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A DEFORMATION THEOREM IN THE NONCOMPACT NONSMOOTH SETTING AND ITS APPLICATIONS

GIANNI ARIOLI

Abstract. We build a deformation for a continuous functional defined on a Banach space and invariant with respect to an isometric action of a noncompact group. Under these assumptions the Palais-Smale condition does not hold. When the functional is also invariant with respect to the action of a compact Lie group, we prove that the deformation can be chosen to be equivariant with respect to the same action. In the second part of the paper a system of periodic quasilinear partial differential equations invariant under the action of some compact Lie group is considered. Using the deformation technique developed in the first part, we prove the existence of infinitely many solutions.

1. Introduction

In the last decade the nonsmooth critical point theory introduced by Degiovanni and his collaborators [11, 14, 16] has been applied to a large variety of nonlinear differential problems whose Lagrangian functional lacks differentiability. The core of the theory is the concept of *weak slope*, which plays the part of the norm of the derivative, although it is defined also for lower semi-continuous functionals. One defines the critical points as points with zero weak slope and proves that they provide weak solutions for the differential equation. The beauty of the theory consists in the fact that most topological tools used in critical point theory, such as the mountain pass theorem, the saddle point theorem, Morse theory and index theory, can be applied up to small adaptations. All these tools rely on the existence of a deformation in the regions where there are no critical points. Accordingly to [14, 16], in order to build a deformation a certain compactness is required on the functional, more precisely the nonsmooth version of the well-known Palais-Smale condition. On the other hand, if a functional is invariant under the action of a noncompact topological group, a typical example being $\mathbb{Z}^n$, then the Palais-Smale condition cannot hold. Functionals which are invariant under the action of $\mathbb{Z}^n$ are very common: as an example consider the problem of looking for homoclinic solutions in the theory of dynamical system. It is in that setting that a weaker version of the Palais Smale condition has been introduced, namely the $\overline{PS}$ condition

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(see [18] and also [4] where the symmetry with respect to a compact Lie group has been considered). It turns out that the $\overline{PS}$ condition is sufficient to build a deformation under suitable circumstances. The proof relies in following the negative gradient flow and proving that it does not come to an end, unless it comes close to a critical point. Due to the nonexistence of the gradient, it is not possible to extend directly the proof to the nonsmooth setting. In the first part of this paper we take a different approach and we build a deformation for continuous functionals on a Banach space, invariant with respect to some isometric action of a noncompact group. In order to achieve this goal, we build a deformation by using the tools of the abstract nonsmooth critical point theory, without recurring to the (possible) integral form of the functional. This approach is very natural in this context, and provides a flexible abstract result.

In the second part of the paper we apply this technique to a system of m quasilinear differential equations defined and periodic in $\mathbb{R}^n$ (and therefore invariant with respect to the natural isometric action of $\mathbb{Z}^n$ on $\mathbb{R}^n$). We prove the existence of a (weak) solution in the general case, and the existence of infinitely many (weak) solutions when the system is also invariant under the action of some compact Lie group G on $\mathbb{R}^m$. A similar system has been considered in [3] in the case where the nonlinear part of the functional is compact; that case excludes the periodic setting.

We remark that the simpler case of a single equation, symmetric with respect to the action of $\mathbb{Z}_2$, comes as a corollary, and indeed it requires weaker assumptions.

2. Equivariant nonsmooth critical point theory

The following definitions provide the basic tools of the nonsmooth critical point theory in the equivariant setting introduced in [14, 16] and are given for the convenience of the reader. For a detailed introduction we refer to the above cited papers and to [11]. We remark that the non equivariant definitions correspond exactly to the following ones, if one just forgets about the invariant-equivariant requirements. The main definitions concerning representation theory are summarized in Section 8.

Definition 2.1. *Let (X, d) be a metric $G-$space, let $I \in C(X, \mathbb{R})$ be an invariant function and let $x \in X$. We denote by $|d_G I|(x)$ the supremum of the $\sigma \in [0, +\infty)$ such that there exist an invariant neighborhood U of x, $\delta > 0$ and a continuous map $\mathcal{H} : U \times [0, \delta] \longrightarrow X$ such that for all $y \in U$ and for all $t \in [0, \delta]$*

$$d(\mathcal{H}(y, t), y) \le t, \qquad I(\mathcal{H}(y, t)) \le I(y) - \sigma t$$

and $\mathcal{H}(\cdot, t)$ is equivariant for all $t \in [0, \delta]$; the extended real number $|d_G I|(x)$ is called the equivariant weak slope *of I at x.*

Definition 2.2. *Let (X, d) be a metric $G-$space, let $I \in C(X, \mathbb{R})$ be an invariant function; a point $x \in X$ is said to be $G-$critical for I if $|d_G I|(x) = 0$. A real number c is said to be a $G-$critical value for I if there exists $x \in X$ such that $I(x) = c$ and $|d_G I|(x) = 0$.*

The equivariant version of the compactness condition of Palais-Smale has been defined in this context (see [16]):

Definition 2.3. *Let (X, d) be a metric $G-$space, let $I \in C(X, \mathbb{R})$ be an invariant function. A sequence $\{x_m\} \subset X$ is called a $G-$Palais-Smale sequence ($G-(PS)$) if there exists $K > 0$ such that $|I(x_m)| \le K$ and $|d_G I|(x_m) \to 0$.*

I satisfies the $G-(PS)$ condition if all its $G-(PS)$ sequences are precompact.

The following definition provides a tool to relate the weak slope to the norm of the directional derivatives, when X is a Banach space.

Definition 2.4. *Let X be a Banach space, let $I \in C(X, \mathbb{R})$ and let Y be a dense subspace of X. If the directional derivative of I exists for all x in X in all the directions $y \in Y$ we say that I is weakly Y-differentiable and we call weak Y-slope in x the extended real number*

$$\|I'_Y(x)\|_* := \sup\{I'(x)[\phi] : \ \phi \in Y, \ \|\phi\|_X = 1\}.$$

Consider now the Hilbert space $H = H^1(\mathbb{R}^n, \mathbb{R}^m)$, i.e. the closure of $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ (the space of smooth vector functions with compact support in $\mathbb{R}^n$) with respect to the norm induced by the scalar product $(\psi, \phi) = \int_{\mathbb{R}^n} \langle D_i\psi, D_i\phi\rangle + \langle \psi, \phi\rangle$. The following theorem provides the connection between the equivariant nonsmooth critical point theory and applications to an invariant continuous functional defined on H. In order to do so, we extend to the equivariant setting Theorem 1.5 in [10], which states that the weak slope provides an upper limit for the $C_c^\infty(\mathbb{R}^n)$-slope.

Theorem 2.1. *Let $\mathbb{R}^m$ be a representation space for G, let $J : H \to \mathbb{R}$ be defined by*

$$J(u) = \int_{\mathbb{R}^n} L(x, u, \nabla u)dx,$$

where $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}$ satisfies the following assumptions:

$L(x, s, \xi)$ is measurable with respect to x for all $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$
$L(x, s, \xi)$ is of class C^1 with respect to (s, ξ) for a.e. $x \in \mathbb{R}^n$

and there exist $h_1 \in L^1(\mathbb{R}^n, \mathbb{R})$, $h_2 \in L^1_{loc}(\mathbb{R}^n, \mathbb{R})$, $h_3 \in L^\infty_{loc}(\mathbb{R}^n, \mathbb{R})$ and $c > 0$ such that for all $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$ and a.e. $x \in \mathbb{R}^n$ the following inequalities hold:

$$\begin{aligned} |L(x, s, \xi)| &\le h_1(x) + c(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ \left|\tfrac{\partial L}{\partial s}(x, s, \xi)\right| &\le h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ \left|\tfrac{\partial L}{\partial \xi}(x, s, \xi)\right| &\le h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2). \end{aligned}$$

Furthermore L is invariant with respect of the action of G on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{mn}$ defined by

$$(x, s, (\xi_1, \dots, \xi_m)) \mapsto (x, gs, (g\xi_1, \dots, g\xi_m)).$$

Then J is continuous, invariant with respect to the action of G on H defined by $g(u)(x) := g(u(x))$, weakly $C_c^\infty(\Omega)-$differentiable and for all $u \in H$ we have

$$|d_G J|(u) \ge \|J'_{C_c^\infty}(u)\|_*. \tag{2.1}$$

In particular, if u is a $G-$critical point of J, then the Euler equation

$$div\left(\frac{\partial L}{\partial \xi}(x, u, \nabla u)\right) - \frac{\partial L}{\partial s}(x, u, \nabla u) = 0$$

is satisfied in distributional sense.

Proof. By the Sobolev embedding theorem J is continuous,

$$J'(u)[v] = \int_{\mathbb{R}^n} \left[\frac{\partial L}{\partial \xi}(x, u, \nabla u)\nabla v + \frac{\partial L}{\partial s}(x, u, \nabla u)v\right] dx$$

for all $u \in H$ and all $v \in C_c^\infty(\mathbb{R}^n)$ and the functional $\{u \mapsto J'(u)[v]\}$ is continuous for all $v \in C_c^\infty(\mathbb{R}^n)$. The invariance of J is straightforward. Choose $u_0 \in H$; if $\|J'_{C_c^\infty}(u_0)\|_* = 0$ there is nothing to prove, otherwise let $0 < \sigma < \|J'_{C_c^\infty}(u_0)\|_*$.

