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Homogenization of a nonlinear degenerate parabolic differential equation *

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Abstract

In this article, we study the homogenization of the nonlinear degenerate parabolic equation

$$\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon) - \operatorname{div} a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) = f(x,t),$$

with mixed boundary conditions(Neumann and Dirichlet) and obtain the limit equation as $\varepsilon \to 0$. We also prove corrector results to improve the weak convergence of ∇u_ε to strong convergence.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary and let $\mathrm{T} > 0$ be a constant. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where it is assumed that Γ_1 has positive Hausdorff measure, $H^{n-1}(\Gamma_1)$. We will denote $\Omega \times [0,T]$ by Ω_T, and $\Gamma_i \times [0,T]$ by $\Gamma_{i,T}$, $i = 1, 2$. We consider the following initial-boundary value problem

$$\begin{gathered}
\partial_t b(\tfrac{x}{\varepsilon}, u_\varepsilon) - \operatorname{div}\ a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) = f(x,t) \quad \text{in } \Omega_T, \\
a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).\nu = 0 \quad \text{on } \Gamma_{2,T}, \\
u_\varepsilon = g \quad \text{on } \Gamma_{1,T}, \\
u_\varepsilon(x,0) = u_0 \quad \text{in } \Omega.
\end{gathered} \tag{1.1}$$

whose diffusion term is a monotone operator. Regarding the existence, uniqueness and regularity results for the above problem, which we will refer to as (P_ε), we refer the reader to [2].

We are interested in the asymptotic behaviour of the problem (P_ε) as $\varepsilon \to 0$. The homogenization of such equations with $b(y,s) \equiv s$ or $b(y,s)$ linear in s has been studied quite widely (cf. [3, 5, 4, 7, 6, 16, 9, 15]). However, the case where b is nonlinear has not been studied so much. Recently, H. Jian (cf. [10]) studied this problem for $b(y,s)$ of the form $b(s)$, assumed to be continuous and nondecreasing in s and satisfying the monotonicity condition. It was shown, under

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an *a priori* assumption on the boundedness of the sequence u_ε in $L^\infty(\Omega_T)$, that the homogenized equation corresponding to this problem is

$$\begin{gathered} \partial_t b(u) - \operatorname{div} A(u, \nabla u) = f(x,t) \quad \text{in } \Omega_T, \\ A(u, \nabla u).\nu = 0 \quad \text{on } \Gamma_{2,T}, \\ u = g \quad \text{on } \Gamma_{1,T}, \\ u(x,0) = 0 \quad \text{in } \Omega \end{gathered} \tag{1.2}$$

for a suitable function A. That is, the solutions u_ε of the in-homogeneous problem converge in some sense to a solution u of the homogeneous problem. They first obtain a uniform bound, with respect to ε, on ∇u_ε in $L^p(\Omega_T)$ and hence on $\partial_t b(u_\varepsilon)$ and $a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon)$ in an appropriate dual space. Thus, the sequences $\partial_t b(u_\varepsilon)$, $a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon)$ each have a weak $*$ limit in that space, but to complete the analysis these limits have to be identified as $\partial_t b(u)$ and $A(u, \nabla u)$, respectively. A crucial link in showing this was the fact that $b(u_\varepsilon)$ converges strongly to $b(u)$ in some $L^q(\Omega_T)$ and this in turn was used to prove the strong convergence of u_ε to u in some $L^r(\Omega_T)$ (note that we cannot conclude the strong convergence of u_ε to u from the uniform bound on the sequence ∇u_ε in $L^p(\Omega_T)$ because the time derivative is not involved, but this information is hidden in the boundedness of $\partial_t b(u_\varepsilon)$). This is then used to identify the limits of the sequences $\partial_t b(u_\varepsilon)$ and $a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon)$.

However, for the class of problems that we consider, $b(\frac{x}{\varepsilon}, u_\varepsilon)$ can be expected to have only a weak limit in any $L^q(\Omega_T)$. This does not help in proving the strong convergence of u_ε to a u in any $L^r(\Omega_T)$, which is crucially needed for identifying the weak limits of the sequences $\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon)$ and $a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon)$. However, we are able to prove directly that $u_\varepsilon \to u$ in some $L^r(\Omega_T)$ by adapting a technique found in [2]. From this we can prove that $\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon)$ has as its weak limit $\partial_t \overline{b}(u)$. Here, $\overline{b}(s)$ denotes the average of $b(y,s)$ in the variable y in the unit cell $Y = [0,1]^n$. Interestingly, we show that $b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u)$ strongly converges to 0 in any $L^q(\Omega_T)$, $0 < q < \infty$, which yields the strong convergence of $b(u_\varepsilon)$ to $b(u)$ when b is independent of the variable y. The diffusion term in the homogenized problem is the same as in [10], viz. $\operatorname{div} A(u, \nabla u)$ (cf. Theorem (2.3)), but we identify this using the method of two-scale convergence. We also use the two-scale convergence method to prove the corrector results.

We prove corrector results under the strong monotonicity assumption on a which in turn, yields a corrector result for the work of H. Jian. That is, we construct suitable strong approximations for ∇u_ε.

The layout of the paper is as follows. In Section 2, we give the weak formulation for the problem (P_ε). Then, we state our main results viz. Theorem 2.3 and Theorem 2.5. In Section 3, we prepare the ground for homogenization by obtaining some *a priori* estimates and by proving the strong convergence of u_ε to some u (for a subsequence) in some $L^r(\Omega_T)$. In Section 4, we prove our main theorems.

2 Assumptions and Main Results

For $p > 1$, p^* will denote the conjugate exponent $p/(p-1)$. Let V be the space, $\{v \in W^{1,p}(\Omega) : v = 0 \text{ on } \Gamma_1\}$ and let V^* be the dual of V. Let $E = L^p(0,T;V)$ and let $W^{1,p}_{per}(Y)$ be the space of elements of $W^{1,p}(Y)$ having the same trace on opposite faces of Y. We say, $u_\varepsilon \in g + E$ is a weak solution of the problem (P_ε) if it satisfies:

$$b(\frac{x}{\varepsilon}, u_\varepsilon) \in L^\infty(0,T;L^1(\Omega)), \partial_t b(\frac{x}{\varepsilon}, u_\varepsilon) \in L^{p^*}(0,T;V^*)\,, \tag{2.1}$$

that is

$$\int_0^T < \partial_t b(\frac{x}{\varepsilon}, u_\varepsilon), \xi(x,t) > \,dt + \int_{\Omega_T} (b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u_0))\partial_t \xi \,dx\,dt = 0 \tag{2.2}$$

for all $\xi \in E \cap W^{1,1}(0,T;L^\infty(\Omega))$ with $\xi(T) = 0$ and

$$\int_0^T < \partial_t b(\frac{x}{\varepsilon}, u_\varepsilon), \xi(x,t) > \,dt + \int_{\Omega_T} a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).\nabla\xi(x,t)\,dx\,dt$$
$$= \int_{\Omega_T} f(x,t)\xi(x,t)\,dx\,dt \tag{2.3}$$

for all $\xi \in E$.

For the existence of a solution for the weak formulation we make the following assumptions (cf. [2]).

(A1) The function b(y,s) is continuous in y and s, Y-periodic in y and non-decreasing in s and $b(y,0) = 0$.

