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L^1 stability of conservation laws for a traffic flow model *

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Abstract

We establish the L^1 well-posedness theory for a system of nonlinear hyperbolic conservation laws with relaxation arising in traffic flows. In particular, we obtain the continuous dependence of the solution on its initial data in L^1 topology. We construct a functional for two solutions which is equivalent to the L^1 distance between the solutions. We prove that the functional decreases in time which yields the L^1 well-posedness of the Cauchy problem. We thus obtain the L^1-convergence to and the uniqueness of the zero relaxation limit.

We then study the large-time behavior of the entropy solutions. We show that the equilibrium shock waves are nonlinearly stable in L^1 norm. That is, the entropy solution with initial data as certain L^1-bounded perturbations of an equilibrium shock wave exists globally and tends to a shifted equilibrium shock wave in L^1 norm as $t \to \infty$. We also show that if the initial data ρ_0 is bounded and of compact support, the entropy solution converges in L^1 to an equilibrium N-wave as $t \to +\infty$.

1 Introduction

We establish the L^1 well-posedness theory for a system of nonlinear hyperbolic conservation laws with relaxation arising in traffic flows. In particular, we obtain the continuous dependence of the solution on its initial data in L^1 topology, the L^1-convergence to and the uniqueness of the zero relaxation limit. We then show that the equilibrium shock waves are nonlinearly stable in L^1 norm. The L^1 topology is natural from point view of the conservation laws. The well-posedness problem in the L^1 topology for nonlinear conservation laws has been studied, see Bressan, Liu and Yang [2], Liu and Yang [13]. L^1-stability of shock waves in scalar conservation laws has been studied, see Freistühler and Serre [4], Mascia and Natalini [14], Natalini [15].

The system of nonlinear hyperbolic conservation laws with relaxation we study was derived as a nonequilibrium continuum model of traffic flows by Zhang

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[22], also see Li and Zhang [10]. The main purpose of the model is to address the anisotropic feature of traffic flows. The resulting hyperbolic system with relaxation is marginally stable.

The model is the following

$$\rho_t + (\rho v)_x = 0 \tag{1.1}$$

$$v_t + (\frac{1}{2}v^2 + g(\rho))_x = \frac{v_e(\rho) - v}{\tau} \tag{1.2}$$

with initial data

$$(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x)). \tag{1.3}$$

It is assumed that

$$\rho_0(x) \geq \delta_0 > 0 \tag{1.4}$$

for some $\delta_0 > 0$. g is the anticipation factor satisfying

$$g'(\rho) = \rho(v_e'(\rho))^2. \tag{1.5}$$

$\tau > 0$ is the relaxation time. Equation (1.1) is a conservation law for ρ. (1.2) is a rate equation for v, which is not a conservation of momentum as in fluid flow equations. The anticipation factor g in (1.2) compare to pressure in the momentum equation. It describes drivers' car-following behavior. The right hand side of (1.2) is the relaxation term. Let

$$h(\rho, v) = \frac{v_e(\rho) - v}{\tau}. \tag{1.6}$$

When the state is in equilibrium, the system of equations (1.1) (1.2) is reduced to the equilibrium equation

$$\rho_t + (\rho v_e(\rho))_x = 0 \tag{1.7}$$

with initial data

$$\rho(x,0) = \rho_0(x) > 0. \tag{1.8}$$

It is assumed that the equilibrium velocity $v_e(\rho)$ is a decreasing function of ρ, $v_e'(\rho) < 0$. It is also assumed that $v_e(0) = v_f$ and $v_e(\rho_j) = 0$ where v_f is the free flow speed and ρ_j is the jam concentration. The equilibrium flux $q(\rho) = \rho v_e(\rho)$ is assumed to be a concave function of ρ

$$q''(\rho) = \rho v_e''(\rho) + 2v_e'(\rho) < 0. \tag{1.9}$$

The equilibrium characteristic speed is

$$\lambda_*(\rho) = q'(\rho) = v_e(\rho) + \rho v_e'(\rho). \tag{1.10}$$

For the traffic flow model (1.1) (1.2), the characteristic speeds are

$$\lambda_1(\rho, v) = \rho v_e'(\rho) + v < -\rho v_e'(\rho) + v = \lambda_2(\rho, v) \tag{1.11}$$

and the right eigenvectors of the Jacobian of the flux are

$$r_i(\rho, v) = (1, (-1)^{i-1} v_e'(\rho))^T, \quad i = 1,\ 2.$$

The system is strictly hyperbolic provided $\rho > 0$. Furthermore, each characteristic field is genuinely nonlinear

$$\nabla\lambda_i(\rho, v) \cdot r_i(\rho, v) = (-1)^{i-1} q''(\rho) \neq 0, \quad i = 1,\ 2$$

where the concavity of q is assumed, see (1.9).

On the equilibrium curve $v = v_e(\rho)$, a marginal stability condition

$$\lambda_1 = \lambda_* < \lambda_2 \tag{1.12}$$

is satisfied. Thus there is no diffusion in the process of relaxation for the traffic flow model (1.1) (1.2). (1.12) is a direct consequence of the anisotropic feature of traffic flows.

In Li [9], using a generalized Glimm scheme, we obtained global existence of solution of (1.1) (1.2) (1.3) for initial data of bounded total variation, of bounded oscillations and of small distance to the equilibrium curve. We also showed that a sequence of the solutions obtained for the relaxed system converge to a solution of the equilibrium equation (1.7) as the relaxation parameter goes to zero.

In the current paper, we study the continuous dependence of the solution on its initial data in L^1 topology. The uniqueness of solutions is a corollary of the continuous dependence of the solution on its initial data. We construct a functional for two solutions such that it is equivalent to the L^1 distance between the two solutions and it is time-decreasing. The construction makes use of the L^1 contraction semigroup property for the scalar conservation laws, Keyfitz [5], Kruzkov [6], Lax [8] and the exponential decay property of the source term. We show an L^1-contractive property of the entropy solution operator in the Riemann invariant coordinate. For general systems of conservation laws, there is no such a property, see [20].

