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On the solvability of nonlocal pluriparabolic problems *

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Abstract

The aim of this paper is to prove existence, uniqueness, and continuous dependence upon the data of solutions to mixed problems for pluriparabolic equations with nonlocal boundary conditions. The proofs are based on a priori estimates established in non-classical function spaces and on the density of the range of the operator generated by the studied problems.

1 Introduction

In this paper, we study a class of second order pluriparabolic equations with nonlocal conditions. The aim is to proof existence, uniqueness, and continuous dependence of generalized solutions.

Evolution problems with nonlocal boundary conditions have received attention in several papers. Most of the papers were directed to second order parabolic equations, particularly to heat conduction equations; see, for instance, Cannon et al. [16]-[19], Kamynin [21], Ionkin [20], Yurchuk [27], Benouar-Yurchuk [1], Bouziani-Benouar [9], Bouziani [2]-[4] and Mesloub-Bouziani [22]-[23]. For similar problems, related to other equations, we refer the reader to Bouziani [5]-[8], Bouziani-Benouar [10] and Pulkina [24]-[25]. Mixed problems with nonlocal boundary conditions or with nonlocal initial conditions were studied in Yurchuk [28], Byszewski et al. [11]-[16], and Bouziani [7]-[8].

The presence of integral terms in the boundary conditions can greatly complicate the application of standard functional and numerical techniques. The main tool used in this paper is the introduction of a new function space in which we can establish an a priori estimate.

This paper is outlined as follows: In Section 2, we give notation, the statement of two problems, and the basic assumptions. Section 3 is devoted to the introduction of the function spaces to be used in the rest of the paper. In Section 3, we present abstract formulations of the posed problems and make precise

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the concept of solution of the problems. In Section 4, we establish a priori estimates which are derived to show the uniqueness and continuous dependence of the solutions upon the data. Finally, we prove the existence of the solutions in Section 5.

2 Notation and statement of problems

Throughout this paper we use the following notation:

$$t = (t_1, \dots, t_n), \quad \tau = (\tau_1, \dots, \tau_n), \quad t^{i,0} = (t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n),$$
$$t^{i,\tau} = (t_1, \dots, t_{i-1}, \tau_i, t_{i+1}, \dots, t_n), \quad t^{i,T} = (t_1, \dots, t_{i-1}, T_i, t_{i+1}, \dots, t_n),$$
$$dt^i = dt_1 \dots dt_{i-1} dt_{i+1} \dots dt_n \quad (i = 1, \dots, n),$$

$\Omega = (a,b) \subset \mathbb{R}$, with $a, b < \infty$, $I_i = (0, T_i)$, where $T_i < \infty$ $(i = 1, \dots, n)$, $I = \prod_{i=1}^n I_i$,

$$\Delta_i = I_1 \times \dots \times I_{i-1} \times I_{i+1} \times \dots \times I_n,$$
$$Q_i = \Omega \times I_1 \times \dots \times I_{i-1} \times I_{i+1} \times \dots \times I_n,$$
$$Q_{i,0} = I_1 \times \dots \times I_{i-1} \times \{0\} \times I_{i+1} \times \dots \times I_n$$

We consider the following problem: Given the data σ, p, Φ_i $(i = 1, \dots, n)$, E, G and H, find a function $v(x,t)$ satisfying the pluriparabolic equation

$$(\mathcal{L}v)(x,t) := \sum_{i=1}^{n} \frac{\partial v}{\partial t_i} - \frac{1}{\sigma(x)} P(t) v = g(x,t), \quad \text{for } (x,t) \in \Omega \times I, \tag{1}$$

where

$$P(t)v = \frac{\partial}{\partial x}(p(J_x \sigma, t) \frac{\partial v}{\partial x})$$

with $J_x \sigma$ the primitive of σ, satisfying the conditions

$$(\ell_i v)(x,t) := v(x, t^{i,0}) = \Phi_i(x, t^i), \quad \text{for } (x,t) \in Q_{i,0} \ (i = 1, \dots, n), \tag{2}$$

$$\int_\Omega \sigma(x) v(x,t) dx = E(t) \quad \text{for } \ t \in I, \tag{3}$$

and satisfying one of the conditions

$$\int_\Omega x \sigma(x) v(x,t) dx = G(t) \quad \text{for } \ t \in I, \tag{4}$$

$$\frac{\partial v(a,t)}{\partial x} = H(t) \quad for \; t \in I. \tag{4'}$$

If $\sigma(x_0) = 0$ for all x_0 in $\overline{\Omega}$, then (1) is called **a singular pluriparabolic equation**.