(A2) There exists a constant $\theta > 0$ such that for every δ and R with $0 < \delta < R$, there exists $C(\delta,R) > 0$ such that

$$|b(y,s_1) - b(y,s_2)| > C(\delta,M)|s_1 - s_2|^\theta \tag{2.4}$$

for all $y \in Y$ and $s_1, s_2 \in [-R,R]$ with $\delta < |s_1|$.

(A3) The mapping $(y,s,\mu,\lambda) \mapsto a(y,s,\mu,\lambda)$ defined from $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ is measurable in (y,s) and continuous in (μ,λ). Further, it is assumed that there exists three positive constants α, r, τ_0 so that for all (y,s,μ,λ) and all $\mu_1,\mu_2 \in \mathbb{R}$ and $\lambda,\lambda_1,\lambda_2 \in \mathbb{R}^n$ one has,

$$a(y,s,\mu,\lambda)(\lambda) \geq \alpha|\lambda|^p \tag{2.5}$$
$$(a(y,s,\mu,\lambda_1) - a(y,s,\mu,\lambda_2))(\lambda_1 - \lambda_2) > 0, \quad \forall\,\lambda_1 \neq \lambda_2 \tag{2.6}$$
$$|a(y,s,\mu,\lambda)| \leq \alpha^{-1}(1 + |\mu|^{p-1} + |\lambda|^{p-1}) \tag{2.7}$$
$$|a(y,s,\mu_1,\lambda) - a(y,s,\mu_2,\lambda)$$
$$\leq \alpha^{-1}|\mu_1 - \mu_2|^r(1 + |\mu_1|^{p-1-r} + |\mu_2|^{p-1-r} + |\lambda|^{p-1-r}) \tag{2.8}$$

Also it is assumed that $a(y,s,\mu,\lambda)$ is $Y - \tau_0$ periodic in (y,s) for all (μ,λ).

(A4) Assume $g \in L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(\Omega_T)$, $\partial_t g \in L^1(0,T;L^\infty(\Omega))$, $u_0 \in L^\infty(\Omega)$, and $f \in L^{p^*}(\Omega_T)$.

(A5) For all $y, s, \mu, \lambda_1, \lambda_2$,

$$(a(y,s,\mu,\lambda_1) - a(y,s,\mu,\lambda_2))(\lambda_1 - \lambda_2) \geq \alpha|\lambda_1 - \lambda_2|^p. \tag{2.9}$$

Remark 2.1 It is to be noted that (A5) implies the conditions (2.5) and (2.6) in (A3), which alone are sufficient to guarantee the existence of a solution to the weak formulation of (P_ε) and for its homogenization. (A5) will be used only in proving the corrector result.

Remark 2.2 The prototype for b is a function of the form $c(y)|s|^k \operatorname{sgn}(s)$ for some positive real number k and continuous and Y-periodic function, $c(.)$, which is non-vanishing on Y.

We now state our main theorems.

Theorem 2.3 *Let u_ε be a family of solutions of (P_ε). Assume that there is a constant $C > 0$, such that*

$$\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega_T)} \leq C \tag{2.10}$$

Under, the assumptions (A1)-(A4), there exists a subsequence of ε, still denoted by ε, such that for all q with $0 < q < \infty$, we have,

$$\begin{aligned} u_\varepsilon &\rightarrow u \text{ strongly in } L^q(\Omega_T) & (2.11)\\ \nabla u_\varepsilon &\rightharpoonup \nabla u \text{ weakly in } L^p(\Omega_T) & (2.12)\\ b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u) &\rightarrow 0 \text{ strongly in } L^q(\Omega_T) & (2.13)\\ b(\frac{x}{\varepsilon}, u_\varepsilon) &\rightarrow \bar{b}(u) \text{ weakly in } L^q(\Omega_T) \text{ for } q > 1, & (2.14) \end{aligned}$$

and u solves,

$$\begin{gathered} \partial_t \bar{b}(u) - \operatorname{div} A(u, \nabla u) = f \quad \text{in } \Omega_T, \\ A(u, \nabla u).\nu = 0 \quad \text{on } \Gamma_{2,T}, \\ u = g \quad \text{on } \Gamma_{1,T}, \\ u(x,0) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{2.15}$$

where $\bar{b}$ and A are defined below by (2.16)-(2.17).

Remark 2.4 Of course, the assumption (2.10) is true in special cases (see [12] Ch. 5) and it is reasonable on physical grounds (see [10]).

The functions $\bar{b}$ and A are defined by

$$\bar{b}(s) = \int_Y b(y,s)\,dy \tag{2.16}$$

$$A(\mu,\lambda) = \frac{1}{\tau_0}\int_0^{\tau_0}\int_Y a(y,s,\mu,\lambda+\nabla\Phi_{\mu,\lambda}(y,s))\,dy\,ds \tag{2.17}$$

where $\Phi_{\mu,\lambda} \in L^p(0,\tau_0;W^{1,p}_{per}(Y))$ solves the periodic boundary value problem

$$\int_0^{\tau_0}\int_Y a(y,s,\mu,\lambda+\nabla\Phi_{\mu,\lambda}(y,s)).\nabla\phi(y,s)\,dy\,ds = 0 \tag{2.18}$$

for all $\phi \in L^p(0,\tau_0;W^{1,p}_{per}(Y))$. For the existence of solutions to (2.18), we refer the reader to Corollary 1.8, Ch. 3 of [11]. It can be shown that $A(\mu,\lambda)$: $\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies

$$|A(\mu,\lambda)| \le \beta^{-1}(1+|\mu|^{p-1}+|\lambda|^{p-1}) \tag{2.19}$$

$$(A(\mu,\lambda_1)-A(\mu,\lambda_2)).(\lambda_1-\lambda_2) > 0,\ \forall\lambda_1\neq\lambda_2, \tag{2.20}$$

$$A(\mu,\lambda).\lambda \ge \beta|\lambda|^p \tag{2.21}$$

for a positive constant β which depends only on α, n, p, τ_0(cf. Lemmas 2.4-2.6 in [8]).

Note that in (2.12) we only have weak convergence of ∇u_ε in L^p. We construct some correctors for ∇u_ε which will improve the weak convergence (2.12) to strong convergence. Such results are known as corrector results in the literature of homogenization and are very useful in numerical computations. Let $u(x,t)$ be as in Theorem 2.3 and let $U_1 \in L^p(\Omega_T\times(0,\tau_0);W^{1,p}_{per}(Y))$ be the solution of the variational problem,

$$\int_{\Omega_T}\int_Y\int_0^{\tau_0} a(y,s,u,\nabla_x u+\nabla_y U_1(x,t,y,s)).\nabla_y\Psi(x,t,y,s) = 0, \tag{2.22}$$

for all $\Psi \in L^p(\Omega_T\times(0,\tau_0);W^{1,p}_{per}(Y))$. It will be seen that there is such a function U_1. The statement of the corrector result is as follows.

Theorem 2.5 *Let u_ε, u be as in Theorem 2.3 and let U_1 be as defined above. We assume all the assumptions in Theorem 2.3 and furthermore, the strong monotonicity assumption (A5). Then, if u, U_1 are sufficiently smooth, i.e. belong to $C^1(\Omega_T)$ and $C(\Omega_T;C_{per}(0,\tau_0)\times C^1_{per}(Y))$, then*

$$u_\varepsilon - u - \varepsilon U_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}) \ \to\ 0 \text{ strongly in } L^p(\Omega_T) \text{ and,} \tag{2.23}$$

$$\nabla u_\varepsilon - \nabla u - \nabla_y U_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}) \ \to\ 0 \text{ strongly in } L^p(\Omega_T). \tag{2.24}$$

Remark 2.6 Note that we are not claiming $u_\varepsilon - u - \varepsilon U_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}) \to 0$ strongly in $L^p(0,T;W^{1,p}(\Omega))$ as we do not have the full gradient of U_1 with respect to x in (2.24).