We show the L^1 stability of the equilibrium shock waves. That is, the entropy solution with initial data as certain L^1-bounded perturbations of an equilibrium shock wave exists globally and tends to a shifted equilibrium shock wave in L^1 norm as $t \to \infty$.

We then show that if the initial data ρ_0 is bounded and of compact support, the entropy solution converges in L^1 to an equilibrium N-wave as $t \to +\infty$.

Uniqueness issues do not seem to have been systematically studied in conjunction with higher order models. In general, the zero relaxation limit is highly singular because of shock and initial layers. In [15], Natalini obtained the uniqueness of the zero relaxation for semilinear systems of equations with relaxation. The uniqueness problem for the quasilinear case remains open. For the quasilinear system of equations (1.1) (1.2), we obtain the L^1-convergence to and the uniqueness of the zero relaxation limit. We prove the uniqueness of the zero relaxation by using the property that the solution depends on its data continuously, the fact that the signed distance $-v_e(\rho) + v$ of $(\rho,\ v)$ to the equilibrium curve is one of the Riemann invariants and that it decays in τ exponentially. The relaxation limit models dynamic limit from the continuum nonequilibrium processes to the equilibrium processes. Typical examples for the limit include gas flow near thermal-equilibrium and phase transition with small transition time. There has been a large literature on the mathematical theory of relaxation, see Chen, Levermore and Liu [3], Liu [12], Natalini [15].

The plan of the paper is the following: In Section 2, we give the preliminaries including a brief derivation of the traffic flow model. In Section 3, we establish the L^1-contractivity property for solutions of (1.1) (1.2). Asymptotic behavior of solutions is studied in Section 4. In Section 5, we obtain the L^1-convergence to and the uniqueness of the zero relaxation limit. In Section 6, we give the conclusions.

2 Preliminaries

Zhang's traffic flow model (1.1) (1.2) was derived based on the physical assumption that the time needed for a *following* vehicle to assume a certain speed is determined by *leading* vehicles, see [10] [22]. For $\tau > 0$ and $\Delta x > 0$,

$$\frac{dx}{dt}(t+\tau) = v_e(\rho(x+\Delta x, t)).$$

To leading order

$$v + \tau\frac{dv}{dt} = v_e(\rho(x,t)) + \Delta x \rho_x v_e'(\rho(x,t)).$$

That is

$$\frac{dv}{dt} = \frac{v_e(\rho(x,t)) - v}{\tau} + \frac{\Delta x}{\tau}\rho_x v_e'(\rho(x,t)). \tag{2.1}$$

Letting

$$\frac{\Delta x}{\tau} = -(\lambda_*(\rho) - v_e(\rho)) = -\rho v_e'(\rho),$$

the relative wave propagating speed to the car speed at the equilibrium, we obtain the anticipation factor which expresses the effect of drivers reacting to conditions downstream. The minus sign on the right hand side comes from the fact that the behavior of the driver is determined by leading vehicles.

We assume that the equilibrium velocity $v_e(\rho)$ is a linear function of ρ

$$v_e(\rho) = -a\rho + b, \quad a,\ b > 0 \tag{2.2}$$

as in [9]. Under assumption (2.2),

$$g(\rho) = \frac{a^2}{2}\rho^2 \tag{2.3}$$

and

$$q(\rho) = \rho v_e(\rho) = -a\rho^2 + b\rho \tag{2.4}$$

where the flux q is a quadratic function which corresponds to the flux of the classical PW(Payne-Whitham) model. Therefore the case that the equilibrium velocity is linear is an important nontrivial case in traffic flow. The assumption has also been used by other authors, see, for example, Lattanzio and Marcati [7]. The right eigenvectors of the Jacobian of the flux of (1.1) (1.2) are constant vectors

$$r_i(\rho, v) = (1, (-1)^i a)^T, \quad i = 1,\ 2.$$

Thus both the rarefaction curves and shock curves are straight lines. Furthermore, the shock wave curves coincide with the rarefaction wave curves, $S_i(u_0) = R_i(u_0)$, $i = 1, 2$.

Multiplying (1.1) (1.2) on the left with the jth left eigenvector, $l_j(\rho, v) = ((-1)^{j-1}v_e'(\rho), 1)^T$, $j = 1,\ 2$, of the Jacobian of the flux, we have that

$$(-v_e(\rho) - v)_t + \lambda_1(-v_e(\rho) - v)_x = -h(\rho, v) \tag{2.5}$$

$$(-v_e(\rho) + v)_t + \lambda_2(-v_e(\rho) + v)_x = h(\rho, v) \tag{2.6}$$

where h is defined in (1.6). The Riemann invariants r and s are

$$r(\rho, v) = -v_e(\rho) - v \tag{2.7}$$

$$s(\rho, v) = -v_e(\rho) + v. \tag{2.8}$$

From (2.6) we see that one of the Riemann invariants is the signed distance $-v_e(\rho) + v$ of $(\rho,\ v)$ to the equilibrium curve. Noting (1.6), (1.11) and (2.2), we have

$$r_t - \left(\frac{1}{2}r^2 + br\right)_x = \frac{s}{\tau} \tag{2.9}$$

$$s_t + \left(\frac{1}{2}s^2 + bs\right)_x = -\frac{s}{\tau}. \tag{2.10}$$

The initial data is obtained from (1.3)

$$r(x, 0) = r_0(x) \tag{2.11}$$

$$s(x, 0) = s_0(x). \tag{2.12}$$

3 The Cauchy problem

For a scalar balance law

$$u_t + f(x, t, u)_x = g(x, t, u) \tag{3.1}$$

with initial data

$$u(x, 0) = u_0(x), \tag{3.2}$$

Kruzkov [6] defined generalized solutions of problem (3.1) (3.2).

