a $\mu \ge 1$ such that for all $\tau \in (0, T)$ it holds*

$$\sup\{f(x) : x \in [0, \tau]\} \le \mu \inf\{f(x) : x \in [\tau, T]\}. \tag{2.1}$$

Proof. Assume that f is quiet at zero and there is a $T > 0$ such that for each positive integer μ there is a point $\tau_\mu \in (0, T)$, with

$$\sup\{f(x) : x \in [0, \tau]\} > \mu \inf\{f(x) : x \in [\tau, T]\}.$$

This implies that there are sequences (x_μ), (y_μ), with $0 < x_\mu \le \tau_\mu \le y_\mu$ and $f(x_\mu) > \mu f(y_\mu)$ for all μ. Since f is bounded on $[0,T]$, taking limits as $\mu \to +\infty$, we get $f(y_\mu) \to 0$, thus $y_\mu \to 0$ and, so, also $x_\mu \to 0$. These facts contradict to our assumption.

For the "if" part of the proof, we assume that there are sequences (x_n), (y_n), with $0 \le x_n < y_n$ and $x_n \to 0$, $y_n \to 0$ and, moreover,

$$\lim_{n\to\infty} \frac{f(x_n)}{f(y_n)} = +\infty. \tag{2.2}$$

Set $T := max\{y_n\}$. Then, by assumption, there is a $\mu \ge 1$ such that for all $\tau \in (0,T)$ it holds (2.1). From (2.2) there is an index n_0 such that $f(x_{n_0}) \ge (\mu+1)f(y_{n_0})$. Set $\tau := x_{n_0}$ and observe that

$$\begin{aligned}\sup\{f(x) : x \in [0,\tau]\} &\ge f(x_{n_0}) \ge (\mu+1)f(y_{n_0})\\ &\ge (\mu+1)\inf\{f(x) : x \in [\tau,T]\},\end{aligned}$$

contradicting to 2.1. The proof is complete. □

Remark. We observe that if $f(0) > 0$, then f is quiet at zero. Indeed, for each $T > 0$ the real number

$$\mu := \frac{\sup\{f(x) : x \in [0,T]\}}{\inf\{f(x) : x \in [0,T]\}}$$

works in (2.1) for all $\tau \in (0,T)$.

Also, if f is a nondecreasing in a right neihborhood of zero, then it is quiet at zero. To see this we assume that f is nondecreasing on an interval $[0,\delta]$ and consider two sequences $(x_n),(y_n)$, with $0 \le x_n < y_n$ for all n and $x_n \to 0$, $y_n \to 0$. These sequences belong eventually in the interval $(0,\delta]$, hence, by the monotonicity of f, $f(x_n) \le f(y_n)$ for all large n. This proves that the function f is quiet at zero.

Example. We give a simple example of a function, which is not quiet at zero. Consider sequences (x_n), (y_n) such that $x_n \to 0$, $y_n \to 0$, with $0 < x_n < y_n < x_{n-1} < y_{n-1} < \dots < x_1 < y_1$ and $\lim \frac{f(x_n)}{f(y_n)} = +\infty$. Then, define a new sequence (z_n) with $z_{2n} = x_n$ and $z_{2n+1} = y_n$. Consider the continuous function f, defined on the interval $[0,+\infty)$, whose the graph passes from the points $(\frac{1}{n}, z_n)$, it is linear in between and vanishes at zero. It is not hard to see that f is not quiet at zero.

This example may justify why the function f is named "quiet". If we discuss in details the behavior of f which is not quiet at zero, then we can observe rapid oscillations close to zero. We mean that as the argument approaches to zero from the right, the rate of successive maximum and minimum of f is not bounded, though the limit might be zero. So, f has a kind of a singular motion to the zero.

3. The assumptions and some lemmas

In the sequel we shall denote by $\mathbb{R}$ the real line, by $\mathbb{R}^+$ the interval $[0,+\infty)$ and by I the interval $[0,1]$. Then $C(I)$ will denote the space of all continuous functions $x : I \to \mathbb{R}$. Let $C_0^1(I)$ be the space of all functions $x : I \to \mathbb{R}$, whose the first derivative x' is absolutely continuous on I and $x(0) = 0$. This is a Banach space when it is furnished with the norm $\| \ \|$ defined by

$$\|x\| := \sup\{|x'(t)| : t \in I\}, \ \ x \in C_0^1(I).$$

Also we denote by $L_1^+(I)$ the space of all functions $x : I \to \mathbb{R}^+$ which are Lebesgue integrable on I, endowed with its usual norm $\|\cdot\|_1$.