3 Preliminaries

We first obtain *a priori* bounds under the assumption (2.10). From now on, C will denote a generic positive constant which is independent of ε.

Lemma 3.1 *Let u_ε be a family of solutions of* (P_ε) *and assume that (2.10) holds. Then,*

$$\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^p(\Omega_T)} \leq C \tag{3.1}$$

$$\sup_\varepsilon \|a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon)\|_{L^{p^*}(\Omega_T)} \leq C \tag{3.2}$$

$$\sup_\varepsilon \|\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon)\|_{E^*} \leq C \tag{3.3}$$

Proof: Define the function $B(.,.) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$B(y, s) = b(y, s)s - \int_0^s b(y, \tau)\, d\tau \tag{3.4}$$

The following identity can be deduced as in Lemma 1.5 of Alt and Luckhaus [2].

$$\begin{aligned}
&\int_\Omega (B(\frac{x}{\varepsilon}, u_\varepsilon(x, T)) - B(\frac{x}{\varepsilon}, u_0))\, dx \\
&= \int_0^T \langle \partial_t b(\frac{x}{\varepsilon}, u_\varepsilon), (u_\varepsilon - g)\rangle dt - \int_{\Omega_T} (b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u_0))\, \partial_t g\, dx\, dt \\
&\quad + \int_\Omega (b(\frac{x}{\varepsilon}, u_\varepsilon(T)) - b(\frac{x}{\varepsilon}, u_0))g(T)\, dx
\end{aligned} \tag{3.5}$$

Therefore, using (2.3) we obtain,

$$\begin{aligned}
&\int_\Omega B(\frac{x}{\varepsilon}, u_\varepsilon(x, T))\, dx + \int_{\Omega_T} a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).\nabla u_\varepsilon\, dx\, dt \\
&= \int_\Omega B(\frac{x}{\varepsilon}, u_0)\, dx + \int_{\Omega_T} a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).\nabla g\, dx\, dt \\
&\quad + \int_{\Omega_T} f(x, t)(u_\varepsilon - g)\, dx\, dt - \int_{\Omega_T} (b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u_0))\, \partial_t g\, dx\, dt \\
&\quad + \int_\Omega (b(\frac{x}{\varepsilon}, u_\varepsilon(T)) - b(\frac{x}{\varepsilon}, u_0))g(T)\, dx.
\end{aligned} \tag{3.6}$$

Notice that due to (2.10), (A1), (A3) and (A4) we obtain

$$\begin{aligned}
&\int_\Omega B(\frac{x}{\varepsilon}, u_\varepsilon(x, T))\, dx + \int_{\Omega_T} a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).\nabla u_\varepsilon\, dx\, dt \\
&\qquad \leq C + C\|\nabla u_\varepsilon\|_{p,\Omega_T}^{p-1} \|g\|_{p,\Omega_T}
\end{aligned} \tag{3.7}$$

Therefore, as B is non-negative from its definition, we get using (A3) again that,

$$\alpha \|\nabla u_\varepsilon\|_{p,\Omega_T}^p \leq C + C\|\nabla u_\varepsilon\|_{p,\Omega_T}^{p-1} \tag{3.8}$$

for all ε. This implies (3.1). Then, (3.2) follows from (3.1) and (A3), while (3.3) follows from (3.1), (3.2) and the weak formulation (2.3). ♢

As a consequence of (2.10) and the above lemma, we immediately conclude that, for a subsequence of ε(to be denoted by ε again),

$$u_\varepsilon \rightharpoonup u \text{ weakly * in } L^\infty(\Omega_T), \tag{3.9}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega_T) \tag{3.10}$$

$$b(\frac{x}{\varepsilon}, u_\varepsilon) \rightharpoonup b^* \text{ weakly * in } L^\infty(\Omega_T) \tag{3.11}$$

$$\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon) \rightharpoonup w \text{ weakly * in } E^*, \tag{3.12}$$

$$a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup A^*(x,t) \text{ weakly in } L^{p^*}(\Omega_T) \tag{3.13}$$

for some $b^* \in L^\infty(\Omega_T)$, $w \in E^*$ and $A^* \in L^{p^*}(\Omega_T)$. The task is to identify these quantities and obtain the limit equation. We now prove that, for a subsequence, u_ε converges *a.e.* to u in Ω_T and this will form a crucial part of the present analysis. This will be found useful in identifying b^*, w and A^*.

Lemma 3.2 *There exists a continuous, increasing function ω on $\mathbb{R}^+$ with $\omega(0) = 0$, such that, given any $C > 0$, $\delta > 0$, if v_1, v_2 are any two functions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\|v_i\|_{\infty,\Omega} \le C$, $i = 1, 2$, satisfying*

$$\int_\Omega (b(\frac{x}{\varepsilon}, v_1) - b(\frac{x}{\varepsilon}, v_2))(v_1 - v_2)\, dx \le \delta \quad \forall \varepsilon > 0$$

then

$$\int_\Omega |b(\frac{x}{\varepsilon}, v_1) - b(\frac{x}{\varepsilon}, v_2)|\, dx \le \omega(\delta) \quad \forall \varepsilon > 0.$$

Proof: By the *a priori* bounds for v_1, v_2 in L^∞, we can restrict b to the domain $Y \times [-C, C]$, where it is uniformly continuous. Now,

$$\begin{aligned}
&\int_\Omega |b(\frac{x}{\varepsilon}, v_1) - b(\frac{x}{\varepsilon}, v_2)|\, dx \\
&= \int_{\Omega \cap \{|v_1 - v_2| < \delta^{\frac{1}{2}}\}} |b(\frac{x}{\varepsilon}, v_1) - b(\frac{x}{\varepsilon}, v_2)|\, dx \\
&\quad + \int_{\Omega \cap \{|v_1 - v_2| \ge \delta^{\frac{1}{2}}\}} |b(\frac{x}{\varepsilon}, v_1) - b(\frac{x}{\varepsilon}, v_2)|\, dx \\
&\le \omega_b(\delta^{\frac{1}{2}}) m(\Omega) + \delta^{-\frac{1}{2}} \int_\Omega (b(\frac{x}{\varepsilon}, v_1) - b(\frac{x}{\varepsilon}, v_2))(v_1 - v_2)\, dx \\
&\le \omega_b(\delta^{\frac{1}{2}}) m(\Omega) + \delta^{\frac{1}{2}}
\end{aligned}$$

where ω_b is the modulus of continuity function for b. Thus, we obtain the lemma by taking $\omega(t) \doteq \omega_b(t^{\frac{1}{2}}) m(\Omega) + t^{\frac{1}{2}}$. ♢