Next we formulate the main assumptions:

A1: There are positive constants c_i, $(i = 0, 1, 2, 3)$ such that

$$c_0 \leq p(J_x\sigma, t) \leq c_1, \quad \left| \frac{\partial p(J_x\sigma, t)}{\partial t_i} \right| \leq c_2 \quad (i = 1, \dots, n),$$
$$\left| \frac{\partial p(J_x\sigma, t)}{\partial J_x\sigma} \right| \leq c_3 \quad \text{for } (x, t) \in \Omega \times I.$$

A2: $x \to \sigma(x)$ is a positive continuous function on Ω, such that $\sigma(x) \leq c_4$, where c_4 is a positive constant.

A3: $g \in C(\Omega \times I, \mathbb{R})$, $\Phi_i \in C^1(Q_i, \mathbb{R})$ $(i = 1, \dots, n)$, $E, G, H \in C^1(I, \mathbb{R})$,

$$\int_\Omega \sigma(x)\Phi_i(x, t^i)dx = E(t^{i,0}), \quad \text{for } t^{i,0} \in \Delta_i \ (i = 1, \dots, n),$$

$$\int_\Omega x\sigma(x)\Phi_i(x, t^i)dx = G(t^{i,0}), \quad \text{for } t^{i,0} \in \Delta_i \ (i = 1, \dots, n),$$

$$\left[\frac{\partial \Phi_i(a, t^i)}{\partial x} = H(t^{i,0}), \text{ respectively} \right] \quad \text{for } t^{i,0} \in \Delta_i \ (i = 1, \dots, n)$$

and $\Phi_i = \Phi_j$ $(i \neq j;\ i, j = 1, \dots, n)$.

Problem (1)-(3) and (4') can be viewed as a generalization of that in Bouziani [8], where the author studied a similar problem for the case of a second order pluriparabolic equation with $n = 2$, $\sigma(x) = 1$, $\Omega = (0, b)$ and with the function p satisfying, in addition of Assumption A1, other supplementary assumptions. The proof in [8] is based on an a priori estimate, which is established by taking the inner product in L^2-space of the considered equation by the integro-differential operator

$$(Mu)(x, t_1, t_2) := 2(b - x)(\Im_x \frac{\partial u}{\partial t_1} + \Im_x \frac{\partial u}{\partial t_2}) - p(x, t_1, t_2)\frac{\partial u}{\partial x},$$

where the definition of $\Im_x$ is similar to the definition of $\Im_x$ from Section 3.1. The results of this paper are also continuations of those obtained by the author et al. in [2]-[7], [9]-[10] and [22]-[23].

Let us, now, reformulate problem (1)-(4) [respectively, (1)-(3) and (4')] with non-homogeneous boundary conditions (3), (4) [respectively, (3), (4')] as a problem with homogeneous boundary conditions, by introducing a new unknown function $u(x, t)$ defined as follows:

$$u(x, t) = v(x, t) - U(x, t), \quad \text{for } (x, t) \in \Omega \times I,$$

where

$$\begin{aligned} U(x, t) \quad &= \quad \frac{E(t)}{(b - a)\sigma(x)} + \frac{6(2G(t) - (b + a)E(t))}{(b - a)^4\sigma(x)} \\ &\quad \times (3(x - a)^2 - 2(b - a)(x - a)) \quad \text{for } (x, t) \in \Omega \times I\,; \end{aligned}$$

respectively

$$\begin{aligned} U(x,t) &= (x-a)H(t)+\frac{(x-a)^2}{\int_\Omega (x-a)^2\sigma(x)dx} \\ &\quad \times (E(t)-H(t)\int_\Omega (x-a)\sigma(x)dx) \text{ for } (x,t)\in\Omega\times I\,. \end{aligned}$$

Then, we have to find a function $u(x,t)$, such that

$$(\mathcal{L}u)(x,t)=g(x,t)-(\mathcal{L}U)(x,t)=: f(x,t) \text{ for } (x,t)\in\Omega\times I, \tag{5}$$

$$\begin{aligned} (\ell_i u)(x,t) &= \Phi_i(x,t^i)-(\ell_i U)(x,t) \\ &= :\varphi_i(x,t^i)(i=1,\dots,n)\text{for } (x,t)\in Q_{i,0}, \end{aligned} \tag{6}$$

$$\int_\Omega \sigma(x)u(x,t)dx=0 \quad \text{for } t\in I, \tag{7}$$

$$\int_\Omega x\sigma(x)u(x,t)dx=0 \quad \text{for } t\in I, \tag{8}$$

$$\left[\frac{\partial u(a,t)}{\partial x}=0,\text{ respectively}\right] \quad \text{for } t\in I. \tag{8'}$$

3 Preliminaries

Function spaces

In this subsection, we introduce and study certain fundamental function spaces. For this purpose, let us denote by $C_0(\Omega)$ the space of the continuous functions with compact support in Ω. We define on $C_0(\Omega)$ the bilinear form $((.,.))_\sigma$, given by

$$((u,\theta))_\sigma := \int_\Omega \Im_x(\sigma u)\cdot\Im_x(\sigma\theta)dx, \tag{9}$$

where

$$\Im_x\eta := \int_a^x \eta(\xi,.)d\xi.$$

The bilinear form, defined by (9), is a scalar product in $C_0(\Omega)$ for which $C_0(\Omega)$ is not complete. Thus we are led to introduce its completion.