Now consider the problem (1.1), (1.2), (1.3). By a solution of this problem we mean a function $x \in C_0^1(I)$ satisfying equation (1.1) for almost all $t \in I$ and condition (1.3).

As the functions appeared in this problem we assume the following:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, with $f(x) > 0$, when $x > 0$ and quiet at zero.

(H2) The functions p, q belong to $C(I)$ and they are such that $p > 0$, $q \geq 0$ and $\sup\{q(s) : \eta \leq s \leq 1\} > 0$. It is clear that without loss of generality we can assume that $p(1) = 1$.

(H3) The function $g : I \to \mathbb{R}$ is increasing and such that

$$g(\eta) = 0 < g(\eta+) =: b_0.$$

(H4) It holds: $\int_\eta^1 \frac{1}{p(s)} dg(s) < 1$.

Last assumption implies that the quantity

$$\alpha := \left(1 - \int_\eta^1 \frac{1}{p(s)} dg(s)\right)^{-1}$$

is a real number.

As we indicated in [25], the problem (1.1), (1.2), (1.3) is equivalent to the operator equation $x = Ax$, $x \in C_0^1(I)$, where A is the operator defined by

$$Ax(t) := \alpha P(t) \int_\eta^1 \Phi(f(x))(s) dg(s) + \int_0^t \Phi(f(x))(s) ds. \tag{2.2}$$

Here the functions P and Φ are defined by

$$P(t) := \int_0^t \frac{1}{p(s)} ds, \ \ t \in I$$

and

$$(\Phi y)(t) := \frac{1}{p(t)} \int_t^1 q(s) y(s) ds, \ \ t \in I, \ \ y \in C(I)$$

and the constants H, θ are given by

$$H := \int_\eta^1 \Phi(1)(s) dg(s) \ \text{ and } \ \theta := \frac{p(0)}{\alpha H + \|q\|_1}.$$

It is clear that A is a completely continuous operator.

In the sequel we shall do use of the function

$$f_s(w) := \sup\{f(x) : x \in [0, w]\},$$

for which we assume the following:

(H5) There exists a point $v > 0$ such that $f_s(v) \leq \theta v$.

Now we set

$$\mathbb{X} := \left\{x \in C_0^1(I) : x \geq 0, x' \geq 0, \ \ x \text{ is concave and } \ \|x\| \leq v.\right\}$$

and we give the following auxiliary results:

Lemma 3.1. *It holds $A\mathbb{X} \subset \mathbb{X}$.*

Proof. Let $x \in \mathbb{X}$. Then $Ax(t) \geq 0$, $(Ax)'(t) \geq 0$, and $(Ax)''(t) = -q(t)f(x(t)) \leq 0$ for all $t \in I$. Also, $x \in \mathbb{X}$ implies that $0 \leq x(t) \leq v$ for all $t \in I$. Then

$$\begin{aligned}\|Ax\| = (Ax)'(0) &= \frac{\alpha}{p(0)} \int_\eta^1 \Phi(f(x))(s)dg(s) + \frac{1}{p(0)} \int_0^1 q(s)f(x(s))ds \\ &\leq f_s(v)\left[\frac{\alpha H}{p(0)} + \frac{1}{p(0)} \int_0^1 q(s)ds\right] \\ &\leq \theta v \left[\frac{\alpha H + \|q\|_1}{p(0)}\right] = v.\end{aligned}$$

□

Lemma 3.2. *There exists a $\lambda_v > 0$ such that for all $x \in \mathbb{X}$ it holds*

$$\int_\eta^1 \Phi(f(x))(s)dg(s) \geq \frac{b_0}{\lambda_v} \int_0^1 q(s)f(x(s))ds.$$

Proof. From the assumption $(H3)$ we have

$$g(s) \geq b_0, \quad s \in (\eta, 1]. \tag{3.1}$$

Let $x \in \mathbb{X}$. Then x is nondecreasing and $\|x\| \leq v$. Since f is quiet at zero, for the number $T_v := v$, there is a $\mu_v \geq 1$ such that (2.1) holds for all $\tau \in (0, T_v)$. Hence (2.1) also holds for the real number $\tau := x(\eta) < \|x\| \leq v$. Therefore we have