Then there exists $v_0 \in C_c^\infty(\mathbb{R}^n)$, $||v_0|| = 1$, such that $J'(u_0)[v_0] > \sigma$, and there exists $\delta > 0$ such that $J'(u)[v_0] > \sigma$ for all $u \in B_{2\delta}(u_0)$. Let $\eta : H \to [0,1]$ be a smooth function satisfying $\eta(u) = 1$ if $u \in B_\delta(u_0)$ and $\eta(u) = 0$ if $u \notin B_{2\delta}(u_0)$. Let $U = \mathcal{O}_G(B_\delta(u_0))$ and define the function $v : U \to H$ by

$$v(u) = \alpha(u) \int_G g v_0 \eta(g^{-1}u) dg,$$

where dg is the Haar measure and $\alpha(u) = \left[\int_G \eta(g^{-1}u)dg\right]^{-1}$; by Fubini Theorem $||v(u)|| = ||v|| = 1$. Define a continuous map $\mathcal{H} : U \times [0, \delta] \longrightarrow H$ by

$$\mathcal{H}(u, t) = u - tv(u).$$

The map $\mathcal{H}(\cdot, t)$ is equivariant for all $t \in [0, \delta]$, because

$$v(g_0 u) = \int_G g v_0 \eta(g^{-1} g_0 u) dg = \int_G g_0 \tilde{g} v_0 \eta(\tilde{g}^{-1} u) d\tilde{g} = g_0 v(u)$$

so $\mathcal{H}(gu, t) = gu - tv(gu) = g(u - tv(u))$. Since $||v(u)|| = 1$, then $||\mathcal{H}(u, t) - u|| \le t$. By the invariance of L and Fubini Theorem it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, u, \nabla u) v(u) dx &= \alpha(u) \int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, u, \nabla u) \int_G g v_0 \eta(g^{-1}u) dg dx \\
&= \alpha(u) \int_G \eta(g^{-1}u) \int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, u, \nabla u) g v_0 dx dg \\
&= \alpha(u) \int_G \eta(g^{-1}u) \int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, g^{-1}u, g^{-1}\nabla u) v_0 dx dg
\end{aligned}$$

and analogously

$$\int_{\mathbb{R}^n} \frac{\partial L}{\partial \xi}(x, u, \nabla u) \nabla(v(u)) dx = \alpha(u) \int_G \eta(g^{-1}u) \int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, g^{-1}u, g^{-1}\nabla u) \nabla v_0 dx dg.$$

For all $u \in B_{2\delta}(u_0)$ we have

$$\int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, u, \nabla u) v_0 + \frac{\partial L}{\partial s}(x, u, \nabla u) v_0 dx = J'(u)[v_0] > \sigma$$

and for all $u \notin B_{2\delta}(u_0)$ and $\eta(u) = 0$, therefore

$$\begin{aligned}
&J'(u)[v(u)] \\
&= \int_{\mathbb{R}^n} \frac{\partial L}{\partial \xi}(x, u, \nabla u) \nabla(v(u)) + \frac{\partial L}{\partial s}(x, u, \nabla u) v(u) dx \\
&= \alpha(u) \int_G \eta(g^{-1}u) \int_{\mathbb{R}^n} \frac{\partial L}{\partial s}(x, g^{-1}u, g^{-1}\nabla u) \nabla v_0 + \frac{\partial L}{\partial s}(x, g^{-1}u, g^{-1}\nabla u) v_0 dx dg \\
&\ge \sigma \alpha(u) \int_G \eta(g^{-1}u) dg = \sigma,
\end{aligned}$$

$J(\mathcal{H}(u, t)) - J(u) = tJ'(u - sv_0)[v_0]$ for some $s \in [0, t]$ and $J(\mathcal{H}(u, t)) \le J(u) - \sigma t$. By the definition of equivariant weak slope we have $|d_G J|(u) \ge \sigma$ and by the arbitrariness of σ we obtain (2.1). $\square$

3. The deformation theorem

In this section we build a deformation in the abstract setting of a Banach space with an isometric action of a discrete topological group. Let B be a Banach space, let $\mathbb{D}$ be a discrete topological group acting on B with a linear isometry $* : \mathbb{D}\times B \to B$, $(k,u)\mapsto k*u$.

Definition 3.1. *For $l\in\mathbb{N}$, $\bar{k}=(k^1,\dots,k^l)\in\mathbb{D}^l$ and $\bar{u}=(u^1,\dots,u^l)\in B^l$, let*

$$\bar{k}*\bar{u} := \sum_i k^i * u^i.$$

Given a sequence $\{\bar{k}_m\}\subset\mathbb{D}^l$, we say that $\{\bar{k}_m\}$ diverges or $\bar{k}_m\to\infty$ if for all $i\neq j$ the sequence $\{k^i_m(k^j_m)^{-1}\}$ does not admit any convergent subsequence (i.e. $\{k^i_m(k^j_m)^{-1}\}$ does not admit any constant subsequence).

In the following, $J : B\to\mathbb{R}$ will denote a continuous functional, invariant with respect to the action of $\mathbb{D}$. The set $K := \{u\in B : |dJ|(u)=0\}$ is the (nonsmooth) critical set. We also assume that

$$\begin{gathered} K\setminus\{0\} = \mathcal{O}_{\mathbb{D}}(\mathcal{K}),\ \text{where } \mathcal{K} \text{ is a compact set,}\\ \mathcal{K}\cap(k*\mathcal{K})=\emptyset \text{ whenever } k\neq 1_{\mathbb{D}} \end{gathered} \tag{3.1}$$

and

$$\lim_{m\to\infty} ||\bar{k}_m*\bar{u}||^2 = \sum_i ||u^i||^2 \tag{3.2}$$

for all $l\in\mathbb{N}$, all diverging sequences $\{\bar{k}_m\}\subset\mathbb{D}^l$ and all $\bar{u}=(u^1,\dots,u^l)\in B^l$.

Lemma 3.1. *There exists $r_0>0$ such that*

$$d(k*\mathcal{K},\mathcal{K}) := \min_{(v,w)\in\mathcal{K}^2} \|k*v-w\| \geq 3r_0$$

for all $k\in\mathbb{D}\setminus\{0\}$ and $||u||>2r_0$ for all $u\in K\setminus\{0\}$. Also note that $\mathcal{O}_{\mathbb{D}}(u)$ is closed for all $u\in\mathcal{K}$.

Proof. First note that $\alpha := \min_{u\in K\setminus\{0\}} ||u||^2 > 0$ because $\mathcal{K}$ is compact and $0\notin\mathcal{K}$. Take $(v,w)\in\mathcal{K}^2$; by (3.2) $\|k*v-w\|^2\leq\frac{\alpha}{2}$ only for a finite number of $k\in\mathbb{D}$. By the compactness of $\mathcal{K}$ we also have $\min_{(v,w)\in\mathcal{K}^2}\|k*v-w\|\leq\frac{\alpha}{2}$ only for a finite number of $k\in\mathbb{D}$. Since $\mathcal{K}$ is disjoint to $k*\mathcal{K}$ for all $k\neq 1_{\mathbb{D}}$ and it is compact we easily infer the result. ☐

Remark 3.1. *If the conditions in (3.1) are **not** satisfied, then the critical set consists of infinitely many points, up to the $\mathbb{D}$ symmetry and up to the symmetry with respect to any compact Lie group, therefore there is no need to prove a multiplicity theorem. The condition in (3.2) ensures that the action of $\mathbb{D}$ separates in some sense the points of B; it could be replaced with a weaker (but less natural) condition.*

Whenever a functional is periodic, hence invariant with respect to the action of $\mathbb{D}=\mathbb{Z}^n$, the Palais-Smale condition cannot hold. In the smooth setting it is often the case that a Palais-Smale sequence consists, up to a subsequence, to the sum of translated sequences each of which converges, up to translations. We call this type of Palais-Smale sequences *representable* (see the Definition below for a precise statement) and in Section 5 we prove that Palais-Smale sequences are representable even in the nonsmooth setting, when proper assumptions are taken.

Definition 3.2. *A Palais-Smale sequence* $\{u_m\} \subset B$ *at level* $c > 0$ *is said to be* representable *if there exist an integer* l, $\bar{u} = (u^1, \dots, u^l) \in \mathcal{K}^l$ *and a diverging sequence* $\{\bar{k}_m\} \subset \mathbb{D}^l$ *such that, up to a subsequence,*

$$\|u_m - \bar{k}_m * \bar{u}\| \to 0$$

and

$$\sum_{i=1}^{l} J(u^i) = c.$$

For all $k \in \mathbb{D}$ and $b > a > 0$ let

$$\mathcal{U}_a = \bigcup_{k \in \mathbb{D}} \{w \in B : d(w, k * \mathcal{K}) < a\}$$
$$\mathcal{T}_{a,b}(k) = \{w \in B : d(w, k * \mathcal{K}) \in (a, b)\},$$

where $d(v, A) = \inf_{w \in A} \|v - w\|$. In order to prove a deformation lemma, we give a lower bound for $|dJ|$ in a suitable set; for this purpose we adapt the proof of Lemma 5.1 in [4] to the present context.

Lemma 3.2. *Assume that all PS sequences at some level* $c > 0$ *are representable. The following statements hold:*

(a) $\mu_\rho := \inf\{|dJ|(u) : u \in \mathcal{U}_{r_0} \setminus \bar{\mathcal{U}}_\rho\} > 0$ *for all* $\rho \in (0, r_0)$.

(b) *If* $\{u_m\}$ *is a Palais-Smale sequence such that* $\|u_{m+1} - u_m\| < \frac{r_0}{2}$ *and* $J(u_m)$ *is bounded away from* 0 *for all* m, *then there exists* $k \in \mathbb{D}$ *such that* $d(u_m, k * \mathcal{K}) \to 0$.