Let $\Pi_T = R \times [0, T]$. Let $u_0(x)$ be a bounded measurable function such that $|u_0(x)| \leq M_0$ on R.

Definition A bounded measurable function $u(x,t)$ is called a generalized solution of problem (3.1) (3.2) in Π_T if:

i) for any constant k and any smooth function $\phi(x,t) \geq 0$ which is finite in Π_T (the support of ϕ is strictly in Π_T), if the following inequality holds,

$$\int\int_{\Pi_T} \{|u(x,t)-k|\phi_t + \operatorname{sign}(u(x,t)-k)[f(x,t,u(x,t)) - f(x,t,k)]\phi_x -$$

$$- \operatorname{sign}(u(x,t)-k)[f_x(x,t,u(x,t)) - g(x,t,u(x,t))]\} dxdt \geq 0; \tag{3.3}$$

ii) there exists a set E of zero measure on $[0,T]$, such that for $t \in [0,T] \backslash E$, the function $u(x,t)$ is defined almost everywhere in R, and for any ball $K_r = \{|x| \leq r\}$

$$\lim_{t \to 0} \int_{K_r} |u(x,t) - u_0(x)| dx = 0.$$

Inequality (3.3) is equivalent to condition E in [17], if (u_-, u_+) is a discontinuity of u and v is any number between u_- and u_+, then

$$\frac{f(x,t,u_+) - f(x,t,u_-)}{u_+ - u_-} \leq \frac{f(x,t,v) - f(x,t,u_-)}{v - u_-}. \tag{3.4}$$

Remark In the case that f is strictly convex (or concave) in u and $u_- \neq u_+$, the strict inequality in (3.4) holds.

The following results on the existence and uniqueness of the generalized solution of problem (3.1) (3.2) are due to Kruzkov [6].

Uniqueness follows from the following result on the stability of the solutions relative to changes in the initial data in the norm of L^1.

For any $R > 0$ and $M > 0$, we set

$$N_M(R) = \max_{K_R \times [0,T] \times [-M,M]} |f_u(x,t,u)|$$

and let κ be the cone $\{(x,t) : |x| \leq R - Nt, 0 \leq t \leq T_0 = \min\{T, RN^{-1}\}\}$. Let S_τ designate the cross-section of the cone κ by the plane $t = \tau$, $\tau \in [0, T_0]$.

Theorem 3.1 *(Kruzkov) Assume that: i) $f(x,t,u)$ and $g(x,t,u)$ are continuously differentiable in the region $\{(x,t) \in \Pi_T, -\infty < u < +\infty\}$; ii) $f_x(x,t,u)$ and $f_t(x,t,u)$ satisfy Lipschitz condition in u. Let $u(x,t)$ and $v(x,t)$ be generalized solutions of problem (3.1) (3.2) with bounded measurable initial data $u_0(x)$ and $v_0(x)$, respectively, where $|u(x,t)| \leq M$ and $|v(x,t)| \leq M$ almost everywhere in $K_R \times [0,T]$. Let $\gamma = \max g_u(x,t,u)$ in the region $(x,t) \in \kappa$ and $|u| \leq M$. Then for almost all $t \in [0, T_0]$*

$$\int_{S_t} |u(x,t) - v(x,t)| dx \leq e^{\gamma t} \int_{S_0} |u_0(x) - v_0(x)| dx. \tag{3.5}$$

Theorem 3.2 *(Kruzkov) Assume that: i) $f(x,t,u)$ is three times continuously differentiable; ii) $f_u(x,t,u)$ is uniformly bounded for $(x,t,u) \in D_M = \Pi_T \times [-M,M]$; iii) $f_x(x,t,u) - g(x,t,u)$ is twice continuously differentiable and uniformly bounded for $(x,t,u) \in D_M$, where*

$$\sup_{(x,t)\in\Pi_T} |f_x(x,t,0) - g(x,t,0)| \le c_0, ;$$

$$\sup_{(x,t)\in\Pi_T, -\infty<u<\infty} [-f_{xu}(x,t,u) + g_u(x,t,u)] \le c_1;$$

iv) $u_0(x)$ is an arbitrary bounded measurable function in R. Then a generalized solution of problem (3.1) (3.2) exists.

For the initial value problem (2.10) (2.12), our goal is to obtain *a priori* global bounds of the solutions and thus obtain the global existence and uniqueness of the solutions.

First, we have the following result on the global stability of the solutions relative to changes in the initial data in the norm of L^1 which implies the uniqueness of the solutions.

Theorem 3.3 *If $s_1(x,t)$ and $s_2(x,t)$ are generalized solutions of problem (2.10) (2.12) with bounded measurable initial data $s_{10}(x)$ and $s_{20}(x)$ such that $s_{10} - s_{20} \in L^1$. Then for almost all $t > 0$*

$$\int_{S_t} |s_1(x,t) - s_2(x,t)| dx \le e^{-\frac{t}{\tau}} \int_{S_0} |s_{10}(x) - s_{20}(x)| dx. \tag{3.6}$$

Proof Applying Theorem 3.1 to two solutions, s_1 and s_2, of equation (2.10) and noting that $\gamma = -\frac{1}{\tau} < 0$, we obtain (3.6). ♢

We obtain a global bound for the generalized solutions of (2.10) for bounded measurable data (2.12).

Theorem 3.4 *Generalized solutions to the initial value problem (2.10) (2.12) are bounded almost everywhere and the bounds depend only on their initial data.*

Proof Let s be a generalized solution of equation (2.10). Applying Theorem 3.3 to two solutions, s and 0, of equation (2.10), we conclude that the generalized solutions are bounded almost everywhere or all $t > 0$. ♢

It can be checked that all conditions in Theorem 3.2 are satisfied by equation (2.10). Therefore we have the following global existence result.

Theorem 3.5 *A unique generalized solution of problem (2.10) (2.12) exists globally.*

Now we turn to solve the initial value problem (2.9) with bounded measurable data (2.11).