Lemma 3.3 *Let u_ε be the solution of (1.1). Then, the sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ is relatively compact in $L^\theta(\Omega_T)$, where θ is as in (A2). As a result, there is a subsequence of u_ε such that,*

$$u_\varepsilon \to u \text{ a. e. in } \Omega_T \tag{3.14}$$

Proof: *Step 1*: Using the arguments from [10], it can be shown that

$$h^{-1}\int_0^{T-h}\int_\Omega (b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t)))(u_\varepsilon(t+h)-u_\varepsilon(t))\,dx\,dt\le C$$

for some constant C which is independent of ε and h.
Step 2: We show that

$$\int_0^{T-h}\int_\Omega |b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t))|\,dx\,dt\to 0$$

as $h\to 0$, uniformly with respect to ε. Set, for $R>0$ and large,

$$\begin{aligned}E_{\varepsilon,R} &= \{t\in(0,T-h): \|u_\varepsilon(t+h)\|_{W^{1,p}(\Omega)}+\|u_\varepsilon(t)\|_{W^{1,p}(\Omega)}+\|g\|_{W^{1,p}(\Omega)}\\ &\quad +h^{-1}\int_\Omega (b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t))).(u_\varepsilon(t+h)-u_\varepsilon(t))\,dx>R\}\end{aligned}$$

From the estimate in Step 1, it follows that $m(E_{\varepsilon,R})\le C/R$. Set $E'_{\varepsilon,R}$ to be the complement of $E_{\varepsilon,R}$ in $(0,T-h)$. Hence, for $t\in E'_{\varepsilon,R}$, by Lemma 3.2, we have

$$\int_\Omega |b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t))|\,dx<\omega(hR). \tag{3.15}$$

Therefore,

$$\begin{aligned}&\int_0^{T-h}\int_\Omega |b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t))|\\ &= \int_{E_{\varepsilon,R}}\int_\Omega |b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t))|\\ &\quad +\int_{E'_{\varepsilon,R}}\int_\Omega |b(\frac{x}{\varepsilon},u_\varepsilon(t+h))-b(\frac{x}{\varepsilon},u_\varepsilon(t))|\\ &\le C/R+T\,\omega(hR)\end{aligned}$$

for all ε,R and h. Now, choose R large, fixed so that C/R is as small as we please and then let $h\to 0$ to complete the proof of Step 2.
Step 3: By assumption (A2), it follows from Step 2 that

$$\int_0^{T-h}\int_\Omega |u_\varepsilon(t+h)-u_\varepsilon(t)|^\theta\,dx\,dt\to 0 \text{ as } h\to 0 \tag{3.16}$$

uniformly with respect to ε.
Step 4: In this crucial step, we demonstrate the relative compactness of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^\theta(\Omega_T)$. This is an argument to reduce it to the time independent case. Set,

$$v_\varepsilon(x,t)=\begin{cases} u_\varepsilon(x,t) & \text{if } t\in(0,T-h)\setminus E_{\varepsilon,R}\\ 0 & \text{if } t\in E_{\varepsilon,R}\cup[T-h,T]\end{cases} \tag{3.17}$$

Choose, h so that T is an integral multiple of h. We have,

$$\begin{aligned}
&\frac{1}{h}\int_0^h\int_0^T\int_\Omega |u_\varepsilon(t)-\sum_{i=1}^{T/h}\chi_{((i-1)h,ih)}v_\varepsilon((i-1)h+s)|^\theta\,dx\,dt\,ds\\
&= \frac{1}{h}\sum_{i=1}^{T/h}\int_{(i-1)h}^{ih}\int_{(i-1)h}^{ih}\int_\Omega |u_\varepsilon(t)-v_\varepsilon(s)|^\theta\,dx\,dt\,ds\\
&\le \frac{1}{h}\int_{-h}^{h}\int_{max(0,-s)}^{min(T,T-s)}\int_\Omega |u_\varepsilon(t)-v_\varepsilon(s+t)|^\theta\,dx\,dt\,ds\\
&\le Sup_{|s|\le h}\int_{max(0,-s)}^{min(T,T-s)}\int_\Omega |u_\varepsilon(t)-u_\varepsilon(s+t)|^\theta\,dx\,dt\\
&\quad +\int_{E_{\varepsilon,R}\cup(T-h,T)}\int_\Omega |u_\varepsilon(t)|^\theta\,dx\,dt\\
&\le T\,w(hR)+C/R
\end{aligned}$$

which can be taken small, say less than μ (for all ε), by fixing h small and $R=h^{-\frac{1}{2}}$. Therefore, there exists $s_\varepsilon\in(0,h)$ such that

$$\int_{\Omega_T}|u_\varepsilon(t)-\sum_{i=1}^{T/h}\chi_{((i-1)h,ih)}v_\varepsilon((i-1)h+s_\varepsilon)|^\theta\,dx\,dt$$

is small uniformly in ε. Note that the sequences $\{v_\varepsilon((i-1)h+s_\varepsilon)\}_{\varepsilon>0}$ are independent of time. Therefore, it is enough to show that $\{v_\varepsilon((i-h)+s_\varepsilon)\}_{\varepsilon>0}$ are relatively compact sequences in $L^\theta(\Omega_T)$ for $i=1,...,T/h$. But, this follows from the compact inclusion of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ as these sequences are bounded in $W^{1,p}(\Omega)$ (by the definition of $E_{\varepsilon,R}$) for each i. $\diamondsuit$

We end the section by recalling a fact which is quite useful in periodic homogenization. Let f be a function in $L^q_{loc}(\mathbb{R}^n;C_{per}(Y))$. Then we have the following lemma.

Lemma 3.4 *The oscillatory function $f(\frac{x}{\varepsilon},x)$ converges weakly in $L^q_{loc}(\mathbb{R}^n)$ to $\int_Y f(y,x)\,dy$, for all $q>1$.*

4 Homogenization and Correctors

First, we prove (2.11), (2.13) and (2.14) using Lemma 3.3 and Lemma 3.4. Then, we identify b^* and A^* given by (3.13). Finally, we prove that u satisfies the homogenized equation (2.15).

By the *a priori* bound (2.10) and (3.14), it follows by the Lebesgue dominated convergence theorem that

$$u_\varepsilon\to u \text{ strongly in } L^q(\Omega_T)\,, \tag{4.1}$$

for all q with $0<q<\infty$. Thus, we have shown (2.11) and we have the following proposition.

Proposition 4.1 *We have,*

$$b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u) \to 0 \text{ strongly in } L^q(\Omega_T)\ \forall\, q,\ 0 < q < \infty.$$

Proof: By the *a priori* bound (2.10), it is enough to consider the function b on $Y \times [-M, M]$ for a large $M > 0$. As b is continuous, it is uniformly continuous on $Y \times [-M, M]$. Therefore, given $h_0 > 0$, there exists a $\delta > 0$ such that,

$$|b(y, s) - b(y', s')| < h_0,$$

whenever $|y - y'| + |s - s'| < \delta$.