Proof. (a) It follows from Lemma 3.1 that $\mathcal{U}_{r_0} \setminus \bar{\mathcal{U}}_\rho = \bigcup_{k \in \mathbb{D}} \mathcal{T}_{\rho, r_0}(k)$ and the sets $\mathcal{T}_{\rho, r_0}(k)$ are disjoint. If $\mu_\rho = 0$ for some ρ, then by the $\mathbb{D}-$invariance of the functional there exists a sequence $\{u_m\} \subset B$ such that $u_m \in \mathcal{T}_{\rho, r_0}(0)$ and $|dJ|(u_m) \to 0$. The sequence $\{u_m\}$ is bounded, therefore it is Palais-Smale and representable, so there exist l points $\bar{u} = (u^1, \dots, u^l) \in \mathcal{K}^l$ and a sequence $\{\bar{k}_m\} \subset \mathbb{D}^l$, $\bar{k}_m \to \infty$, such that, up to a subsequence

$$\|u_m - \bar{k}_m * \bar{u}\| \to 0,$$

therefore, for large m we have $\bar{k}_m * \bar{u} \in \mathcal{T}_{\frac{\rho}{2}, 2r_0}(0)$, i.e.

$$\frac{\rho}{2} < d(\bar{k}_m * \bar{u}, \mathcal{K}) < 2r_0.$$

We show that these inequalities lead to a contradiction: indeed, as $\bar{k}_m \to \infty$, up to a subsequence we have $k^i_m \to k^i \in \mathbb{D} \cup \{\infty\}$ for $m \to \infty$ and $k^i \in \mathbb{D}$ for at most one value of i. If $l = 1$ and either $k^1_m \to k^1 \neq 1_{\mathbb{D}}$ or $k^1_m \to \infty$, then $d(\bar{k}_m * \bar{u}, \mathcal{K}) = d(k^1_m * u^1, \mathcal{K}) \geq 3r_0$ for almost all m; if $l = 1$ and $k^1_m \to 1_{\mathbb{D}}$, then $d(\bar{k}_m * \bar{u}, \mathcal{K}) = d(k^1_m * u^1, \mathcal{K}) \to 0$. If $l > 1$, then choose i such that $k^i_m \to \infty$ and note that by (3.2) and Lemma 3.1 we have $\lim_{m \to \infty} d(\bar{k}_m * \bar{u}, \mathcal{K}) \geq \|u^i\| > 2r_0$.

(b) Since $\{u_m\}$ is representable, then there exist l points $(u^1, \dots, u^l) \in \mathcal{K}^l$, a sequence $\{\bar{h}_k\} \subset \mathbb{D}^l$, $\bar{h}_k \to \infty$, and a subsequence $\{u_{m_k}\}$ such that $\|u_{m_k} - \bar{h}_k * \bar{u}\| \to 0$. If $l = 1$, then (b) holds: indeed, since $\|u_{m+1} - u_m\| < \frac{r_0}{2}$, it follows from Lemma 3.1 and (a) that $d(u_m, k^1 * \mathcal{K}) < r_0$ for some k^1 and almost all m. Hence $\bar{h}_k = k^1$ for k large and $u_{m_k} \to k^1 * u^1$.

We show that $l \geq 2$ cannot occur. By (3.2) and Lemma 3.1 we have $d(\bar{h}_m * \bar{z}, \bar{h}_k * \mathcal{K}) \geq 2r_0$ for a fixed arbitrary k and large m, say $m \geq m_k$. So for all k

there exists $h \geq m_k$ such that $d(u_h, \bar{h}_k * \mathcal{K}) > r_0$. Since $\|u_{m+1} - u_m\| < \frac{r_0}{2}$ and $\|u_{m_k} - \bar{h}_k * \bar{u}\| < \frac{r_0}{2}$ whenever k is large, for such k there exists $h_k \geq m_k$ with $d(u_{h_k}, \bar{h}_k * \mathcal{K}) \in (\frac{r_0}{2}, r_0)$. But this is impossible because $\{u_{h_k}\}$ is a PS-sequence and for all sequences $\{\bar{k}_m\} \subset \mathbb{D}^l$ $(l \geq 2)$ such that $\bar{k}_m \to \infty$ as $m \to \infty$ we have

$$\liminf_{m\to\infty} \Big\{ \inf \Big\{ |dJ|(u) : d(u, \bar{k}_m * \mathcal{K}) \in \Big(\frac{r_0}{2}, r_0\Big) \Big\} \Big\} > 0\,.$$

To prove this claim by contradiction, assume that there is a sequence $\{u_m\}$ with $d(u_m, \bar{k}_m * \mathcal{K}) \in \big(\frac{r_0}{2}, r_0\big)$, $|dJ|(u_m) \to 0$ and $\bar{k}_m \to \infty$. Then there exist l' points $(u^1, \dots, u^{l'}) = \bar{u} \in \mathcal{K}^{l'}$ and a sequence $\{\bar{h}_m\}$ such that $\bar{h}_m \to \infty$ and, up to a subsequence, $\|u_m - \bar{h}_m * \bar{u}\| \to 0$. Hence for m large enough $d(\bar{h}_m * \bar{u}, \bar{k}_m * \mathcal{K}) \in \big(\frac{r_0}{4}, 2r_0\big)$. If $l \neq l'$ or $|h^i_m(k^j_m)^{-1}| \to \infty$ for some i and all j, then by (3.2) and Lemma 3.1 we have $\lim_{m\to\infty} d(\bar{h}_m * \bar{u}, \bar{k}_m * \mathcal{K}) \geq \|u^i\| > 2r_0$. So $l = l'$ and up to a subsequence $h^i_m(k^i_m)^{-1} \to a^i \in \mathbb{D}$ for each i (possibly after relabelling the i's in $\bar{k}_m$). Then again by (3.2)

$$\lim_{m\to\infty} d(\bar{h}_m * \bar{u}, \bar{k}_m * \mathcal{K})^2 = \sum_{i=1}^{l} d(a^i * u^i, \mathcal{K})^2.$$

Since the distances on the right-hand side are either 0 or exceed $2r_0$, the above limit is not in $\big(\frac{r_0}{2}, 2r_0\big)$.

So far we have proved that l is necessarily equal to 1 and consequently $\{u_m\}$ has a subsequence $u_{m_k} \to k^1 * u^1$. The same argument shows that any subsequence of $\{u_m\}$ has a subsequence converging to an element of $k^1 * \mathcal{K}$ (with the same k^1). Hence the conclusion. $\square$

Choose $0 < \delta < r_0/5$ and for all $u \in B$ let

$$\sigma(u) = \inf\{|dJ|(v) : v \in B_\delta(u)\}.$$

Lemma 3.3. *If $u \in B$ satisfies $d(u, K) > \delta$ and $\inf_{v\in B_\delta(u)} J(v) > 0$, then $\sigma(u) > 0$. If $u \in B$ satisfies $d(u, K) \leq \delta$, then $\sigma(u) = 0$.*

Proof. If $\sigma(u) = 0$ and $\inf_{v\in B_\delta(u)} J(v) > 0$, then there exists a Palais-Smale sequence $\{v_n\}$ in $B_\delta(u)$ at positive level. By Lemma 3.2(b) v_n converges to a critical point (up to a subsequence), therefore $d(u, K) \leq \delta$. On the other hand, if $d(u, K) \leq \delta$ then there exists a critical point in $B_\delta(u)$, hence $\sigma(u) = 0$. $\square$

Lemma 3.4. *For all $u \in B$ such that $d(u, K) \geq 2\delta$ there exists a continuous map $\mathcal{H}_u : B_\delta(u) \times [0, \delta/2] \to B$ such that*

$$\begin{gathered} ||\mathcal{H}_u(v, t) - v|| \leq t \\ J(\mathcal{H}_u(v, t)) \leq J(v) \\ J(\mathcal{H}_u(v, t)) \leq J(v) - \sigma(u)t \text{ for all } v \in B_{\delta/2}(u) \\ \mathcal{H}_u(v, t) = v \text{ for all } v \notin B_\delta(u). \end{gathered}$$

Proof. It follows from Theorem 2.11 in [14] by replacing δ with $\delta/2$, σ with $\sigma(u)$, setting $C = B_{\delta/2}(u)$ and recalling the definition of $\sigma(u)$. $\square$

Remark 3.2. *By the definition of weak slope, for all $u \in B$ and all $\sigma < |dJ|(u)$ there exists δ_u and a continuous map $\mathcal{H}_u : B_{\delta_u}(u) \times [0, \delta_u] \to B$ such that for all $v \in B_{\delta_u}(u)$ and all $t \in [0, \delta_u]$*

$$\begin{gathered} ||\mathcal{H}_u(v,t) - v|| \le t \\ J(\mathcal{H}_u(v,t)) \le J(v) \\ J(\mathcal{H}_u(v,t)) \le J(v) - \sigma t. \end{gathered}$$

We point out that in the above statement we can choose $\sigma = \sigma(u)$, indeed either $\sigma(u) < |dJ|(u)$ and there is nothing to prove, or $\sigma(u) = |dJ|(u)$, but in this case $|dJ|(v) \ge \sigma(u)$ for all $v \in B_\delta(u)$ and the statement follows again by Theorem 2.11 in [14].

Theorem 3.5. *There exist a continuous map $\eta : X \times [0, +\infty) \to X$ such that for all $u \in X$ and $t \in [0, +\infty)$ we have*

$$\begin{gathered} ||\eta(u,t) - u|| \le t \\ J(\eta(u,t)) \le J(u) \end{gathered}$$

Furthermore, if $u \notin \mathcal{U}_{2\delta}$, then $J(\eta(u,t)) \le J(u) - \sigma(u)$ for all $t \in [0, \delta/5]$.