Theorem 3.6 *If $r_1(x,t)$ and $r_2(x,t)$ are generalized solutions of problem (2.9) (2.11) with bounded measurable initial data $r_{10}(x)$ and $r_{20}(x)$ such that $r_{10} - r_{20} \in L^1$. Then for almost all $t > 0$*

$$\begin{aligned}\int_{S_t} |r_1(x,t) - r_2(x,t)|dx \quad &\leq \int_{S_0} |r_{10}(x) - r_{20}(x)|dx + \\ &+(1 - e^{-\frac{t}{\tau}}) \int_{S_0} |s_{10}(x) - s_{20}(x)|dx.\end{aligned} \tag{3.7}$$

Proof Applying the proof of Theorem 3.1 in [6] to problem (2.9) (2.11) and using (3.6), we obtain (3.7). ♢

Similarly, we obtain a global bound for the generalized solutions of (2.9).

Theorem 3.7 *Generalized solutions to the initial value problem (2.9) (2.11) are bounded almost everywhere and the bounds depend only on their initial data.*

Proof Let ρ_e be the bounded solution to the equilibrium equation (1.7) with initial data (1.8), $r_e = -2v_e(\rho_e)$ and $s_e = 0$. Then r_e is a bounded solution to (2.9) with initial data $-2v_e(\rho_e(x,0))$ and s_e is a solution to (2.10) with initial data zero.

Applying Theorem 3.6 to two solutions r and r_e of (2.9), we obtain the global boundedness of the generalized solution r. ♢

Finally, we have the following.

Theorem 3.8 *A unique generalized solution of problem (2.9) (2.11) exists globally.*

We show an L^1-contractive property of the generalized solutions of (2.9) (2.10) in terms of the Riemann invariants.

Theorem 3.9 *If $(r_1(x,t), s_1(x,t))$ and $(r_2(x,t), s_2(x,t))$ are generalized solutions of (2.9) (2.10) for all x and $t > 0$, with initial data $(r_{10}(x), s_{10}(x))$, $(r_{20}(x), s_{20}(x))$ which are bounded measurable and $r_{10} - r_{20}, s_{10} - s_{20} \in L^1$, then*

$$\begin{aligned}&\|r_1(\cdot,t) - r_2(\cdot,t)\|_{L^1} + \|s_1(\cdot,t) - s_2(\cdot,t)\|_{L^1} \\ &\leq \quad \|r_{10}(\cdot) - r_{20}(\cdot)\|_{L^1} + \|s_{10}(\cdot) - s_{20}(\cdot)\|_{L^1}.\end{aligned} \tag{3.8}$$

Proof Combining the results of Theorem 3.3 and Theorem 3.6, we arrive at our conclusion. ♢

Remark It is interesting to note that there is no contractive property for r, see (3.7). However, there is the contractive property for r and s, see (3.8). This property allows us to investigate the large-time behavior of solutions in next section.

From (2.7) (2.8), it is evident that $\|r_1(\cdot,t) - r_2(\cdot,t)\|_{L^1} + \|s_1(\cdot,t) - s_2(\cdot,t)\|_{L^1}$ is equivalent to the L^1 distance of the two solutions. Thus the L^1 well-posedness theory for the Cauchy problem (1.1) (1.2) (1.3) is established.

Theorem 3.10 *If $(\rho_1(x,t), v_1(x,t))$ and $(\rho_2(x,t), v_2(x,t))$ are generalized solutions of (1.1) (1.2) for all x and $t > 0$, with bounded measurable initial data $(\rho_{10}(x), v_{10}(x))$, $(\rho_{20}(x), v_{20}(x))$ such that $\rho_{10} - \rho_{20}, v_{10} - v_{20} \in L^1$, then*

$$\begin{aligned} &\|\rho_1(\cdot,t) - \rho_2(\cdot,t)\|_{L^1} + \|v_1(\cdot,t) - v_2(\cdot,t)\|_{L^1} \\ \leq\ & C(\|\rho_{10}(\cdot) - \rho_{20}(\cdot)\|_{L^1} + \|v_{10}(\cdot) - v_{20}(\cdot)\|_{L^1}) \end{aligned} \tag{3.9}$$

where C is a constant independent of t and the initial data.

4 Asymptotic Behavior

We study the large-time behavior of the entropy solutions of (1.1) (1.2). We show that the entropy solutions with initial data as certain L^1 bounded perturbations of an equilibrium shock wave exist and tend to a shifted equilibrium shock wave in L^1 norm as $t \to \infty$.

Recall that a steady-state solution is either a constant state on the equilibrium curve, i.e., $(\rho, v) = (\rho_0, v_e(\rho_0))$ or an equilibrium shock wave

$$(\rho_{sh}, v_{sh})(x) = \begin{cases} (\rho_-, v_e(\rho_-)) & x \leq x_0 \\ (\rho_+, v_e(\rho_+)) & x > x_0 \end{cases} \tag{4.1}$$

satisfying the entropy condition $\rho_- \leq \rho_+$. Denote the Riemann invariants of the equilibrium shock wave by $R(x)$ and $S(x)$, see (2.7) (2.8).

We show that the equilibrium shock waves are nonlinearly stable in L^1 norm. The main tools used in the proof are the L^1 contractivity, a result of Kruzkov [6] and the exponentially decay property of the source term, see Theorem 3.3.