$$\begin{aligned}\int_0^1 q(s)f(x(s))ds &= \int_0^\eta q(s)f(x(s))ds + \int_\eta^1 q(s)f(x(s))ds \\ &\leq \sup_{w\in[0,\tau]} f(w) \int_0^\eta q(s)ds + \int_\eta^1 q(s)f(x(s))ds \\ &\leq \frac{\int_0^\eta q(s)ds}{\int_\eta^1 q(s)ds} \frac{\sup_{w\in[0,\tau]} f(w)}{\inf_{w\in[\tau,T_v]} f(w)} \int_\eta^1 q(s)f(x(s))ds \\ &\quad + \int_\eta^1 q(s)f(x(s))ds \\ &\leq \left(\frac{\int_0^\eta q(s)ds}{\int_\eta^1 q(s)ds}\mu_v + 1\right) \int_\eta^1 q(s)f(x(s))ds \\ &= \xi \int_\eta^1 q(s)f(x(s))ds,\end{aligned}$$

where

$$\xi := \frac{\int_0^\eta q(s)ds}{\int_\eta^1 q(s)ds}\mu_v + 1.$$

Next we use (3.1) and get

$$\begin{aligned}\int_0^1 q(s)f(x(s))ds &\le \xi\int_\eta^1 q(s)f(x(s))ds\\ &\le \frac{\xi}{b_0}\int_\eta^1 q(s)f(x(s))g(s)ds\\ &= -\frac{\xi}{b_0}\int_\eta^1 d\left(\int_s^1 q(r)f(x(r))dr\right)g(s)\\ &= \frac{\xi}{b_0}\int_\eta^1\int_s^1 q(r)f(x(r))drdg(s)\\ &\le \frac{\lambda_v}{b_0}\int_\eta^1\frac{1}{p(s)}\int_s^1 q(s)f(x(s))drdg(s),\end{aligned}$$

where

$$\lambda_v := \left(\frac{\int_0^\eta q(s)ds}{\int_\eta^1 q(s)ds}\mu_v + 1\right)\sup_{s\in I} p(s).$$

The proof of the lemma is complete. □

Now we set

$$D := \int_\eta^1 \Phi(P)(s)dg(s),$$

$$b := min\left\{b_0, \frac{\lambda_v H}{\alpha|D\eta p(0) - H|}\right\}$$

and

$$\sigma_v := \frac{\alpha b D p(0)}{\alpha b + \lambda_v}.$$

Lemma 3.3. *It holds*

$$\sigma_v\eta \le H.$$

Proof. If $D\eta p(0) - H > 0$, then we have $b \le \frac{\lambda_v H}{\alpha(D\eta p(0)-H)}$. Solving with respect to H we obtain the result. Also, if $D\eta p(0) - H < 0$, then

$$\sigma_v\eta = \frac{\alpha b p(0)\eta}{\alpha b + \lambda_v}D < \frac{\alpha b H}{\alpha b + \lambda_v} \le H.$$

4. Main results

In this section we present our main results. Let us first define the function

$$f_i(w) := \inf\left\{f(z) : \frac{\eta\sigma_v}{H}w \le z \le w\right\},$$

the cone

$$\mathbb{K} := \left\{x \in C_0^1(I) : x \ge 0, x' \ge 0,\ x \text{ is concave and } \int_\eta^1 \Phi(x)(s)dg(s) \ge \sigma_v\|x\|\right\}$$

and let

$$\rho := \frac{1}{\alpha H}.$$

Theorem 4.1. *Consider the functions f, p, q and g satisfying the assumptions $(H1)-(H5)$ and the following one:*

$(H6)$ *There exists $u > 0$ such that $u < v$ and $f_i(u) \geq \rho u$.*

Then the boundary value problem (1.1), (1.2), (1.3) *admits a solution x such that $u \leq \|x\| \leq v$.*

Proof. Let B_v be the open ball $\{x \in C_0^1(I) : \|x\| < v\}$. We claim that

$$A : \mathbb{K} \cap B_v \to \mathbb{K}.$$

Indeed, let $x \in \mathbb{K} \cap B_v$. Then $x \in \mathbb{K}$ and $x \in \mathbb{X}$. First observe that $Ax \geq 0$, $(Ax)' \geq 0$ and $(Ax)'' \leq 0$. Moreover,

$$\begin{aligned}
\int_\eta^1 \Phi(Ax)(s)dg(s) &\geq \alpha \int_\eta^1 \Phi(P)(s)dg(s) \int_\eta^1 \Phi(f(x))(s)dg(s) \\
&= \alpha D \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
&= \frac{\sigma_v(\alpha b_0 + \lambda_v)}{b_0 p(0)} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
&= \frac{\sigma_v}{p(0)} \left(\alpha + \frac{\lambda_v}{b_0}\right) \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
&= \sigma_v \frac{\alpha}{p(0)} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
&+ \sigma_v \frac{1}{p(0)} \frac{\lambda_v}{b_0} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s).
\end{aligned}$$