Proof. For all $u \in B$ let $\mathcal{H}_u$ be the map as in Lemma 3.4 if $d(u, K) \ge 2\delta$ or as in Remark 3.2 if $d(u, K) < 2\delta$. The proof follows exactly as the proof of Theorem 2.8 in [14]. The function $\tau(u)$ must satisfy $\tau(u) = \delta/5$ for all $u \notin \mathcal{U}_{2\delta}$. □

For all $u \in B$ let $\bar\sigma(u) = \inf_{v \in B} \{\sigma(v) + ||v - u||\}$ and

$$\eta_1(u,t) = \begin{cases} \eta\left(u, \frac{t}{\bar\sigma(u)}\right) & \text{if} \quad d(u,k) \ge 2\delta \\ \eta\left(u, \left(\frac{2d(u,k)}{\delta} - 3\right)\frac{t}{\bar\sigma(u)} + \left(4 - \frac{2d(u,k)}{\delta}\right)t\right) & \text{if} \quad 3\delta/2 \le d(u,k) \le 2\delta \\ \eta(u,t) & \text{if} \quad d(u,k) \le 3\delta/2 \end{cases}$$

and let $\tau_1(u) = \frac{\delta}{5}\bar\sigma(u)$; note that $\bar\sigma$ is Lipschitz continuous, and so is τ_1. We have

$$\begin{aligned} &||\eta_1(u,t) - u|| \le \tfrac{t}{\bar\sigma(u)} \\ &t \le \tau_1(u) \Rightarrow J(\eta_1(u,t)) \le J(u) - t. \end{aligned} \tag{3.3}$$

Define recursively for $m \ge 2$

$$\begin{aligned} &\eta_m(u,t) = \begin{cases} \eta_1(\eta_{m-1}(u, \tau_{m-1}(u)), t - \tau_{m-1}(u)) & \text{if} \quad t \ge \tau_{m-1}(u) \\ \eta_{m-1}(u,t) & \text{if} \quad 0 \le t \le \tau_{m-1}(u) \end{cases} \\ &\tau_m(u) = \tau_{m-1}(u) + \tau_1(\eta_{m-1}(u, \tau_{m-1}(u))). \end{aligned} \tag{3.4}$$

We remark that the definitions of η_m and τ_m match on the boundaries. Furthermore, it is easy to check by induction that the maps η_m and τ_m are continuous. Indeed, by the induction assumption τ_m is continuous, and η_m is also continuous in (t, u) if $t \ne \tau_{m-1}(u)$. The continuity for all t follows by the following easy lemma, whose proof is omitted:

Lemma 3.6. *Let $A_1, A_2 \subset X$ be two closed sets. Let A_3 be a topological space, let $f_i : A_i \to A_3$, $i = 1, 2$, be two continuous functions satisfying $f_1(x) = f_2(x)$ for all $x \in A_1 \cap A_2$. Let $f : A_1 \cup A_2 \to C$ be defined by $f(x) = f_i(x)$ if $x \in A_i$. Then f is continuous.*

The following lemma is the crucial step in the construction of the deformation: it ensures that it is possible to carry on with the deformation η_m as long as the trajectory does not come close to a critical point.

Lemma 3.7. *If $u \in B$ satisfies $\eta_m(u,\tau_m(u)) \notin \mathcal{U}_{2\delta}$, $J(\eta_m(u,\tau_m(u)))$ is bounded away from 0 and $\inf_{v\in B_{2\delta}(\eta_m(u,\tau_m(u)))} J(v) > 0$ for all integers m, then $\lim_{m\to\infty}\tau_m(u) = +\infty$.*

Proof. Assume that $u \in B$ and $\lim \tau_m(u) = c < +\infty$. Let $u_m = \eta_m(u,\tau_m(u))$. By (3.4) if follows that $\tau_1(u_m) \to 0$ and since $u_m \notin \mathcal{U}_{2\delta}$ $\varepsilon_m = \bar\sigma(u_m) \to 0$. By the definition of $\bar\sigma$ there exists $\{w_m\}$ such that $d(u_m,w_m) \le \varepsilon_m$ and $\sigma(w_m) \le \varepsilon_m$; by the definition of σ there exists $\{v_m\}$ such that $d(v_m,w_m) \le \delta$ and $|dJ|(v_m) \le \varepsilon_m$; in particular v_m is a Palais-Smale sequence at positive level and $d(u_m,v_m) \le \delta+\varepsilon_m$. Furthermore

$$
\begin{aligned}
||u_m - u_{m-1}|| &= ||\eta_1(u_{m-1},\tau_m(u)-\tau_{m-1}(u)) - u_{m-1}|| \\
&\le \frac{\tau_m(u)-\tau_{m-1}(u)}{\bar\sigma(u_{m-1})} = \frac{\tau_1(u_{m-1})}{\bar\sigma(u_{m-1})} = \frac{\delta}{5},
\end{aligned}
$$

and, if m is large, $||v_m - v_{m-1}|| \le 2\delta + 2\varepsilon_m + \delta/5 < r_0/2$. Hence by Lemma 3.2(b) v_m converges, up to a subsequence, to a critical point of J, contradicting $u_m \notin \mathcal{U}_{2\delta}$. □

Let $\mu := \inf\{\bar\sigma(u) : u \in \mathcal{U}_{3\delta}\setminus\mathcal{U}_{2\delta}\}$.

Lemma 3.8. $\mu > 0$

Proof. If $\mu = 0$, then there exists $\{u_n\} \subset \mathcal{U}_{3\delta}\setminus\mathcal{U}_{2\delta}$ such that $\bar\sigma(u_n) \to 0$, by the definition of $\bar\sigma$ and σ, and using the same steps as in the proof of Lemma 3.7 there exists a sequence $\{v_n\} \subset \mathcal{U}_{5\delta}\setminus\mathcal{U}_{\delta/2}$ such that $|dJ|(v_n) \to 0$. This contradicts Lemma 3.2(a). □

Choose $0 < \varepsilon < \min\frac{\delta\mu}{4}$ and define $\bar\eta : B\times[0,2\varepsilon] \to B$ by

$$
\bar\eta(u,t) = \lim_{m\to\infty}\eta_m\Big(u, \min\Big\{t, \frac{\mu}{2}(d(u,K)-2\delta)\Big\}\Big).
$$

Lemma 3.9. *The function $\bar\eta$ is well-defined and continuous.*

Proof. Definition. If $u \in \mathcal{U}_{2\delta}$ there is nothing to prove because $\eta_m(u,t) \equiv u$ for all m. If $u \in B$ satisfies $\eta_m(u,\tau_m(u)) \notin \mathcal{U}_{2\delta}$ for all m, then by Lemma 3.7 there exists $\bar m$ such that $\tau_{\bar m}(u) \ge 2\varepsilon$ for all $m \ge \bar m$, hence $\bar\eta(u,t) = \eta_{\bar m}(u,t)$ and the definition is well posed. Assume instead that $u \in B$ satisfies $\eta_m(u,\tau_m(u)) \notin \mathcal{U}_{2\delta}$ for all m smaller than some integer $\bar m$, but $\eta_{\bar m}(u,\tau_{\bar m}(u)) \in \mathcal{U}_{2\delta}$. Then there exists a minimal $t_2 \le \tau_{\bar m}(u)$ such that $\eta_{\bar m}(u,t_2) \in \partial\mathcal{U}_{2\delta}$; furthermore there exists t_1 (possibly 0) such that $\eta_{\bar m}(u,t) \in \mathcal{U}_{3\delta}$ for all $t_1 \le t \le t_2$, and $t_2 - t_1 \le \min\{2\varepsilon, \frac{\mu}{2}(d(u,K)-2\delta)\}$. By (3.3) we have

$$
\begin{aligned}
||\eta_{\bar m}(t_1,u) - \eta_{\bar m}(t_2,u)|| &\le \min\Big\{2\varepsilon, \frac{\mu}{2}(d(u,K)-2\delta)\Big\}\Big(\min_{s\in[s_1,s_2]}\bar\sigma(\eta(s,u))\Big)^{-1} \\
&\le \min\Big\{\frac{\delta}{2}, \frac{d(u,K)-2\delta}{2}\Big\}
\end{aligned}
$$

which is a contradiction because either $u \notin \mathcal{U}_{3\delta}$ and then $||\eta_{\bar m}(t_1,u) - \eta_{\bar m}(t_2,u)|| \ge \delta$, or $u \in \mathcal{U}_{3\delta}$, and then $||\eta_{\bar m}(t_1,u) - \eta_{\bar m}(t_2,u)|| \ge d(u,K)-2\delta$.