Without loss of generality, we set

$$(\rho_0(-\infty), v_0(-\infty)) = (\rho_-, v_e(\rho_-)),\ (\rho_0(+\infty), v_0(+\infty)) = (\rho_+, v_e(\rho_+)). \tag{4.2}$$

Theorem 4.1 *Let the initial data (1.3) be a bounded perturbation of an equilibrium shock wave, satisfy*

$$(\rho_0 - \rho_{sh}, v_0 - v_{sh}) \in (L^\infty)^2 \cap (L^1)^2, \tag{4.3}$$

be of small distance to the equilibrium curve and satisfy further that

$$R(-\infty) \leq r(x,0) \leq R(+\infty). \tag{4.4}$$

Then the global bounded entropy solution of (1.1) (1.2) (1.3) exists and tends to a shifted equilibrium shock wave in L^1 norm as $t \to \infty$,

$$\lim_{t \to +\infty} (\|\rho(\cdot,t) - \rho_{sh}(\cdot + k)\|_{L^1} + \|v(\cdot,t) - v_{sh}(\cdot + k)\|_{L^1}) = 0 \tag{4.5}$$

where the shift k is given by

$$k = \frac{1}{\rho_{sh}(+\infty) - \rho_{sh}(-\infty)} \int_R (\rho(x,0) - \rho_{sh}(x))dx. \tag{4.6}$$

We state a result of Kruzkov [6] before we prove Theorem 4.1.

Consider the initial value problem

$$u_t + (f(u))_x = g \tag{4.7}$$

$$u(x,0) = u_0. \tag{4.8}$$

Let $\Sigma_T = R \times (0,T)$, $[a]_+ = \frac{1}{2}(|a| + a)$ and $H = H(a) = \frac{1}{2}(\operatorname{sgn} a + 1)$, is the Heavyside function.

Theorem 4.2 *Let $u(v)$ be an entropy subsolution(supersolution) of (4.7) (4.8) in Σ_T for the right-hand side $g(h)$ and the initial data $u_0(v_0)$. Fix a, b such that $u, v \in [a,b]$ in Σ_T. Then for each interval (α, β), we have*

$$\begin{aligned}\int_{\alpha+tK}^{\beta-tK} & [u(x,t) - v(x,t)]_+ dx \\ \leq & \int_\alpha^\beta [u_0(x) - v_0(x)]_+ dx + \\ & + \int_0^t ds \int_{\alpha+tK}^{\beta-tK} H(u(x,s) - v(x,s))(g(x,s) - h(x,s))dx\end{aligned} \tag{4.9}$$

where $K \geq L = \|f'(u)\|_{L^\infty(a,b)}, 0 < t < \min\{\tau, T\}$ and $\tau = \dfrac{\beta - \alpha}{2L}$.

Notice that (2.9) (2.10) is a weakly coupled system of quasilinear hyperbolic equations in the sense that it is in diagonal form and the equations are coupled by means of the source term that does not depend on the derivatives of the unknowns, see [16].

Furthermore, we have the quasimonotonicity of the source term $G = (g_1, g_2)^T$ of (2.9) (2.10) in the sense that g_1 is nondecreasing in s and g_2 nondecreasing in r, see [15].

Lemma 4.3 *The source term $G = (g_1, g_2)^T$ of (2.9) (2.10) is quasimonotone.*

Proof From (2.9) (2.10) we have that $g_1(r,s) = \frac{s}{\tau}$ and $g_2(r,s) = -\frac{s}{\tau}$. Consequently, $\frac{\partial g_1}{\partial s} = \frac{1}{\tau} > 0$ and $\frac{\partial g_2}{\partial r} = 0$. The conclusion is proved. $\diamondsuit$

Therefore, solutions of (2.9) (2.10) satisfy a comparison principle, see [15].

Theorem 4.4 *Let U^1 and U^2 be two weak solutions of (2.9) (2.10) in $\mathbb{R} \times (0,T)$ with initial data U_0^1 and U_0^2 respectively. If $U_0^1 \leq U_0^2$ for almost every $x \in \mathbb{R}$, then $U^1 \leq U^2$ for almost every $(x,t) \in \mathbb{R} \times (0,T)$.*

Now we prove Theorem 4.1.

Proof Boundedness of the solution follows directly from the quasimonotone property of the source terms of (2.9) (2.10) and the comparison principle.

Step 1. We first prove the conclusion for initial data (2.11) (2.12) satisfying some additional ordering properties besides (4.3) (4.4). Let

$$R(x) = r(\rho_{sh}, v_{sh}) = -2v_e(\rho_{sh})$$

and

$$S(x) = s(\rho_{sh}, v_{sh}) = 0$$

be stationary solutions of (2.9) and (2.10) respectively. Let $r(x,0)$ satisfy

$$R(x+\gamma) \le r(x,0) \le R(x+\beta), \text{ for all } x \tag{4.10}$$

for some γ and β.

Let

$$r(x,0) = R(x) + \psi_0(x)$$

and

$$r(x,t) = R(x) + \psi(x,t).$$

Applying Theorem 4.2 to solutions of (2.9) and noting (4.10), we have that

$$R(x+\gamma) \le r(x,t) \le R(x+\beta)$$

for all x and for all $t > 0$ or

$$R(x+\gamma) - R(x) \le \psi(x,t) \le R(x+\beta) - R(x) \tag{4.11}$$

for all x and for all $t > 0$. (4.11) follows from (4.9), that the initial data is of small distance (relative to the equilibrium shock wave strength) to the equilibrium curve and that (2.9) has a source term that decays exponentially in t, see (3.6).

Therefore $\{\psi(x,t)\}_{t>0}$ is uniformly bounded by functions in L^1.

We claim that $\{\psi(x,t)\}_{t>0}$ is L^1-equicontinuous. In fact,

$$\begin{aligned}
&\|\psi(\cdot+h,t) - \psi(\cdot,t)\|_{L^1} \\
&\le \|r(\cdot+h,t) - r(\cdot,t)\|_{L^1} + \|R(\cdot+h) - R(\cdot)\|_{L^1} \\
&\le \|r(\cdot+h,0) - r(\cdot,0)\|_{L^1} + (1 - e^{-\frac{t}{\tau}})\|s(\cdot+h,0) - s(\cdot,0)\|_{L^1} + \\
&\quad + \|R(\cdot+h) - R(\cdot)\|_{L^1} \to 0
\end{aligned}$$

as $h \to 0$, uniformly with respect to $t > 0$, where we have used the continuous dependence on data property (3.7) and the condition on the initial data (4.3).