Hence, taking into account Lemma 3.2 we get

$$\begin{aligned}
\int_\eta^1 \Phi(Ax)(s)dg(s) &\geq \sigma_v \frac{\alpha}{p(0)} \int_\eta^1 \frac{1}{p(s)} \int_s^1 q(\theta)f(x(\theta))d\theta dg(s) \\
&+ \sigma_v \frac{1}{p(0)} \int_0^1 q(\theta)f(x(\theta))d\theta \\
&= \sigma_v (Ax)'(0) \\
&= \sigma_v \|(Ax)\|,
\end{aligned}$$

which proves our claim.

Now consider a function $x \in \mathbb{K}$, with $\|x\| = u$. The fact that x is concave implies that

$$\eta x(1) \leq x(\eta) \leq x(r) \leq x(1) \leq \|x\|, \text{ for every } r \in [\eta, 1].$$

So,

$$\begin{aligned}\sigma_v\|x\| &\leq \int_\eta^1 \Phi(x)(s)dg(s)\\ &= \int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)x(\theta)d\theta dg(s)\\ &\leq x(1)\int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)d\theta dg(s)\\ &= x(1)\int_\eta^1 \Phi(1)(s)dg(s)\\ &= x(1)H.\end{aligned}$$

Thus we have

$$x(1) \geq \frac{\sigma_v\|x\|}{H},$$

which implies that

$$x(r) \geq \frac{\eta\sigma_v}{H}\|x\|, \quad r \in [\eta, 1].$$

Therefore, for every $r \in [\eta, 1]$ we have

$$\frac{\eta\sigma_v}{H}\|x\| \leq x(r) \leq \|x\|,$$

where, notice that it also holds $\frac{\eta\sigma_v}{H} \leq 1$, see Lemma 3.3. Then, by assumption $(H6)$, we obtain

$$\begin{aligned}(Ax)'(1) &\geq \alpha\int_\eta^1 \frac{1}{p(s)}\int_s^1 q(\theta)f(x(\theta))d\theta dg(s)\\ &\geq \alpha f_i(u)H\\ &\geq \alpha H\rho u = u.\end{aligned}$$

This means that, if $\|x\| = u$, then $\|Ax\| \geq \|x\|$. Moreover in Lemma 3.1 we have proved that if $\|x\| = v$, then $\|Ax\| \leq \|x\|$.

To complete the proof we set $\Omega_1 := \{x \in C_0^1(I) : \|x\| < u\}$, $\Omega_2 := B_v$ and apply Theorem 1.1.

□

An immediate consequence of this theorem is the following:

Corollary 4.2. *Consider the functions f, q, g satisfying the assumptions $(H1)-(H4)$. Moreover assume that*

(H7) *There exist two two-sided sequences (u_k), (v_k), $k \in \mathbb{Z}$ such that*

$$0 < u_k < v_k < u_{k+1},$$

$$f_i(u_k) \geq \rho u_k \;\; \text{and} \;\; f_s(v_k) \geq \theta v_k$$

for every $k \in \mathbb{Z}$.

Then there exists a sequence x_k, $k \in \mathbb{Z}$ of solutions of the boundary value problem (1.1), (1.2), (1.3), *such that*

$$u_k < \|x_k\| < v_k, \quad k \in \mathbb{Z}.$$

References

[1] R. P. Agarwal and D. O'Regan, *Twin solutions to singular Dirichlet problems*, J. Math. Anal. Appl. **240** (1999), 433–445.

[2] R. P. Agarwal and D. O'Regan, *Twin solutions to singular boundary value problems*, Proc. A.M.S. **128** (2000), 2085–2094.

[3] V. Anuradha, D. D. Hai and R. Shivaji, *Existence results for superlinear semipositone BVP'S*, Proc. A.M.S. **124** (1996), 757–763.

[4] R. Avery, *Existence of multiple positive solutions to a conjugate boundary value problem*, Math. Sci. Rcs. Hot-Line **2** (1998), 1–6.

[5] R. Avery, *A generalization of the Legget-Williams fixed point theorem*, Math. Sci. Rcs. Hot-Line **3** (1999), 9–14.