Continuity. Fix (u,t), assume that $t < \frac{\mu}{2}(d(u,K)-2\delta)$ (the other case is simpler) and let $m(u,t)$ be the lowest integer satisfying $\bar\eta(u,t) = \eta_m(u,t)$. If $\tau_{m(u,t)}(u) > t$, then $\bar\eta = \eta_m$ in a neighborhood of (u,t) and the continuity of $\bar\eta$ at (u,t) follows by the continuity of η_m. If $\tau_{m(u,t)}(u) = t$, take two sequences $t^h \to t$ and $u^h \to u$;

then (possibly neglecting the first terms) either $m(u^h,t^h) = m(u,t)$ or $m(u^h,t^h) = m(u,t) - 1$. To see this, note that $\tau_1(u) > 0$ for all u satisfying $d(u,K) \geq 2\delta$, hence by the continuity of the functions τ_m and η_m we have $\tau_{m-1}(u^h) \leq t^h \leq \tau_{m+1}(u^h)$ for large values of h. If $\tau_{m-1}(u) \leq t^h \leq \tau_m(u)$, then $m(u^h,t^h) = m(u,t) - 1$, otherwise $m(u^h,t^h) = m(u,t)$. The conclusions follows by the continuity of η_m and η_{m-1}. □

Lemma 3.10. *For all $u \in J^{c+\varepsilon} \setminus \mathcal{U}_{3\delta}$ and all $t \in [0, 2\varepsilon]$ satisfying $\bar\eta(u,t) \notin \mathcal{U}_{2\delta}$ we have $J(\bar\eta(u,t)) \leq J(u) - t$ and $||\bar\eta(u,t) - u|| \leq t/\mu_t$, where $\mu_t = \min_{s\in[0,t]} \bar\sigma(\bar\eta(u,s))$.*

Proof. It follows from the definition of $\bar\eta$, (3.3) and (3.4). □

Let $c \geq \inf\{J(u) : u \in K\}$. If $\varepsilon \geq c/2$, change ε to $c/2$.

Lemma 3.11. *There exists a continuous map $f : B \to B$ such that $f(J^{c+\varepsilon} \setminus \mathcal{U}_{3\delta}) \subset J^{c-\varepsilon}$.*

Proof. For all $u \in J^{c+\varepsilon}$ let $f(u) = \bar\eta(u, 2\varepsilon)$. Let $u \in J^{c+\varepsilon} \setminus \mathcal{U}_{3\delta}$; as in Lemma 3.9 we prove that $\bar\eta(u,t) \notin \mathcal{U}_{2\delta}$ for all $t \in [0, 2\varepsilon]$. By Lemma 3.10 we have $J(f(u)) \leq c - \varepsilon$. □

Assume now that G is some compact Lie group, B is a $G-$space and J is invariant with respect to the action of G (besides being invariant with respect to the action of $\mathbb{D}$). Then it is possible to use the equivariant weak slope in all the previous proofs and obtain the same result of Lemma 3.11 with an equivariant deformation, that is the following theorem:

Theorem 3.12. *There exists a continuous $G-$map $f : B \to B$ such that $f(J^{c+\varepsilon} \setminus \mathcal{U}_{3\delta}) \subset J^{c-\varepsilon}$.*

4. Applications to quasilinear elliptic systems

In the sequel we prove multiplicity results for systems of quasilinear elliptic equations defined in $\mathbb{R}^n$ and periodic. More precisely we extend some results proved in [3] to the case where the whole system is periodic.

We consider the following system of m quasilinear elliptic equations in $\mathbb{R}^n$, $n \geq 3$:

$$-D_j(a_{ij}(x,u)D_iu_k) + \frac{1}{2}\frac{\partial a_{ij}}{\partial u_k}(x,u)D_iu_lD_ju_l + b_{jk}(x)u_j = \frac{\partial F}{\partial u_k}(x,u)$$

where $u : \mathbb{R}^n \to \mathbb{R}^m$, the indices i, j run from 1 to n, the indices k, l run from 1 to m and the assumptions on a_{ij}, b_{jk} and F are given below. In the sequel the sum over repeated indices is understood and we assume a more concise vectorial notation by setting $\nabla = \left\{\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_m}\right\}$ and denoting the scalar product in $\mathbb{R}^m$ by $\langle\cdot,\cdot\rangle$; the system takes the form

$$-D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\nabla a_{ij}(x,u)\langle D_iu, D_ju\rangle + b(x)u = \nabla F(x,u). \tag{4.1}$$

In order to prove the existence of weak solutions of (4.1) in a suitable functional space E, we look for critical points of the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \int_{\mathbb{R}^n} \frac{1}{2}a_{ij}(x,u)\langle D_iu, D_ju\rangle + \frac{1}{2}\langle b(x)u, u\rangle - F(x,u). \tag{4.2}$$

This functional is continuous, but not locally Lipschitz continuous if the coefficients a_{ij} depend on u; however, by using the nonsmooth critical point theory developed in [14, 16] it is possible to define critical points in a generalized sense: according to this theory, a critical point u of J solves (4.1) in distributional sense and, a posteriori, it is a weak solution.

In this paper we consider equation (4.1) in the case where $a_{ij}(x,u)$, $b_{jk}(x)$ and $F(x,u)$ are periodic, and therefore the system is invariant under the action of $\mathbb{Z}^n$. We prove a general existence result and a multiplicity result if the equation is also invariant under a suitable action of a compact Lie group. The problems we have to deal with, in order to prove such results, concern the lack of compactness of the functional.

The following condition (A1) is standard in this kind of problems: the matrix $[a_{ij}(x,s)]$ satisfies an ellipticity property and the matrix $[\langle s, \nabla a_{ij}(x,s)\rangle]$ is semipositive definite. More precisely:

(A1) The matrix $[a_{ij}(x,s)]$ satisfies

$$\begin{aligned} &a_{ij} \equiv a_{ji} \\ &a_{ij}(x,s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}) \\ &\nabla a_{ij}(x,s) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m) \\ &a_{ij}(x,\cdot) \in C^1(\mathbb{R}^m) \text{ for a.e. } x \in \mathbb{R}^n \\ &a_{ij}(x,s) = a_{ij}(x+k,s) \text{ for a.e. } x \in \mathbb{R}^n \text{ and all } k \in \mathbb{Z}^n \end{aligned} \tag{4.3}$$

and there exists $\nu_1 > 0$ such that for a.e. $x \in \mathbb{R}^n$, all $s \in \mathbb{R}^m$ and all $\xi \in \mathbb{R}^n$

$$a_{ij}(x,s)\xi_i\xi_j \geq \nu_1|\xi|^2 \tag{4.4}$$

and

$$\langle s, \nabla a_{ij}(x,s)\rangle \xi_i\xi_j \geq 0. \tag{4.5}$$

The assumption (4.5) is the natural extension of a semipositivity condition which is standard in this kind of quasilinear equations. In some sense it means that the ellipticity of the matrix $[a_{ij}(x,s)]$ increases for increasing values of $|s|$.

The following assumption (A2) is a control required on the growth of ellipticity of the matrix $[a_{ij}(x,s)]$ which seems to be necessary in order to have the Palais-Smale condition when $m \geq 2$.

(A2) There exist $K > 0$ and a function $\psi : [0,+\infty) \to [0,+\infty)$ continuously differentiable almost everywhere and satisfying

$$\begin{array}{ll} \text{(i)} & \psi(0) = 0 \text{ and } \lim\limits_{t\to+\infty} \psi(t) = K \\ \text{(ii)} & \psi'(t) \geq 0 \text{ for all } t \in [0,+\infty) \\ \text{(iii)} & \psi' \text{ is non-increasing} \\ \text{(iv)} & \sum\limits_{k=1}^{m} \left|\frac{\partial}{\partial s_k} a_{ij}(x,s)\xi_i\xi_j\right| \leq 2e^{-4K}\psi'(|s|)a_{ij}(x,s)\xi_i\xi_j \text{ for all } s \in \mathbb{R}^m \\ & \text{all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \mathbb{R}^n. \end{array} \tag{4.6}$$

In some sense, ψ is a measure of the growth of ellipticity of the differential operator; we assume that such growth is “not too large”. We refer to [2, 3] for more remarks about (A2).

(B) The matrix $[b_{jk}(x)]$ is symmetric, periodic ($b_{jk}(x) = b_{jk}(x+k)$ for all $k \in \mathbb{Z}^k$ and all $x \in \mathbb{R}$), $b_{jk} \in L^\infty(\mathbb{R}^n, \mathbb{R})$ for all j,k and there exists $\nu_2 > 0$ such that for a.e. $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$

$$b_{jk}(x)\xi_j\xi_k \geq \nu_2|\xi|^2. \tag{4.7}$$

(F1) The function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is measurable with respect to the first n variables and C^1 with respect to the other ones; furthermore we require

$$\begin{array}{ll} F(x,0) = 0 & \text{for a.e. } x \in \mathbb{R}^n \\ F(x,s) = F(x+k,s) & \text{for a.e. } x \in \mathbb{R}^n, \text{ all } k \in \mathbb{Z}^n \text{ and all } s \in \mathbb{R}^m \end{array}$$

(F2) There exists $p \in (2, 2^*)$ and $a > 0$ such that

$$0 < pF(x,s) \le \langle s, \nabla F(x,s)\rangle \qquad \text{for all } s \in \mathbb{R}^m \setminus \{0\} \text{ and for a.e. } x \in \mathbb{R}^n \tag{4.8}$$

and

$$|\nabla F(x,s)| \le \alpha(1 + |s|^{p-1}) \quad \text{for all } s \in \mathbb{R}^m \text{ and for a.e. } x \in \mathbb{R}^n. \tag{4.9}$$

The following assumption is standard for superlinear problems: it relates the properties of the matrix $[a_{ij}]$ with the function F and it is used to prove that Palais-Smale sequences are bounded.