Hence $\{\psi(x,t)\}_{t>0}$ is relatively compact in L^1.

Let B_s be the set of accumulation points of $\{\psi(x,t)\}_{t>s}$ for $s > 0$, then B_s is not empty by compactness. Hence

$$A = R(x) + \cap_{s\ge 0} B_s \ne \emptyset$$

represents the set of all the possible L^1-limit of $r(x,t)$ as $t \to +\infty$.

It is enough to prove that $A = \{R(x+k)\}$ for some k.

Let $a^0(x) \in A$. Then there exist $t_n > 0$ such that $t_n \to +\infty$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \|r(\cdot,t_n) - a^0(\cdot)\|_{L^1} = 0. \tag{4.12}$$

¿From Theorem 3.6, we know that $\|r_1(\cdot,t) - r_2(\cdot,t)\|_{L^1} + e^{-\frac{t}{\tau}}\|s_{10}(\cdot) - s_{20}(\cdot)\|_{L^1}$ decreases in t for any two solutions and hence it admits limit as $t \to +\infty$. Therefore for any h,

$$\lim_{t\to+\infty} \|r(\cdot,t) - R(\cdot+h)\|_{L^1} = c_h \tag{4.13}$$

for some $c_h \geq 0$.

Letting $t_n \to +\infty$ in the above limit, we have

$$\|a^0(\cdot) - R(\cdot+h)\|_{L^1} = c_h.$$

Let $a(x,t)$ be the solution of (2.9) with initial data $a^0(x)$. Then $a(x,t) \in A$ (A is invariant under the flow defined by (2.9)). Therefore for the same reason

$$\|a(\cdot,t) - R(\cdot+h)\|_{L^1} = c_h$$

for any h and any $t > 0$.

Applying the contractive property (3.8) to two solutions $(a(x,t),0)$ and $(R(x+h),0)$ of (2.9) (2.10), we have that

$$\begin{aligned} 0 &= \|a(\cdot,t) - R(\cdot+h)\|_{L^1} - \|a^0(\cdot) - R(\cdot+h)\|_{L^1} \\ &\leq 0 \end{aligned}$$

for any h and any $t > 0$. From the proof of Theorem 3.3 and Theorem 3.6 we know that the above equality holds if and only if $a(x,t) - R(x+h)$ has no shock at the point of sign change for any h and any $t > 0$, see the remark follows (3.4). This shows that there exists some k, satisfying (4.6) due to conservation law (1.1), such that for any $t > 0$

$$a(x,t) = R(x+k).$$

Thus there exists $t_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} \|r(\cdot,t_n) - R(\cdot+k)\|_{L^1} = 0.$$

By (4.13), $\|r(\cdot,t) - R(\cdot+k)\|_{L^1}$ admits a limit as $t \to +\infty$, we conclude that $r(x,t) - R(x+k)$ converges to 0 in L^1 as $t \to +\infty$. By (3.6), $\|s(\cdot,t) - S(\cdot+k)\|_{L^1}$ decays to zero exponentially in t. Thus $\|r(\cdot,t) - R(\cdot+k)\|_{L^1} + \|s(\cdot,t) - S(\cdot+k)\|_{L^1}$ converges to zero. On the other hand, $\|r(\cdot,t) - R(\cdot+k)\|_{L^1} + \|s(\cdot,t) - S(\cdot+k)\|_{L^1}$ is equivalent to the L^1 distance of the two solutions, we arrive at the conclusion (4.5).

Step 2. For general initial data satisfying (4.3) (4.4), we have

$$R(-\infty) \le r(x,0) = R(x) + \psi_0(x) \le R(+\infty),$$

where $\psi_0 \in L^1(R)$. Let $\psi_0^n \in L^1(R)$ be a sequence of functions satisfying the ordering properties (4.10) defined in Step 1 and

$$\lim_{n\to\infty} \|\psi_0 - \psi_0^n\|_{L^1} = 0.$$

Let $r^n(x,t)$ and $s^n(x,t)$ be the solutions of the initial value problem (2.9) (2.10) with $r^n(x,0) = R(x) + \psi_0^n(x)$ and $s^n(x,0) = s(x,0)$ respectively. Then, by Step 1, there exist k_n such that

$$\lim_{t\to\infty} \|r^n(\cdot,t) - R(\cdot + k_n)\|_{L^1} = 0$$

for each n.

By the contractive property (3.8) and that $r^n(\cdot,0)$ and $s^n(\cdot,0)$ are Cauchy sequences in L^1, we deduce that $r^n(\cdot,t)$ and $s^n(\cdot,t)$ are Cauchy sequences for any $t > 0$. Therefore, by letting $t \to \infty$, we obtain that $R(\cdot + k_n)$ is a Cauchy sequence too. So

$$\int_R |R(x+k_n) - R(x+k_m)|dx = (R(+\infty) - R(-\infty))|k_n - k_m|.$$

Thus, for any $\epsilon > 0$, there is an N, such that if $m, n > N$, then

$$|k_n - k_m| < \epsilon.$$

Therefore

$$\lim_{n\to\infty} k_n = k$$

$$\lim_{n\to\infty} \|R(\cdot + k_n) - R(\cdot + k)\|_{L^1} = 0$$

for some k.