Now, since $u_\varepsilon \to u$ a.e in Ω_T, by Egoroff's theorem, given $h_1 > 0$, there exists $E \subset \Omega_T$ such that its Lebesgue measure $m(E) < h_1$ and u_ε converges uniformly to u on $\Omega_T \setminus E \equiv E'$. Therefore, we can find $\varepsilon_1 > 0$ such that

$$\|u_\varepsilon - u\|_{\infty, E'} < \delta\ \forall \varepsilon < \varepsilon_1. \tag{4.2}$$

Therefore, for $\varepsilon < \varepsilon_1$ we have,

$$\begin{aligned}
&\int_{\Omega_T} |b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u)|^q\, dx\, dt \\
&= \int_{E'} |b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u)|^q\, dx\, dt + \int_E |(b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u))|^q\, dx\, dt \\
&\le h_0^q m(\Omega_T) + 2^q \sup(|b|^q)\, m(E) \\
&\le h_0^q\, m(\Omega_T) + 2^q \sup(|b|^q)\, h_1.
\end{aligned}$$

This completes the proof as h_0 and h_1 can be chosen arbitrarily small. ♢

Corollary 4.2 *If* $b(\frac{x}{\varepsilon}, u_\varepsilon) = b(u_\varepsilon)$*, then the above proposition shows that* $b(u_\varepsilon) \to b(u)$ *strongly in* $L^q(\Omega_T)$*, the result of Jian [10].*

Corollary 4.3 *We have the following convergences:*

(i) $b(\frac{x}{\varepsilon}, u_\varepsilon)$ *converges to* $\overline{b}(u)$ *weakly in* $L^q(\Omega_T)$ *for any* $q \in (1, \infty)$ *and hence* $b^* = \overline{b}(u)$.

(ii) $\partial_t b(\frac{x}{\varepsilon}, u_\varepsilon) \rightharpoonup \partial_t \overline{b}(u)$ *weakly * in* E^* *and thus* $w = \partial_t \overline{b}(u)$.

Proof: (i) We can write, $b(\frac{x}{\varepsilon}, u_\varepsilon) = (b(\frac{x}{\varepsilon}, u_\varepsilon) - b(\frac{x}{\varepsilon}, u)) + b(\frac{x}{\varepsilon}, u)$. The result now follows from Proposition 4.1 and Lemma 3.4 and (ii) follows from (i) and (3.12). ♢

Finally, we have to show that $A^* = A(u, \nabla u)$, which can be proved in a manner similar to that in [10]. We present a different proof of this using the method of *two-scale convergence*. Besides, some steps of the proof will be used in proving the corrector result. First, we recall the definition and main results concerning the method of two-scale convergence (cf. [1, 13, 14]). We set the period τ_0 in the time variable to be 1, for convenience of notation.

Definition 4.4 Let $1 < q < \infty$. A sequence of functions $v_\varepsilon \in L^q(\Omega_T)$ is said to two-scale converge to a function $v \in L^q(\Omega_T \times Y \times (0,1))$ if

$$\int_{\Omega_T} v_\varepsilon\, \psi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})\,dx\,dt \quad \stackrel{\varepsilon\to 0}{\rightarrow} \quad \int_{\Omega_T}\int_0^1\int_Y v(x,t,y,s)\,\psi(x,t,y,s)\,dy\,ds\,dx\,dt$$

for all $\psi \in L^{q^*}(\Omega_T; C_{per}(Y \times (0,1))$. We write $v_\varepsilon \stackrel{2-s}{\rightarrow} v$.

Remark 4.5 From the definition of two-scale convergence, it is easy to see that if v_ε is a sequence of functions in $L^q(\Omega_T)$ such that $v_\varepsilon \stackrel{2-s}{\rightarrow} v(x,t,y,s)$, then $v_\varepsilon \rightharpoonup \int_0^1\int_Y v(x,t,y,s)\,dy\,ds$ weakly in $L^q(\Omega_T)$.

The following facts about two-scale convergence [1] will be used by us.

Theorem 4.6 *If v_ε is a bounded sequence in $L^q(\Omega_T)$, then there exists a function $v \in L^q(\Omega_T \times Y \times (0,1))$ such that, up to a subsequence, $v_\varepsilon \stackrel{2-s}{\rightarrow} v(x,t,y,s)$.*

Theorem 4.7 *If $v_\varepsilon, \nabla v_\varepsilon$ are bounded sequences in $L^q(\Omega_T)$, then there exist $v \in L^q((0,T)\times(0,1); W^{1,q}(\Omega))$ and $V_1 \in L^q(\Omega_T \times (0,1); W^{1,q}_{per}(Y))$ such that, up to a subsequence,*

$$\begin{aligned} v_\varepsilon &\stackrel{2-s}{\rightarrow} v(x,t,s)\,, \\ \nabla v_\varepsilon &\stackrel{2-s}{\rightarrow} \nabla_x v(x,t,s) + \nabla_y V_1(x,t,y,s). \end{aligned}$$

The following theorem [1] is useful in obtaining the limit of the product of two two-scale convergent sequences. Let $1 < q < \infty$.

Theorem 4.8 *Let v_ε be a sequence in $L^q(\Omega_T)$ and w_ε be a sequence in $L^{q^*}(\Omega_T)$ such that $v_\varepsilon \stackrel{2-s}{\rightarrow} v$ and $w_\varepsilon \stackrel{2-s}{\rightarrow} w$. Further, assume that the sequence w_ε satisfies*

$$\int_{\Omega_T} |w_\varepsilon|^{q^*}(x,t)\,dx\,dt \quad \stackrel{\varepsilon\to 0}{\rightarrow} \quad \int_{\Omega_T}\int_0^1\int_Y |w(x,t,y,s)|^{q^*}\,dy\,ds\,dx\,dt. \tag{4.3}$$

Then,

$$\int_{\Omega_T} v_\varepsilon w_\varepsilon\,dx\,dt \quad \stackrel{\varepsilon\to 0}{\rightarrow} \quad \int_{\Omega_T}\int_0^1\int_Y v(x,t,y,s)w(x,t,y,s)\,dy\,ds\,dx\,dt.$$

Definition 4.9 A sequence w_ε which two-scale converges and satisfies (4.3) is said to be strongly two-scale convergent.

Remark 4.10 An example of a strongly two-scale convergent sequence is the sequence $\psi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ for any $\psi \in L^q_{per}(Y \times (0,1); C(\Omega_T))$.

We will now identify the homogenized problem corresponding to (P_ε) using the two-scale convergence method. To avoid the technicalities, we assume that the Dirichlet boundary data $g = 0$.

Recalling that the solutions u_ε of the problem (P_ε) converges to u strongly in $L^p(\Omega_T)$ and observing that we have (3.1) and (3.2), we conclude using Theorem 4.7 and Theorem 4.6 that

Proposition 4.11 *There exist functions* $U_1 \in L^p(\Omega_T \times (0,1); W^{1,p}_{per}(Y))$ *and* $a_0 \in L^{p^*}(\Omega_T \times Y \times (0,1))$ *such that, up to a subsequence,*

$$\nabla u_\varepsilon \overset{2-s}{\rightarrow} \nabla_x u(x,t) + \nabla_y U_1(x,t,y,s)\,, \tag{4.4}$$

$$a(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) \overset{2-s}{\rightarrow} a_0(x,t,y,s). \tag{4.5}$$

Further, the pair (u, U_1) *satisfies the following two-scale homogenized problem*

$$\begin{aligned}\int_0^T \langle \partial_t \overline{b}(u), \phi\rangle\, dt + \int_{\Omega_T}\int_Y \int_0^1 a_0(x,t,y,s).(\nabla_x\phi \\ +\nabla_y \Phi(x,t,y,s))\, dy\, ds\, dx\, dt &= \int_{\Omega_T} f\,\phi\, dx\, dt\end{aligned} \tag{4.6}$$

for all $\phi \in C_0^\infty(\Omega_T)$ *and* $\Phi \in C_0^\infty(\Omega_T; C^\infty_{per}(Y \times (0,1))$.