(AF) There exists $\gamma \in (0, p-2)$ such that (p as in (F2))

$$\langle s, \nabla a_{ij}(x,s)\rangle \xi_i\xi_j \le \gamma a_{ij}(x,s)\xi_i\xi_j \quad \text{for a.e. } x \in \mathbb{R}^n,\ \text{for all } s \in \mathbb{R}^m \text{ and } \xi \in \mathbb{R}^n\,. \tag{4.10}$$

Theorem 4.1. *Assume (A1), (A2), (AF), (B), (F1) and (F2). Then (4.1) admits a nontrivial weak solution $u \in H$.*

Due to the periodicity of all coefficients, the system is invariant under the action of $\mathbb{Z}^n$ given by $u(x) \mapsto k * u(x) = u(x+k)$. Assume that the system is also invariant with respect to the action of a compact Lie group; in that case we prove the following:

Theorem 4.2. *Assume (A1), (A2), (AF), (B), (F1) and (F2). Assume that all coefficients in (4.1) are equivariant under the action of an admissible compact Lie group. Then (4.1) admits infinitely many geometrically distinct weak solutions in H.*

The case of a single equation is simpler, although the only symmetry that can be considered is with respect to the action of $\mathbb{Z}_2$. In this case we do not need assumption (A2) and we can prove the following:

Theorem 4.3. *Assume (A1), (AF), (B), (F1) and (F2). Assume that $m = 1$, $a_{ij}(x,s) = a_{ij}(x,-s)$ and $F(x,u) = F(x,-u)$. Then (4.1) admits infinitely many geometrically distinct weak solutions in H.*

For the definition of admissibility we refer to Section 8. For examples of nontrivial matrixes a_{ij} satisfying (A1) and (A2) and for examples of admissible representations we refer to [2, 3].

5. Palais-Smale sequences

By assumption (AF) it is possible to evaluate $J'(u)[u]$ for all $u \in H$, hence all Palais-Smale sequences satisfy $|J'(u_m)[u_m]| \le C\|u_m\|$ for some C. The following result was proved in [13] for a single equation:

Lemma 5.1. *All Palais-Smale sequences are bounded.*

Proof. Taking into account the remark above, we can compute $J(u_m)-\frac{1}{p}J'(u_m)[u_m]$, and taking into account (4.8) and (4.10) we have

$$\frac{p-2-\gamma}{2p}\int_{\mathbb{R}^n} a_{ij}(x,u_m)\langle D_iu_m, D_ju_m\rangle + \langle b(x)u_m, u_m\rangle \le C(\|u_m\|+1)$$

and the result follows by (4.4) and (4.7). □

By $\omega \subset\subset \mathbb{R}^n$ we denote an open regular bounded subset ω of $\mathbb{R}^n$.

Lemma 5.2. *Assume (A1) and (A2), let $\{u^h\}\subset H$ be a bounded sequence and set*

$$w^h = -D_j(a_{ij}(x,u^h)D_iu^h) + \frac{1}{2}\nabla a_{ij}(x,u^h)\langle D_iu^h, D_ju^h\rangle.$$

If $\{w^h\}\subset H^{-1}$ and it is strongly convergent to some w in $H^{-1}(\omega)$ for all $\omega\subset\subset\mathbb{R}^n$, then, up to a subsequence, $\{u_m\}$ converges strongly in $H^1(\omega)$ for all $\omega\subset\subset\mathbb{R}^n$.

Proof. Since $\{u^h\}$ is bounded, then $u^h \rightharpoonup u$ for some u up to a subsequence. Each component u^h_l satisfies (2.5) in [7] and since Theorem 2.1 in the same paper can be extended to unbounded domains, we infer that $D_iu^h_l \to D_iu_l$ a.e. in $\mathbb{R}^n$ for all $l=1,\dots,m$ (see also [15]). As in [3] (see Lemma 6.1) for all $v\in H\cap L^\infty$ we have

$$\int_{\mathbb{R}^n} a_{ij}(x,u)\langle D_iu, D_jv\rangle + \frac{1}{2}\int_{\mathbb{R}^n}\langle\nabla a_{ij}(x,u), v\rangle\langle D_iu, D_ju\rangle = w[v] \tag{5.1}$$

where $w[v]$ represents the duality product between $w\in H^{-1}$ and $v\in H$.

Now choose $\omega\subset\subset\mathbb{R}^n$ and a positive smooth cut-off function $\chi:\mathbb{R}^n\to[0,1]$ with compact support Ω and such that $\chi=1$ on ω. From (5.1) and by a density argument, we have

$$\int_{\mathbb{R}^n} a_{ij}(x,u)\langle D_iu, D_j(\chi u)\rangle + \frac{1}{2}\langle\nabla a_{ij}(x,u), \chi u\rangle\langle D_iu, D_ju\rangle \ = w[\chi u]$$

and by Fatou's Lemma, we get

$$\liminf_{m\to\infty}\int_{\mathbb{R}^n}\langle\nabla a_{ij}(x,u_m), \chi u_m\rangle\langle D_iu_m, D_ju_m\rangle \ge \int_{\mathbb{R}^n}\langle\nabla a_{ij}(x,u), \chi u\rangle\langle D_iu, D_ju\rangle;$$

therefore,

$$\limsup_{m\to\infty}\int_{\mathbb{R}^n} a_{ij}(x,u_m)\langle D_iu_m, D_j(\chi u_m)\rangle \le \int_{\mathbb{R}^n} a_{ij}(x,u)\langle D_iu, D_j(\chi u)\rangle. \tag{5.2}$$

This implies that $\nabla u_m \to \nabla u$ in $L^2(\omega)$, indeed by (4.4)

$$\begin{aligned}
&\int_\omega |\nabla u_m - \nabla u|^2\\
&\le \frac{1}{\nu}\int_\omega a_{ij}(x,u_m)\langle D_i(u_m-u), D_j(u_m-u)\rangle\\
&\le \frac{1}{\nu}\int_\Omega a_{ij}(x,u_m)\langle D_i(u_m-u), (D_ju_m)\chi\rangle - \frac{1}{\nu}a_{ij}(x,u_m)\langle D_i(u_m-u), (D_ju)\chi\rangle;
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\nu}\int_\Omega a_{ij}(x,u_m)\langle D_i(u_m-u), (D_ju_m)\chi\rangle = \frac{1}{\nu}\int_\Omega a_{ij}(x,u_m)(\langle D_i(\chi(u_m-u)), D_ju_m\rangle\\
- \langle (D_i\chi)(u_m-u), D_ju_m\rangle),
\end{aligned}$$

$$\int_\Omega a_{ij}(x,u_m)\langle D_i(u_m-u), (D_ju)\chi\rangle \to 0$$

because $a_{ij}(x,u_m)D_i(u_m-u) \rightharpoonup 0$ in $L^2(\Omega)$.

$$\int_\Omega a_{ij}(x,u_m)\langle (D_i\chi)(u_m-u), D_j u_m\rangle \to 0$$

because $u_m \to u$ in $L^2(\Omega)$. Hence

$$\begin{aligned}\int_\omega |\nabla u_m - \nabla u|^2 &\le \frac{1}{\nu}\int_\Omega a_{ij}(x,u_m)D_i(\chi(u_m-u))D_j u_m + o(1)\\ &\le \frac{1}{\nu}\int_\Omega a_{ij}(x,u)D_i(\chi u)D_j u - \frac{1}{\nu}a_{ij}(x,u_m)D_i(\chi u)D_j u_m + o(1)\\ &\le o(1)\end{aligned}$$

because of (5.2) and

$$a_{ij}(x,u_m)D_j u_m \rightharpoonup a_{ij}(x,u)D_j u \text{ in } L^2(\Omega)$$

for all $i = 1,\dots,n$. $\square$

The case of a single equation is much simpler and has been already treated in [13] (see Lemma 3), where the following result was proved.

Lemma 5.3. *Assume $m = 1$ and (A1), let $\{u^h\} \subset H$ be a bounded sequence and set*

$$w^h = -D_j(a_{ij}(x,u^h)D_i u^h) + \frac{1}{2}a'_{ij}(x,u^h)D_i u^h D_j u^h.$$

If $\{w^h\} \subset H^{-1}$ and it is strongly convergent to some w in $H^{-1}(\omega)$ for all $\omega \subset\subset \mathbb{R}^n$, then, up to a subsequence, $\{u_m\}$ converges strongly in $H^1(\omega)$ for all $\omega \subset\subset \mathbb{R}^n$.

The following proposition collects the previous results:

Proposition 5.4. *Assume (A1), (AF), (B), (F1), (F2) and either $m = 1$ or (A2). Let $\{u_m\}$ be a Palais-Smale sequence for J. There exists $\bar u \in H$ such that (up to a subsequence)*

- (i) *$u_m \rightharpoonup \bar u$ in H*
- (ii) *$u_m \to \bar u$ in $H^1(\omega)$ for every $\omega \subset\subset \mathbb{R}^n$*
- (iii) *$\bar u$ is a weak solution of (4.1).*

Proof. Note first that $\{u_m\}$ is bounded by Lemma 5.1 and (i) follows. To obtain (ii) it suffices to apply Lemma 5.2 or 5.3 with $\beta_m = \alpha_m + b(x)u_m + \nabla F(x,u_m) \in H^{-1}$ where $\alpha_m \to 0$ in H^{-1}: indeed, if $u_m \rightharpoonup u$ in H^{-1}, then $\beta_m \to \beta$ in $H^{-1}(\omega)$ for all $\omega \subset\subset \mathbb{R}^n$ with $\beta = b(x)u + \nabla F(x,u)$, (recall the compact embedding $H^1(\omega) \subset L^2(\omega)$ and see Theorem 2.2.7 in [11]). Finally, by (5.1) $\bar u$ is a distributional solution of (4.1) and since $D_j(a_{ij}(x,u)D_i u) + b(x)u + \nabla F(x,u) \in H^{-1}$, we also have $\nabla a_{ij}(x,u)\langle D_i u, D_j u\rangle \in H^{-1}$ and the system is solved in a weak sense. $\square$

Remark 5.1. *The functional is invariant with respect to the action of $\mathbb{Z}^n$, therefore the same results of Proposition 5.4 holds for $\{k_m * u_m\}$ whenever $\{k_m\}$ is a sequence in $\mathbb{Z}^n$ and $\{u_m\}$ is a PS sequence.*

6. The representation theorem

To study the behavior of Palais-Smale sequences we need the concentration compactness lemma [17], which we state in a form suitable for our purposes.