[6] R. Avery, J. M. Davis and J. Henderson, *Three symmetric positive solutions for Lidstone problems by a generalization of the Legget-Williams theorem*, Electron. J. Differential Equations **2000, No. 40** (2000), 1–21.

[7] R. Avery and J. Henderson, *Three symmetric positive solutions for a second order boundary value problem*, Appl. Math. Lett. **13** (2000), 1–7.

[8] A. V. Bitsadze, *On the theory of nonlocal boundary value problems*, Soviet Math. Dock. **30** (1964), 8–10.

[9] A. V. Bitsadze and A. A. Samarskii, *Some elementary generalizations of linear elliptic boundary value problems*, Dokl. Akad. Nauk SSSR **185** (1969), 739–740.

[10] G. Bonanno, *Existence of three solutions for a two point boundary value problem*, Appl. Math. Lett. **13** (2000), 53–57.

[11] C.J. Chyan and J. Henderson, *Multiple solutions for $2mth$ Sturm-Liouville boundary value problem*, Comput. Math. Appl. **40** (2000), 231–237.

[12] J. M. Davis, P.W. Eloe and J. Henderson, *Triple positive solutions and dependence on higher order derivatives*, J. Math. Anal. Appl. **237** (1999), 710–720.

[13] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[14] D. R. Dunninger and H. Wang, *Multiplicity of positive solutions for a nonlinear differential equation with nonlinear boundary conditions*, Annales Polonici Math. **LXIX.2** (1998), 155–165.

[15] P. W. Eloe and J. Henderson, *Positive solutions and nonlinear multipoin conjugate eigenvalue problems*, Electronic J. Diff. Equations **1997, No. 8** (1997), 1–11.

[16] L. H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. A.M.S. **120** (1994), 743–748.

[17] L. H. Erbe, S. Hu and H. Wang, *Multiple positive solutions of some boundary value problems*, J. Math. Anal. Appl. **184** (1994), 640–648.

[18] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones,*, Academic Press, Orlando, Fl., 1988.

[19] C. Gupta, *Solvability of a generalized multi-point boundary value problem of mixed type for second order ordinary differential equations*, Proc. Dynamic Dystems and Applications **2** (1995), 215–222.

[20] C. Gupta, S. K. Ntouyas and P. Ch. Tsamatos, *On an m-point boundary value problem for second order ordinary differential equations*, Nonlinear Analysis TMA **23** (1994), 1427–1436.

[21] J. Henderson and H.B. Thompson, *Multiple symmetric positive solutions for a second order boundary value problem*, Proc. A.M.S. **128** (2000), 2373–2379.

[22] J. Henderson and E. R. Kaufmann, *Multiple positive solutions for focal boundary value problems*, Comm. Appl. Anal. **1** (1997), 53–60.

[23] V. Il'in and E. Moiseev, *Nonlocal boundary value problems of the second kind for a Sturm-Liouville operator*, Differential Equations **23** (1987), 979–987.

[24] D. Jiang, *Multiple positive solutions to singular boundary value problems for superlinear higher-order ODEs*, Comput. Math. Appl. **40** (2000), 249–259.

[25] G. L. Karakostas and P. Ch. Tsamatos, *Positive solutions for a nonlocal boundary value problem with increasing response*, Electron. J. Differential Equations **2000, No. 73** (2000), 1-8.

[26] G. L. Karakostas and P. Ch. Tsamatos, *On a nonlocal boundary value problem at resonance*, J. Math. Anal. Appl. (in press).

[27] G. L. Karakostas and P. Ch. Tsamatos, *Existence results for some n-dimensional nonlocal boundary value problems*, J. Math. Anal. Appl. (in press).

[28] G. L. Karakostas and P. Ch. Tsamatos, *Positive solutions of a boundary-value problem for second order ordinary differential equations*, Electron. J. Differential Equations **2000, No. 49** (2000), 1-9.

[29] L. Kong and D. Jiang, *Multiple positive solutions of a nonlinear fourth order periodic boundary value problem*, Annales Polonici Math. **LXIX.3** (1998), 265–270.

[30] M. A. Krasnoselskii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.

[31] R. W. Legget and L.R. Williams, *A fixed point theorem with application to an Infectius disease model*, J. Math. Anal. Appl. **76** (1980), 91–97.

[32] R. Ma, *Multiplicity of positive solutions for second-order three-point boundary value problems*, Comput. Math. Appl. **40** (2000), 193–204.

[33] B. Ricceri, *On a three critical points theorem*, Arch. Math.(Basel) **75** (2000), 220–226.

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