Proof: Existence of U_1, a_0 and the convergence (4.4), (4.5) follow from the previous two-scale convergence theorems and by the estimates (3.1) and (3.2). Note that, we do not get the s dependence in the first term of right hand side of (4.4) because of the strong convergence (4.1). Now, let $\phi \in C_0^\infty(\Omega_T)$ and let $\Phi \in C_0^\infty(\Omega_T; C^\infty_{per}(Y \times (0,1)))$. We take test functions as

$$\phi_\varepsilon = \phi(x,t) + \varepsilon\Phi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$$

in (2.3). Note that,

$$\begin{aligned}\int_0^T \langle \partial_t b(\frac{x}{\varepsilon}, u_\varepsilon), \phi_\varepsilon\rangle\, dt &\overset{\varepsilon\to 0}{\rightarrow} \int_0^T \langle \partial_t \overline{b}(u), \phi\rangle\, dt, \\ \int_{\Omega_T} f\phi_\varepsilon\, dx\, dt &\overset{\varepsilon\to 0}{\rightarrow} \int_{\Omega_T} f\,\phi\, dx\, dt\end{aligned}$$

by Corollary 4.3, (ii) and the strong convergence of ϕ_ε to ϕ in $L^p(\Omega_T)$. Also, using (4.5) and Theorem 4.8 by two-scale convergence of $\nabla\phi_\varepsilon$, we get

$$\begin{aligned}&\int_{\Omega_T} a(\frac{x}{\varepsilon},\frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).\nabla\phi_\varepsilon\, dx\, dt \\ &= \int_{\Omega_T} a(\frac{x}{\varepsilon},\frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon).(\nabla_x\phi(x,t) + \nabla_y\Phi(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}))\, dx\, dt + o(1) \\ &\overset{\varepsilon\to 0}{\rightarrow} \int_{\Omega_T}\int_Y\int_0^1 a_0(x,t,y,s).(\nabla_x\phi(x,t) + \nabla_y\Phi(x,t,y,s))\, dy\, ds\, dx\, dt.\end{aligned}$$

Therefore, letting $\varepsilon \to 0$ in (2.3) with $\xi = \phi_\varepsilon$, we get (4.6). ♢

Remark 4.12 Note that by (4.5),

$$a(\frac{x}{\varepsilon},\frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup \int_Y\int_0^1 a_0(x,t,y,s)\, dy\, ds \text{ weakly in } L^{p^*}(\Omega_T)$$

Therefore, (3.13) implies that $A^*(x,t) = \int_Y \int_0^1 a_0(x,t,y,s)\,dy\,ds$ a. e. in Ω_T. Thus, by setting $\Phi = 0$ in (4.6) we get the homogenized equation

$$\partial_t \overline{b}(u) - \operatorname{div} A^*(x,t) = f \text{ in } \Omega_T, \tag{4.7}$$

The boundary condition can be shown to be, $A^*(x,t).\nu = 0$ on $\Gamma_{2,T}$, by choosing smooth test functions which vanish only $\Gamma_{1,T}$ in the previous proposition. It can be shown that u satisfies the initial condition, $u(x,0) = u_0(x)$, by passing to the limit in (2.2). Thus, in order to complete the homogenization it is enough to show that $A^*(x,t) = A(u, \nabla u)$ where A has been defined in (2.17).

We will first identify a_0. In fact, we prove the following Proposition.

Proposition 4.13 *Let a_0 be given by (4.5) and let (u, U_1) be as in Proposition 4.11. Then,*

$$a_0(x,t,y,s) = a(y,s,u,\nabla_x u(x,t) + \nabla_y U_1(x,t,y,s)) \text{ a. e. in } \Omega_T \times Y \times (0,1).$$

Proof: Let ϕ, Φ be as before. Let $\lambda > 0$ and $\phi_0 \in C_0^\infty(\Omega_T; C_{per}^\infty(Y \times (0,1)))^n$. Set,

$$\begin{aligned} \eta_\varepsilon &\doteq \nabla_x \phi + (\nabla_y \Phi)(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}) + \lambda \phi_0(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}), \qquad (4.8)\\ a_\varepsilon &\doteq a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\nabla u_\varepsilon),\\ d_\varepsilon &\doteq a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\phi,\eta_\varepsilon). \end{aligned}$$

We have,

$$\begin{aligned} J_\varepsilon &\doteq \int_{\Omega_T} (a_\varepsilon - d_\varepsilon).(\nabla u_\varepsilon - \eta_\varepsilon)\\ &= \int_{\Omega_T} a_\varepsilon.\nabla u_\varepsilon - \int_{\Omega_T} d_\varepsilon.\nabla u_\varepsilon - \int_{\Omega_T} a_\varepsilon.\eta_\varepsilon + \int_{\Omega_T} d_\varepsilon.\eta_\varepsilon\\ &\doteq J_{1,\varepsilon} - J_{2,\varepsilon} - J_{3,\varepsilon} + J_{4,\varepsilon} \qquad (4.9) \end{aligned}$$

where $J_{i,\varepsilon}$ denotes the respective terms above for $i = 1, \cdots, 4$. Now,

$$\begin{aligned} J_{1,\varepsilon} &= \int_{\Omega_T} a_\varepsilon.\nabla u_\varepsilon\,dx\,dt\\ &= -\int_0^T \langle \partial_t b(\frac{x}{\varepsilon}, u_\varepsilon), u_\varepsilon\rangle\,dt + \int_{\Omega_T} f\,u_\varepsilon\,dx\,dt\\ &\overset{\varepsilon\to 0}{\to} -\int_0^T \langle \partial_t \overline{b}(u), u\rangle\,dt + \int_{\Omega_T} f\,u\,dx\,dt\\ &= \int_{\Omega_T} A^*(x,t).\nabla_x u\,dx\,dt \qquad (4.10) \end{aligned}$$

where the last equality follows from (4.7). For obtaining the limit of the other terms in the right hand side of (4.9) we will use Theorem 4.8. For this we observe that the continuity assumptions on a and the choice of ϕ, Φ, ϕ_0 imply that the sequence

$$d_\varepsilon \equiv a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \phi, \nabla_x\phi + (\nabla_y\Phi)(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}) + \lambda\phi_0(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon}))$$

is of the form $\psi(t,x,\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ for a $\psi \in L^{p^*}_{per}(Y\times(0,1);C(\Omega_T))$. Thus, d_ε strongly two-scale converges to $a(y,s,\phi,\nabla_x\phi + \nabla_y\Phi + \lambda\phi_0)$. Also, it can be seen that η_ε is strongly two-scale convergent to $\eta(x,t,y,s) \doteq \nabla_x\phi(x,t) + \nabla_y\Phi(x,t,y,s) + \lambda\phi_0(x,t,y,s)$. Thus, from these observations, Theorem 4.8 and (4.10), we obtain

$$\begin{aligned} J_\varepsilon \overset{\varepsilon\to 0}{\to} & \int_{\Omega_T}\int_Y\int_0^1 a_0(x,t,y,s).\nabla_x u\,dy\,ds\,dx\,dt \\ & -\int_{\Omega_T}\int_Y\int_0^1 a(y,s,\phi,\nabla_x\phi+\nabla_y\Phi+\lambda\phi_0).(\nabla_x u+\nabla_y U_1) \\ & -\int_{\Omega_T}\int_Y\int_0^1 a_0(x,t,y,s).(\nabla_x\phi+\nabla_y\Phi+\lambda\phi_0) \\ & +\int_{\Omega_T}\int_Y\int_0^1 a(y,s,\phi,\nabla_x\phi+\nabla_y\Phi+\lambda\phi_0).(\nabla_x\phi+\nabla_y\Phi+\lambda\phi_0) \end{aligned}$$