Lemma 6.1. *Let $\{\rho_k\}$ be a sequence of functions $\rho_k : \mathbb{R}^n \to [0,+\infty)$ such that $\int_{\mathbb{R}^n} \rho_k(x)dx \to \lambda > 0$. Then there exists a subsequence (we still denote it ρ_k) such that one of the following occurs:*

- *Concentration. There exists a sequence* $\{y_k\} \subset \mathbb{R}^n$ *such that for all* $\varepsilon > 0$ *there exist* $R > 0$ *such that* $\liminf_{k\to\infty} \int_{B_R(y_k)} \rho_k(x)dx \geq \lambda - \varepsilon$.
- *Dichotomy. There exists* $\alpha \in (0, \lambda)$ *such that for all* $\varepsilon > 0$ *there exist a sequence* $\{y_k\} \subset \mathbb{R}^n$ *and* $R > 0$ *such that for all* $R' > R$

$$\limsup_{k\to\infty} \left(\left| \int_{B_R(y_k)} \rho_k(x)dx - \alpha \right| + \left| \int_{B^c_{R'}(y_k)} \rho_k(x)dx - (\lambda - \alpha) \right| \right) \leq \varepsilon.$$

- *Vanishing. For all* $R > 0$, $\sup_{y\in\mathbb{R}^n} \int_{B_R(y)} \rho_k(x)dx \to 0$ *as* $k \to \infty$.

Let $\{u_m\}$ be a Palais-Smale sequence for J at positive level. By Lemma 5.1 $\{u_m\}$ is bounded and we may assume that $||u_m||^2 \to \lambda > 0$. We apply the concentration compactness technique on $\rho_m(x) = |\nabla u_m(x)|^2 + |u_m(x)|^2$. In order to exclude the vanishing case, we make use of the following lemma:

Lemma 6.2. *Let* $\{u_m\}$ *be a bounded sequence in* H *satisfying*

$$\lim_{m\to\infty} \sup_{x\in\mathbb{R}^n} \int_{B_r(x_m)} |u_m|^2 = 0$$

for some $r > 0$. *Then* $u_m \to 0$ *in* $L^s(\mathbb{R}^n)$ *for each* $s \in (2, \infty)$, $\int_{\mathbb{R}^n} F(x, u_m) \to 0$ *and* $\int_{\mathbb{R}^n} \nabla F(x, u_m)u_m \to 0$.

Proof. The first part is due to P.L. Lions [17, Lemma I.1], [19, Lemma 1.21].

By (4.9) for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ satisfying

$$|\nabla F(x, u)| \leq \varepsilon|u| + c_\varepsilon|u|^{p-1}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$. Hence by Hölder inequality,

$$\int_{\mathbf{R}^N} |\nabla F(x, u_m)u_m| \leq \varepsilon||u_m||^2_{L^2} + c_\varepsilon||u_m||^p_{L^p}.$$

Since ε was chosen arbitrarily, $||u_m||^2_{L^2} \leq ||u_m||^2$ and $u_m \to 0$ in L^p by the first part of the Lemma, it follows that

$$\int_{\mathbf{R}^N} \nabla F(x, u_m)u_m \to 0$$

and by a similar argument,

$$\int_{\mathbf{R}^N} F(x, u_m) \to 0.$$

□

Lemma 6.3. *Vanishing cannot occur.*

Proof. If by contradiction $\{u_m\}$ is a vanishing sequence, then taking into account Lemma 6.2 we have

$$2J(u_m) - J'(u_m)[u_m] = -\frac{1}{2} \int_{\mathbb{R}^n} \langle \nabla a_{ij}(x, u_m), u_m\rangle \langle D_i u_m, D_j u_m\rangle + o(1).$$

We conclude by noting that $2J(u_m) - J'(u_m)[u_m] \to c > 0$ and

$$\limsup -\frac{1}{2} \int_{\mathbb{R}^n} \langle \nabla a_{ij}(x, u_m), u_m\rangle \langle D_i u_m, D_j u_m\rangle + o(1) \leq 0$$

because of (4.5). □

Lemma 6.4. *If concentration holds, then there exists $\{k_m\} \subset \mathbb{Z}^n$ such that $k_m * u_m \to u \neq 0$, up to a subsequence.*

Proof. By the definition of concentration, there exists a sequence $\{y_k\} \subset \mathbb{R}^n$ such that for all $\varepsilon > 0$ there exist $R > 0$ such that $\liminf_{k\to\infty} \int_{B_R(y_k)} |\nabla u_m(x)|^2 + |u_m(x)|^2 \geq \lambda - \varepsilon$. For each m let k_m be the closest $n-$ple of integers to y_m, and let ε_m be a vanishing positive sequence. Then for all $\varepsilon > 0$ there exist $R > 0$ such that

$$\liminf_{k\to\infty} \int_{B_R(0)} |\nabla(k_m * u_m(x))|^2 + |k_m * u_m(x)|^2 \geq \lambda - \varepsilon.$$

By Lemma 5.4 $k_m * u_m \to u$ in $H^1(B_R(0))$, hence $||u|| \geq \lambda - \varepsilon$. By the arbitrariness of ε we infer $||k_m * u_m|| \to ||u||$, hence $k_m * u_m \to u$ in H. □

The dichotomy case is more delicate. Let $\{k_m\} \subset \mathbb{Z}^n$, $\{R_m\} \subset [0, +\infty)$ and $\alpha_1 \in (0, \lambda)$ be such that

$$\left| \int_{B_{R_m}} k_m * \rho_m(x)dx - \alpha_1 \right| \leq \frac{1}{m} \text{ and } \left| \int_{B^c_{2R_m}} k_m * \rho_m(x)dx - (\lambda - \alpha_1) \right| \leq \frac{1}{m} \tag{6.1}$$

(taking a subsequence if necessary).

Lemma 6.5. *If the dichotomy case occurs and $\{k_m\}$, $\{R_m\}$ and α_1 are as in (6.1), then $k_m * u_m \rightharpoonup u_1 \neq 0$ and u_1 is a weak solution of (4.1).*

Proof. By Remark 5.1 the PS sequence $k_m * u_m$ is also a Palais-Smale sequence, therefore it weakly converges to some solution of (4.1) u_1. Arguing as in Lemma 6.4 we have $||u_1|| = \alpha_1 > 0$. □

Remark 6.1. *So far we have proved that a Palais-Smale sequence yields at least a single solution of the quasilinear system; indeed vanishing cannot occur, and both concentration and dichotomy provide us with a nontrivial solution.*

If we want instead to apply the deformation technique developed in the previous sections in order to obtain a multiplicity result, we need a more detailed study of Palais-Smale sequences; more precisely we need an accurate description of the "missing norm" $(\lambda - \alpha_1)$. To this purpose we want to get rid of the part of $\{k_m * u_m\}$ which strongly converges to u. The standard technique used in this setting consists in showing that the sequence $v_m := k_m * u_m - u$ is also Palais-Smale, therefore one can start over the concentration-compactness procedure. This fails for quasilinear equations, therefore a different approach must be taken.

Let $\varphi_m : \mathbb{R}^n \to [0, 1]$ be a sequence of smooth functions satisfying $\varphi(x) = 0$ for all $x \in B_{R_m}$ and $\varphi(x) = 1$ for all $x \notin B_{2R_m}$. Let $v_m(x) = \varphi_m(x)(k_m * u_m(x))$: we have $||v_m||^2 \to \lambda - \alpha > 0$ and we can apply again the concentration compactness lemma to the sequence $\rho^1_m(x) = |\nabla v_m(x)|^2 + |v_m(x)|^2$.

Lemma 6.6. *Let $\{\rho^1_m(x)\}$ be defined as above. Vanishing cannot occur.*

Proof. If this were the case, then $\int_{\mathbb{R}^n} F(x, v_m) \to 0$ and $\int_{\mathbb{R}^n} \nabla F(x, u_m)v_m \to 0$. We have

$$2J(v_m) - J'(u_m)[v_m] = I_1 - I_2 - I_3 + o(1)$$

where

$$I_1 = \int_{\mathbb{R}^n} a_{ij}(x, v_m)\langle D_i v_m, D_j v_m\rangle,$$
$$I_2 = \int_{\mathbb{R}^n} a_{ij}(x, u_m)\langle D_i u_m, D_j v_m\rangle,$$
$$I_3 = \frac{1}{2}\int_{\mathbb{R}^n} \langle \nabla a_{ij}(x, u_m), v_m\rangle\langle D_i u_m, D_j u_m\rangle.$$

Note that $v_m(x) = 0$ for all $x \in B_{R_m}$, $v_m(x) = u_m(x)$ for all $x \notin B_{2R_m}$ and by the dichotomy definition there exists $c > 0$ such that

$$\int_{R\le|x|\le 2R} a_{ij}(x, v_m)\langle D_i v_m, D_j v_m\rangle \le \frac{c}{m}$$

and

$$\int_{R\le|x|\le 2R} a_{ij}(x, u_m)\langle D_i u_m, D_j v_m\rangle \le \frac{c}{m},$$

therefore $|I_1 - I_2| \le \frac{2c}{m}$. $I_3 \ge 0$ because of (4.5) and v_m is defined as u_m times a positive scalar function. Finally $J'(u_m)[v_m] \to 0$ because $\{v_m\}$ is bounded, therefore $J(v_m) \to c \le 0$. On the other hand

$$J(v_m) = \int_{\mathbb{R}^n} \frac{1}{2} a_{ij}(x, v_m)\langle D_i v_m, D_j v_m\rangle + o(1) \ge c||v_m||^2 + o(1),$$

therefore $v_m \to 0$ contradicting $||v_m||^2 \to \lambda - \alpha > 0$. □

Lemma 6.7. *If concentration or dichotomy holds, then $v_m \rightharpoonup v \not\equiv 0$ up to translations and subsequences and v is a weak solution.*

Proof. Assume that dichotomy holds (the concentration case is simpler) and let $\{k^1_m\} \subset \mathbb{Z}^n$ and R^1_m be such that $\left|\int_{B_{R^1_m}} k^1_m * \rho^1_m(x)dx - \alpha_2\right| \le \frac{1}{m}$. By the definition of dichotomy we have $k^1_m \to \infty$, therefore $(k_m + k^1_m) * u_m \rightharpoonup v$ and v is a weak solution by Remark 5.1. Furthermore, arguing as in Lemma 6.4 we have $||v|| = \alpha_1$. □

Since there exists $c > 0$ such that $||u|| \ge c$ for all $u \in K$, then by iterating this procedure for a finite number of times we finally end up with a sequence strongly convergent to 0. The following theorem collects all the results we obtained so far.