Finally,

$$\|r(\cdot,t) - R(\cdot + k)\|_{L^1} \le \|r(\cdot,t) - r^n(\cdot,t)\|_{L^1} +$$

$$+ \|r^n(\cdot,t) - R(\cdot + k_n)\|_{L^1} + \|R(\cdot + k_n) - R(\cdot + k)\|_{L^1}.$$

Therefore, we conclude that $r(x,t)$ converges to $R(x+k)$ in L^1 as $t \to +\infty$. By (3.6), $\|s(\cdot,t) - S(\cdot + k)\|_{L^1}$ decays to zero exponentially in t. Thus $\|r(\cdot,t) - R(\cdot + k)\|_{L^1} + \|s(\cdot,t) - S(\cdot + k)\|_{L^1}$ converges to zero. Since $\|r(\cdot,t) - R(\cdot + k)\|_{L^1} + \|s(\cdot,t) - S(\cdot + k)\|_{L^1}$ is equivalent to the L^1 distance of the two solutions, we arrive at the conclusion that the entropy solution of (1.1) (1.2) (1.3) exists globally and tends to a shifted equilibrium shock wave in L^1 norm as $t \to \infty$, where the shift k satisfies (4.6) due to conservation law (1.1). ♢

Now we consider bounded compact support initial data ρ_0 in (1.3). We show that the entropy solution converges in L^1 to an equilibrium N-wave as $t \to +\infty$.

An N-wave of a scalar conservation law (3.1) is

$$N(x,t)=\begin{cases} \frac{1}{k}\left(\frac{x}{t}-c\right) & -(pkt)^{\frac{1}{2}}<x-ct<(qkt)^{\frac{1}{2}} \\ 0 & \text{otherwise,} \end{cases} \tag{4.14}$$

where p, q, c and k are constants. Let $u_0(x)$ be the initial data with compact support, then $c=f'(0)$, $k=f''(0)$

$$p=-2\min_x \int_{-\infty}^{x} u_0(y)dy\,, q=2\max_x \int_{x}^{\infty} u_0(y)dy. \tag{4.15}$$

The entropy solution of (3.1) decays in L^1 to an N-wave uniformly at a rate $t^{-\frac{1}{2}}$.

Theorem 4.5 *Let $(\rho_0, v_0) \in (L^\infty)^2 \cap (L^1)^2$ and ρ_0 have compact support. Let (ρ, v) be the bounded unique entropy solution of (1.1) (1.2) (1.3). Then $\rho(x,t)$ decays in L^1 norm to the N-wave $N(x,t)$ determined by initial data ρ_0 and*

$$\|\rho(\cdot,t)-N(\cdot,t)\|_{L^1} \le Ct^{-\frac{1}{2}} \tag{4.16}$$

for t large and some constant $C>0$. $v \to v_e(\rho)$ in L^1 norm as $t \to +\infty$.

Proof Consider two entropy solutions, (ρ, v) and $(\rho_e, v_e(\rho_e))$, of (1.1) (1.2), where ρ_e is the unique entropy solution of the equilibrium equation (1.7) with initial data ρ_0. Applying Theorem 3.3 to these two solutions, we have

$$\|s_1(\cdot,t)-s_2(\cdot,t)\|_{L^1} \le e^{-\frac{t}{\tau}}\|s_{10}(\cdot)-s_{20}(\cdot)\|_{L^1} \to 0 \tag{4.17}$$

as $t \to +\infty$, where $s_1=-v_e(\rho)+v$ and $s_2=0$.

We claim that $\{\rho(x,t)-\rho_e(x,t)\}_{t>0}$ is L^1-equicontinuous. In fact,

$$\begin{aligned} &\|\rho(\cdot+h,t)-\rho_e(\cdot+h,t)-\rho(\cdot,t)+\rho_e(\cdot,t)\|_{L^1} \\ &\le \|\rho(\cdot+h,t)-\rho(\cdot,t)\|_{L^1}+\|\rho_e(\cdot+h,t)-\rho_e(\cdot,t)\|_{L^1} \\ &\le C(\|r(\cdot+h,t)-r(\cdot,t)\|_{L^1}+\|s(\cdot+h,t)-s(\cdot,t)\|_{L^1})+ \\ &\quad +\|\rho_e(\cdot+h,t)-\rho_e(\cdot,t)\|_{L^1} \\ &\le C(\|r(\cdot+h,0)-r(\cdot,0)\|_{L^1}+\|s(\cdot+h,0)-s(\cdot,0)\|_{L^1})+ \\ &\quad +\|\rho_0(\cdot+h)-\rho_0(\cdot)\|_{L^1} \to 0 \end{aligned}$$

uniformly with respect to $t>0$ as $h \to 0$, where we have used the continuous dependence on data property (3.8). Hence $\{\rho(x,t)-\rho_e(x,t)\}_{t>0}$ is relatively compact in L^1. Let A be the set of accumulation points of $\{\rho(x,t)-\rho_e(x,t)\}_{t>0}$, then $A \subset L^\infty \cap L^1$ is not empty by compactness. Let $\phi(x) \in A$, then $\phi(x)$ is of compact support and there exists a sequence t_n such that $t_n \to +\infty$ as $n \to +\infty$ and

$$\|\rho(\cdot,t_n)-\rho_e(\cdot,t_n)-\phi(\cdot)\|_{L^1} \to 0.$$

Letting $t_n \to +\infty$ in (1.1) and noting (4.17) and the uniform boundedness of (ρ, v), we have that $\rho(x,t_n)$ solves (1.7) asymptotically. Noticing that ρ_e is the unique entropy solution of (1.7) with data ρ_0 and that $\phi(x)$ is of compact support, we deduce that $\phi(x) = 0$ a.e.. That is $A = \{0\}$. Since every convergent sequence of $\rho(x,t) - \rho_e(x,t)$ converges to a same limit 0, therefore, we have

$$\|\rho(\cdot,t) - \rho_e(\cdot,t)\|_{L^1} \to 0 \quad \text{as } t \to +\infty\,.$$

On the other hand, $\rho_e(x,t)$ decays in L^1 to the N-wave $N(x,t)$ determined by the initial data at a rate $t^{-\frac{1}{2}}$ as $t \to +\infty$. Furthermore, ρ decays to the N-wave also at a rate $t^{-\frac{1}{2}}$ as $t \to +\infty$.