Note that by setting $\phi = 0$ in (4.6) we get,

$$\int_{\Omega_T}\int_Y\int_0^1 a_0(x,t,y,s).\nabla_y\Phi(x,t,y,s)\,dy\,ds\,dx\,dt = 0 \tag{4.11}$$

for any $\Phi \in C_0^\infty(\Omega_T; C^\infty_{per}(Y\times(0,1)))$. Thus, the above limit can be rewritten as

$$\begin{aligned} \lim_{\varepsilon\to 0} J_\varepsilon = & \int_{\Omega_T}\int_Y\int_0^1 a_0(x,t,y,s).(\nabla_x u - \nabla_x\phi - \lambda\phi_0)\,dy\,ds\,dx\,dt \\ & -\int_{\Omega_T}\int_Y\int_0^1 a(y,s,\phi,\nabla_x\phi+\nabla_y\Phi+\lambda\phi_0) \\ & \times(\nabla_x u+\nabla_y U_1-\nabla_x\phi-\nabla_y\Phi-\lambda\phi_0). \end{aligned}$$

Now, letting $\phi \to u$ strongly in $L^p(0,T;V)$ and $\Phi \to U_1$ in $L^p(\Omega_T\times(0,1);W^{1,p}_{per}(Y))$ strongly we get,

$$\lim_{\substack{\phi\to u\\ \Phi\to U_1}} \lim_{\varepsilon\to 0} J_\varepsilon = \int_{\Omega_T}\int_Y\int_0^1 (a(y,s,u,\nabla_x u+\nabla_y U_1+\lambda\phi_0) - a_0(x,t,y,s)).\lambda\phi_0, \tag{4.12}$$

where we have used the continuity properties of a. On the other hand,

$$J_\varepsilon = \int_{\Omega_T}(a_\varepsilon - a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\eta_\varepsilon)).(\nabla u_\varepsilon - \eta_\varepsilon)\,dx\,dt$$

$$\begin{aligned}&+\int_{\Omega_T}(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\eta_\varepsilon)-a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\phi,\eta_\varepsilon)).(\nabla u_\varepsilon-\eta_\varepsilon)\,dx\,dt\\ \doteq\;& L_{1,\varepsilon}+L_{2,\varepsilon},\end{aligned}$$

where $L_{i,\varepsilon}$, $i=1,2$ denotes the respective terms above. By the monotonicity assumption (2.6), $L_{1,\varepsilon}\geq 0$. Therefore, $J_\varepsilon \geq L_{2,\varepsilon}$. Now, by (2.8) and generalized Hölder's inequality,

$$\begin{aligned}|L_{2,\varepsilon}| &\leq \int_{\Omega_T}|a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\eta_\varepsilon)-a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\phi,\eta_\varepsilon)|.|\nabla u_\varepsilon-\eta_\varepsilon|\,dx\,dt\\ &\leq \alpha^{-1}\|u_\varepsilon-\phi\|_p^r(m(\Omega_T)^{\frac{p-1-r}{p}}+\|u_\varepsilon\|_p^{p-1-r}+\|\phi\|_p^{p-1-r}\\ &\quad+\|\eta_\varepsilon\|_p^{p-1-r})\|\nabla u_\varepsilon-\eta_\varepsilon\|_p.\end{aligned}$$

Therefore,

$$\begin{aligned}J_\varepsilon &\geq L_{2,\varepsilon}\\ &\geq -\alpha^{-1}\|u_\varepsilon-\phi\|_p^r(m(\Omega_T)^{\frac{p-1-r}{p}}+\|u_\varepsilon\|_p^{p-1-r}+\|\phi\|_p^{p-1-r}\\ &\quad+\|\eta_\varepsilon\|_p^{p-1-r})\|\nabla u_\varepsilon-\eta_\varepsilon\|_p\\ &\geq -\alpha^{-1}\|u_\varepsilon-\phi\|_p^r(C+\|\phi\|_p^{p-1-r}+\|\eta_\varepsilon\|_p^{p-1-r})(C+\|\eta_\varepsilon\|_p)\end{aligned}$$

since $u_\varepsilon,\nabla u_\varepsilon$ are bounded in $L^p(\Omega_T)$. We now use the fact that η_ε is strongly two-scale convergent to η, defined before, to obtain the limit as $\varepsilon\to 0$ in the above inequality and we get

$$\lim_{\varepsilon\to 0}J_\varepsilon \;\geq\; -\alpha^{-1}\|u-\phi\|_p^r(C+\|\phi\|_p^{p-1-r}+\|\eta\|_p^{p-1-r})(C+\|\eta\|_{p,\Omega_T\times Y\times(0,1)}).$$

Now letting $\phi\to u$ and $\Phi\to U_1$ as before, we get

$$\lim_{\substack{\phi\to u\\ \Phi\to U_1}}\lim_{\varepsilon\to 0}J_\varepsilon\geq 0. \tag{4.13}$$

Therefore, from (4.12) and (4.13), we get

$$\int_{\Omega_T}\int_Y\int_0^1(a(y,s,u,\nabla_x u+\nabla_y U_1+\lambda\phi_0)-a_0(x,t,y,s)).\lambda\phi_0\,dy\,ds\,dx\,dt\geq 0 \tag{4.14}$$

for all $\lambda>0$ and for all $\phi_0\in C_0^\infty(\Omega_T;C_{per}^\infty(Y\times(0,1)))^n$. Dividing the above inequality and letting $\lambda\to 0$, we get using the continuity of a, that

$$\int_{\Omega_T}\int_Y\int_0^1(a(y,s,u,\nabla_x u+\nabla_y U_1)-a_0(x,t,y,s)).\phi_0\,dy\,ds\,dx\,dt\geq 0 \tag{4.15}$$

for all $\phi_0\in C_0^\infty(\Omega_T;C_{per}^\infty(Y\times(0,1)))^n$. By the density of these functions in $L^p(\Omega_T\times Y\times(0,1))^n$, we get $a_0(x,t,y,s)=a(y,s,u,\nabla_x u(x,t)+\nabla_y U_1(x,t,y,s))$ a.e. in $\Omega_T\times Y\times(0,1)$. ♢

Proof of Theorem 2.3: The proof follows from Proposition 4.11, Remark 4.12, Proposition 4.13, (2.17) and (4.11). ♢

We now prove corrector results. First, we prove a certain corrector result without any smoothness assumption on (u, U_1). Then we deduce Theorem 2.5 from this corrector result.

Let $\delta > 0$ and choose $\phi \in C_0^1(\Omega_T)$, $\Phi \in C_0(\Omega_T; C_{per}(0,1) \times C_{per}^1(Y))$ approximating u, U_1 respectively, viz.

$$\|\phi - u\|_{L^p(0,T;W^{1,p}(\Omega))} \le \delta \tag{4.16}$$

$$\|\Phi - U_1\|_{L^p(\Omega_T \times (0,1); W_{per}^{1,p}(Y))} \le \delta\,.. \tag{4.17}$$

Define, η_ε as in (4.8) with $\lambda = 0$. Then we have the following lemma.

Lemma 4.14 *Let $\delta > 0$ be fixed. Fix ϕ, Φ as above. Under the strong monotonicity assumption (A5), we have*

$$\limsup_{\varepsilon \to 0} \|\nabla u_\varepsilon - \eta_\varepsilon\|_{p,\Omega_T} \le O(\delta^{\frac{r_0}{p}}) \tag{4.18}$$

where $r_0 = min(r, 1)$.