Theorem 6.8. *All Palais-Smale sequences $\{u_m\} \subset H$ at level $c > 0$ are* representable.

The following proposition is a trivial consequence of the definition of the action of $\mathbb{Z}^m$ on H and of diverging sequence:

Theorem 6.9. *For all $l \in \mathbb{N}$, all diverging sequences $\{\bar{k}_m\} \subset (\mathbb{Z}^m)^l$ and all $\bar{u} = (u^1, \dots, u^l) \in H^l$*

$$\lim_{m\to\infty} \|\bar{k}_m * \bar{u}\|^2 = \sum_j \|u^j\|^2.$$

7. Proofs of the theorems

We first show that the functional J has a mountain pass structure.

Proposition 7.1. (1) $J(0) = 0$

(2) *For all finite dimensional subspaces V of H there exists $R > 0$ such that $J(x) \leq 0$ for all $x \in V$, $||x|| \geq R$*
(3) *There exists $\rho > 0$ such that $J(x) > 0$ for all x, $||x|| = \rho$*

Proof. 1. Obvious.

2. Since the function F is superquadratic at $+\infty$, then for all $x \in H \setminus \{0\}$ we have $\lim_{t\to\infty} J(tx) = -\infty$; the result follows by compactness.
3. For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $0 \leq F(x,s) \leq \varepsilon s^2 + C_\varepsilon s^{2^*}$, hence, by (4.4) we have $J(u) \geq C_1\|u\|^2 - C_2\|u\|^{2^*}$. The proof follows. □

Proposition 7.2. *Choose a function $e \in H \setminus \{0\}$ such that $e \in C_c^\infty$ and $J(te) \leq 0$ for all $t \geq 1$ (such function exists by Proposition 7.1), set*

$$\Gamma = \{\gamma \in C([0,1]; H),\ \gamma(0) = 0,\ \gamma(1) = e\}$$

and

$$b = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t));$$

then $b > 0$ and J admits a Palais-Smale sequence $\{u_m\}$ at level b.

Proof. It follows by Proposition 7.1 and the nonsmooth mountain pass Lemma in [16]. □

We can now prove Theorem 4.1: by Propositions 7.2 we have a Palais-Smale sequence at positive level, and by Remark 6.1 it weakly converges to a nontrivial solution of (4.1) (up to translations and a subsequence).

In order to prove our multiplicity result Theorem 4.2 we apply the symmetric critical point theory presented developed by Bartsch. The following theorem is an adaptation of Theorem 2.25 in [5].

Theorem 7.3. *Let G be a compact Lie group and let X be a $G-$Hilbert space. Consider an admissible representation of G on $\mathbb{R}^m$, and assume that $X = \bigoplus_i X_i$ where $X_i \simeq \mathbb{R}^m$ and each X_i is isomorphic to $\mathbb{R}^m$ as a representation of G. Let $I : X \to \mathbb{R}$ be a continuous $G-$invariant functional, $I(0) = 0$; assume moreover that there exists an integer p such that*

(i) *There exist $\rho, \beta > 0$ such that $I(x) \geq \beta$ for all $x \in \partial B_\rho$.*
(ii) *For every integer k there exists $R_k > \rho$ such that $I(x) \leq 0$ for all $x \in \partial B_{R_k} \bigcap \bigoplus_{i=1}^k X_i$.*

Let

$$B_k = \left\{ x \in \bigoplus_{i=1}^k X_i : ||x|| \leq R_k \right\},$$

let Γ_k be the set of $G-$maps of the form $h : B_k \to X$ satisfying $h(x) = x$ if $||x|| = R_k$ and let

$$c_k = \inf_{h\in\Gamma_k} \sup_{x\in B_k} I(x)$$

Then $c_k \to +\infty$.

By Proposition 7.1 the assumptions of Theorem 7.3 are satisfied. We need to prove that $\{c_k\}$ is a sequence of critical values. First note that, if (3.1) does not hold, then there is nothing to prove (see also Remark 3.1). Condition (3.2) is satisfied, (see Proposition 6.9), and by Theorem 6.8 all Palais-Smale sequences are representable. Then all results of Section 3 hold, in particular Theorem 3.12. By the existence of an equivariant deformation and the definition of c_k it is a standard procedure to show that the levels c_k are critical and Theorem 4.2 is proved.

The proof of Theorem 4.3 follows exactly the same steps: only recall that the action of $\mathbb{Z}_2$ on H considered has no fixed points, therefore it is admissible (see the following section), and assumption (A2) is not required for a single equation.

8. Representation theory

For the convenience of the reader we recall here some definitions and known results from representation theory which are used in this paper; more information can be found in [5, 9, 12]. In the following, G denotes some compact Lie group.

Definition 8.1. *A compact Lie group G is* solvable *if there exists a sequence $G_0 \subset G_1 \subset \ldots \subset G_r = G$ of subgroups of G such that G_0 is a maximal torus of G, G_{i-1} is a normal subgroup of G_i and $G_i/G_{i-1} \cong \mathbb{Z}/p_i$, $1 \le i \le r$. Here the p_i's are prime numbers.*

Remark 8.1. If G is abelian, then G is isomorphic to the product of a torus with a finite abelian group [9, Corollary I.3.7]. In particular, all abelian compact Lie groups are solvable.

Definition 8.2. *A G-space is a topological space E together with a continuous action*

$$G \times E \to E \qquad (g, x) \mapsto gx$$

satisfying $(gh)x = g(hx)$ and $ex = x$ for all $g, h \in G$ and all $x \in E$, where $e \in G$ denotes the unit element.

Definition 8.3. *Let E and $\tilde{E}$ be two G-spaces. A subset B of E is said to be* invariant *if $gB \subset B$ for all $g \in G$. A functional $I : E \to \mathbb{R}$ is said to be* invariant *if $I(gx) = I(x)$ for all $g \in G$ and all $x \in E$. A map $f : E \to \tilde{E}$ is said to be* equivariant *if $f(gx) = gf(x)$ for all $g \in G$ and all $x \in E$. A continuous equivariant map between two G-spaces is called a G-map.*

Definition 8.4. *Let E be a G-space. A homotopy $\eta : [0,1] \times E \to E$ is called a G-homotopy if $\eta(t, \cdot)$ is a G-map for all $t \in [0,1]$.*

Definition 8.5. *Let E be a $G-$space provided with a Hilbert structure given by the scalar product $\langle \cdot, \cdot \rangle$. If the group action is linear and preserves the scalar product, i.e. $\langle gx, gy \rangle = \langle x, y \rangle$ for all $x, y \in E$ and all $g \in G$, then E is called a $G-$Hilbert space.*

Definition 8.6. *The space $E^G := \{x \in E : gx = x \text{ for all } g \in G\}$ is called the* fixed point space *of (the representation of) G.*

Definition 8.7. *The* orbit *of x is defined by $\mathcal{O}_G(x) := \{gx : g \in G\}$. We say that $x, y \in E$ are* geometrically distinct *if $y \notin \mathcal{O}_G(x)$.*

First we choose a representation of a compact Lie group G in H: given a representation of G in $\mathbb{R}^m$ (i.e. a finite dimensional G-space which we identify with $\mathbb{R}^m$) the natural choice is $g(u)(x) := g(u(x))$. We have to restrict our choice of representations as follows:

Definition 8.8. *Let V be a finite-dimensional G-space. V is called* admissible *if for each open, bounded and invariant neighborhood $\mathcal{U}$ of 0 in V^k $(k \geq 1)$ and each equivariant map $f : \overline{\mathcal{U}} \to V^{k-1}$, $f^{-1}(0) \cap \partial\mathcal{U} \neq \emptyset$.*

Remark 8.2. The admissibility of a representation space consists substantially in requiring that a generalized Borsuk-Ulam theorem holds. In [5, Theorem 3.7] it is proved that V is admissible if and only if there exist subgroups $K \subset H$ of G such that K is normal in H, H/K is solvable, $V^K \neq 0$ and $V^H = 0$. It follows that, if G is solvable, then any finite-dimensional representation space V with $V^G = \{0\}$ is admissible. Furthermore, all admissible representations have a trivial fixed point space.

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Gianni Arioli
Dipartimento di Scienze e T.A., C.so Borsalino 54, 15100 Alessandria Italy
E-mail address: `gianni@unipmn.it` `http://www.mfn.unipmn.it/~gianni`