5 Unique Zero Relaxation Limit

Uniqueness issues do not seem to have been systematically studied in conjunction with higher order models.

In general, the zero relaxation limit is highly singular because of shock and initial layers. In [15], Natalini obtained the uniqueness of the zero relaxation for semilinear systems of equations with relaxation. The uniqueness problem for the quasilinear case remains open. For the quasilinear system of equations (1.1) (1.2), we show that the entropy solutions of (1.1) (1.2) (1.3) converge in L^1 norm to the unique entropy solution of the equilibrium equation (1.7) (1.8) as the relaxation parameter τ goes to zero. The limit models dynamic limit from the continuum nonequilibrium processes to the equilibrium processes. We proved the uniqueness of the zero relaxation limit by using the property that the solution depends on its data continuously, the fact that the signed distance $-v_e(\rho) + v$ of $(\rho,\ v)$ to the equilibrium curve is one of the Riemann invariants and that it decays in τ exponentially.

We denote the solutions to (1.1) (1.2) (1.3) as (ρ^τ, v^τ) for each $\tau > 0$ and ρ the unique entropy solution of the equilibrium equation (1.7) (1.8).

Theorem 5.1 *Let (ρ^τ, v^τ) be the global bounded entropy solution of (1.1) (1.2) (1.3) with $(\rho_0, v_0) \in (L^\infty)^2$ and $v_0 - v_e(\rho_0) \in L^1$. Then (ρ^τ, v^τ) converges in $(L^1)^2$ to $(\rho, v_e(\rho))$ as $\tau \to 0$ for any $t > 0$. Moreover, ρ is the unique entropy solution of the equilibrium equation (1.7) (1.8).*

Proof Let (ρ^τ, v^τ) be the unique entropy solution of (1.1) (1.2) (1.3). Let ρ be the unique entropy solution of the equilibrium equation (1.7) (1.8).

Applying (3.6) to the two solutions (ρ^τ, v^τ) and $(\rho, v_e(\rho))$, we have that

$$\|s_1^\tau(\cdot,t) - s_2(\cdot,t)\|_{L^1} \le e^{-\frac{t}{\tau}}\|s_{10}(\cdot) - s_{20}(\cdot)\|_{L^1} \to 0 \tag{5.1}$$

as $\tau \to 0$ for $t > 0$, where $s_1^\tau = -v_e(\rho^\tau) + v^\tau$ and $s_2 = 0$. Therefore

$$\| - v_e(\rho^\tau) + v^\tau\|_{L^1} \to 0 \tag{5.2}$$

as $\tau \to 0$ for $t > 0$.

Applying Theorem 3.9 to two solutions, (r_1^τ, s_1^τ) and (r_2, s_2), of (2.9) (2.10), where $r_1^\tau(\rho^\tau, v^\tau) = -v_e(\rho^\tau) - v^\tau$ and $r_2(\rho, v_e(\rho)) = -2v_e(\rho)$, we have that the

$(L^1)^2$ distance between these two solutions is uniformly bounded with respect to the relaxation parameter τ.

We claim that $\{r_1^\tau(x,t) - r_2(x,t)\}_{\tau>0}$ is L^1-equicontinuous in x and locally L^1 Lipschitz continuous in t. The L^1-equicontinuity in x is obtained by using the continuous dependence on data property (3.8), see the proof of Theorem 4.5. The locally L^1 Lipschitz continuous in t for $t > t_\tau = \tau \ln \frac{1}{\tau}$ is a direct consequence of finite speed of propagation of the elementary waves and the exponential decay in τ of the source terms of (2.9) (2.10), see [6] [15]. Therefore, there is a sequence τ_n such that $\tau_n \to 0$ as $n \to +\infty$ and that $r_1^{\tau_n}(x,t) - r_2(x,t)$ converges to a function for each $t > 0$. Combining with (5.1), we have that as $n \to +\infty$, $\rho^{\tau_n}(x,t) - \rho(x,t)$ converges to a function denoted as $\phi(x,t)$ for $t > 0$. It can be checked that $\phi(x,t) \in L^\infty \cap L^1$ for all $t > 0$. Noticing that ρ^{τ_n} and ρ have the same initial data (1.3) (1.8), we have that $\phi(x,0) = 0$.

Letting $\tau_n \to 0$ in (1.7) and noting the uniform boundedness of $(\rho^{\tau_n}, v^{\tau_n})$ and (5.1), we derive that the limit $\phi(x,t) = 0$ a.e.. Therefore

$$\|\rho^{\tau_n}(\cdot,t) - \rho(\cdot,t)\|_{L^1} \to 0$$

as $n \to +\infty$. Since every convergent sequence of $\rho^\tau(x,t) - \rho(x,t)$ converges to a same limit 0, we conclude that $\rho^\tau - \rho$ converges to 0 in L^1 as $\tau \to 0$. This and (5.2) prove the theorem.

6 Conclusions

For a nonequilibrium continuum traffic flow model, which was derived based on the assumption that drivers respond with a delay to changes of traffic conditions in front of them, we established the L^1 well-posedness theory for the Cauchy problem. We obtained the continuous dependence of the solution on its initial data in L^1 topology. We constructed a functional for two solutions which is equivalent to the L^1 distance between the solutions. We proved that the functional decreases in time which yields the L^1 well-posedness of the Cauchy problem.

We also showed that the equilibrium shock waves are nonlinearly stable in L^1 norm. That is, the entropy solution with initial data as certain L^1 bounded perturbations of an equilibrium shock wave exists globally and tends to a shifted equilibrium shock wave in L^1 norm as $t \to \infty$. We then showed that if the initial data ρ_0 is bounded and of compact support, the entropy solution converges in L^1 to an equilibrium N-wave as $t \to +\infty$. We finally showed that the solutions for the relaxed system converge in the L^1 norm to the unique entropy solution of the equilibrium equation as the relaxation time goes to zero.

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