Proof: We will use some of the calculations from Proposition 4.13. For that we observe that the regularity that we have now taken for ϕ, Φ would have been sufficient in the proof of that proposition also. Let J_ε be as in the proof of Proposition 4.13. We have, by the strong monotonicity condition (A5),

$$\begin{aligned} \alpha \|\nabla u_\varepsilon - \eta_\varepsilon\|_{p,\Omega_T}^p &\le \int_{\Omega_T} (a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) - a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \eta_\varepsilon)).(\nabla u_\varepsilon - \eta_\varepsilon)\,dx\,dt \\ &\doteq K_\varepsilon \end{aligned} \tag{4.19}$$

Now,

$$\begin{aligned} K_\varepsilon &= \int_{\Omega_T} ([a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon) - a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \phi, \eta_\varepsilon)] \\ &\quad + [a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \phi, \eta_\varepsilon) - a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u, \eta_\varepsilon)] \\ &\quad + [a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u, \eta_\varepsilon) - a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, \eta_\varepsilon)]).(\nabla u_\varepsilon - \eta_\varepsilon)\,dx\,dt \\ &\le J_\varepsilon + \alpha^{-1}\|u - \phi\|_p^r[(m(\Omega_T)^{\frac{p-1-r}{p}} + \|u\|_p^{p-1-r} + \|\phi\|_p^{p-1-r} + \|\eta_\varepsilon\|_p^{p-1-r}) \\ &\quad \times (\sup_\varepsilon \|\nabla u_\varepsilon\|_p + \|\eta_\varepsilon\|_p)] \\ &\quad + \alpha^{-1}\|u_\varepsilon - u\|_p^r[(m(\Omega_T)^{\frac{p-1-r}{p}} + \|u_\varepsilon\|_p^{p-1-r} + \|u\|_p^{p-1-r} + \|\eta_\varepsilon\|_p^{p-1-r}) \\ &\quad \times (\sup_\varepsilon \|\nabla u_\varepsilon\|_p + \|\eta_\varepsilon\|_p)] \\ &\le J_\varepsilon + C\delta^r(C + \|\eta_\varepsilon\|_p^{p-1-r})(C + \|\eta_\varepsilon\|_p) \\ &\quad + C\|u - u_\varepsilon\|_p^r(C + \|u_\varepsilon\|_p^{p-1-r} + \|\eta_\varepsilon\|_p^{p-1-r})(C + \|\eta_\varepsilon\|_p). \end{aligned}$$

Letting $\varepsilon \to 0$ we get,

$$\begin{aligned}\limsup_{\varepsilon\to 0} K_\varepsilon &\le \lim_{\varepsilon\to 0} J_\varepsilon + C\delta^r(C + \|\nabla_x\phi + \nabla_y\Phi\|_p^{p-1-r}).(C + \|\nabla_x\phi + \nabla_y\Phi\|_p)\\ &\le \lim_{\varepsilon\to 0} J_\varepsilon + C\delta^r,\end{aligned}$$

where the last constant C is independent of δ for $0 < \delta \le 1$, as the norms of ϕ, Φ are close to the norms of u, U_1 respectively. Also, we know from the proof of Proposition 4.13 that

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon &= \int_{\Omega_T}\int_Y\int_0^1 a(y,s,u,\nabla_x u + \nabla_y U_1).(\nabla_x u - \nabla_x\phi)\,dy\,ds\,dx\,dt\\ &\quad - \int_{\Omega_T}\int_Y\int_0^1 a(y,s,\phi,\nabla_x\phi + \nabla_y\Phi).(\nabla_x u - \nabla_x\phi + \nabla_y U_1 - \nabla_y\Phi)\\ &\le C\|\nabla_x u - \nabla_x\phi\|_p(1 + \|u\|_p^{p-1} + \|\nabla_x u\|_p^{p-1} + \|\nabla_y U_1\|_p^{p-1})\\ &\quad + C(\|\nabla_x u - \nabla_x\phi\|_p + \|\nabla_y U_1 - \nabla_y\Phi\|_p)\\ &\quad \times(1 + \|\phi\|_p^{p-1} + \|\nabla_x\phi\|_p^{p-1} + \|\nabla_y\Phi\|_p^{p-1})\\ &\le C\delta\end{aligned}$$

by the choice of ϕ, Φ. Thus,

$$\limsup_{\varepsilon\to 0}\|\nabla u_\varepsilon - \eta_\varepsilon\|_p^p \le \lim_{\varepsilon\to 0} J_\varepsilon + C\delta^r \le C(\delta + \delta^r) \le C\delta^{r_0}$$

for $0 < \delta \le 1$. Hence the lemma. $\diamondsuit$

Under the stronger continuity assumption on a, viz.

$$|a(y,s,\mu,\lambda_1) - a(y,s,\mu,\lambda_2)| \le |\lambda_1 - \lambda_2|^r(1 + |\mu|^{p-1-r} + |\lambda_1|^{p-1-r} + |\lambda_2|^{p-1-r}) \tag{4.20}$$

for all $(y, s, \mu, \lambda_1, \lambda_2)$, we have the following corollary.

Corollary 4.15 *Assume (4.20). Then, we have*

$$\lim_{\varepsilon\to 0}\|a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\nabla u_\varepsilon) - a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,\eta_\varepsilon)\|_{p^*} \le \delta^{\frac{r_0^2}{p}} \tag{4.21}$$

Proof: Note that,

$$\begin{aligned}&\int_{\Omega_T}|a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\nabla u_\varepsilon) - a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,\eta_\varepsilon|^{p^*}\,dx\,dt\\ &\le C\int_{\Omega_T}|a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\nabla u_\varepsilon) - a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,\nabla u_\varepsilon)|^{p^*}\,dx\,dt\\ &\quad + C\int_{\Omega_T}|a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,\nabla u_\varepsilon) - a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,\eta_\varepsilon)|^{p^*}\,dx\,dt\\ &\le C\|u_\varepsilon - u\|_{p,\Omega_T}^{\frac{r}{p}} + \|\nabla u_\varepsilon - \eta_\varepsilon\|_{p,\Omega_T}^{\frac{r}{p}}(C + \|\eta_\varepsilon\|_{p,\Omega_T}^{p-1-r})\end{aligned}$$

where in the last inequality we have used (2.8) and (4.20) and the fact that the sequences u_ε, ∇u_ε are bounded. Letting $\varepsilon \to 0$, using Theorem 2.5, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\|a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u_\varepsilon,\nabla u_\varepsilon)-a(\frac{x}{\varepsilon},\frac{t}{\varepsilon},u,\eta_\varepsilon)\|_{p^*} &\le \delta^{\frac{r_0^2}{p}}(C+\|\nabla_x\phi+\nabla_y\Phi\|_p^{p-1-r})\\ &\le C\delta^{\frac{r_0^2}{p}}.\end{aligned}$$

This completes the proof. ♢

Proof of Theorem 2.5: If u, U_1 are sufficiently smooth we can take $\phi = u$ and $\Phi = U_1$ in the proof of the previous lemma and (2.24) follows as we can take $\delta \equiv 0$. The convergence in (2.23) is obvious from the strong convergence of u_ε to u in $L^p(\Omega_T)$. ♢

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