Definition 3.1 We denote by $B^1_{2,\sigma}(\Omega)$ a completion of $C_0(\Omega)$ for the scalar product defined by (9), called the **weighted Bouziani space**.

Thus, we have the following result:

Proposition 3.1 *The space $B^1_{2,\sigma}(\Omega)$ is a Hilbert space for the scalar product*

$$(u,\theta)_{B^1_{2,\sigma}(\Omega)} = ((u,\theta))_\sigma,$$

with the associated norm

$$\|w\|_{B^1_{2,\sigma}(\Omega)} = \|\Im_x(\sigma w)\|_{L^2(\Omega)}\,.$$

Remark 3.1 If $\sigma(x) = 1$, then the space $B^1_{2,1}(\Omega)$ coincides with $B^1_2(\Omega)$, which was firstly introduced by the author in [4] and [5].

Definition 3.2 Let m be a nonnegative integer and let $1 \le p < \infty$. We define the space $B^m_{p,\sigma}(\Omega)$ to be a completion of $C_0(\Omega)$ in the norm

$$\|w\|_{B^m_{p,\sigma}(\Omega)} := \|\Im^m_x(\sigma w)\|_{L^p(\Omega)}\,,$$

and for $p = 2$, we define an inner product by

$$(u,\theta)_{B^m_{2,\sigma}(\Omega)} := (\Im^m_x(\sigma u), \Im^m_x(\sigma\theta))_{L^2(\Omega)}.$$

Definition 3.3 We denote by $L^2_\sigma(\Omega)$, the Hilbert space of weighted square integrable functions with the inner product

$$(u,\theta)_{L^2_\sigma(\Omega)} := (\sigma u, \theta)_{L^2(\Omega)},$$

and with the associated norm

$$\|w\|_{L^2_\sigma(\Omega)} := \left\|\sqrt{\sigma} w\right\|_{L^2(\Omega)}.$$

Let X be a Hilbert space with a norm $\|.\|_X$.

Definition 3.4 i) We denote by $L^2(I,X)$ the set of all measurable abstract functions

$$I \ni t \to u(.,t) \in X$$

such that

$$\|u\|^2_{L^2(I,X)} := \int_I \|u(.,t)\|^2_X \, dt < \infty.$$

ii) Let $C(\overline{I}_i, X)(i = 1,\ldots,n)$ be the set of all continuous functions

$$\overline{I}_i \ni t_i \to u(\ldots,t_i,\ldots) \in X \quad (i = 1,\ldots,n),$$

with

$$\|u\|_{C(\overline{I}_i,X)} := \max_{t_i\in\overline{I}_i} \|u(\ldots,t_i,\ldots)\|_X < \infty \ (i = 1,\ldots,n).$$

Proposition 3.2 *i) $L^2(I,X)$ is a Hilbert space.*
ii) $C(\overline{I}_i, X)$ $(i = 1,\ldots,n)$ are Banach spaces.

Abstract formulation

Let us reformulate problem (5)-(8) [(5)-(7) and (8'), respectively] as the problem of solving the abstract equation

$$Lu = \{f, \varphi_1, \ldots, \varphi_n\}\,,$$

where L is the operator given by the formula

$$Lu := \{\mathcal{L}u, \ell_1 u, \ldots, \ell_n u\}\,.$$

We consider L as an unbounded operator with the domain $D(L)$ consisting of all functions u belonging to $L^2(I, B^1_{2,\sigma}(\Omega))$, such that $\frac{\partial u}{\partial t_i} \in L^2(I, B^1_{2,\sigma}(\Omega))$ $(i = 1, \dots, n)$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2} \in L^2(I, B^1_{2,\sigma}(\Omega))$ and such that u satisfies conditions (7) and (8) [(7) and (8'), respectively].

Let B be the Hilbert space obtained by completing of the domain $D(L)$ in the norm

$$\|u\|_B = \Big\{ \|P(t)u\|^2_{L^2(I,B^1_{2,\sigma}(\Omega))} + \sum_{i=1}^{n} \|u\|^2_{C(\overline{I}_i, L^2(\Delta_i, L^2_\sigma(\Omega)))} \Big\}^{1/2}.$$

By completing of the set $D(L)$ with respect to the norm

$$\|u\|_B = \Big\{ \|u\|^2_{L^2(I,H^1(\Omega))} + \sum_{i=1}^{n} \|u\|^2_{C(\overline{I}_i, L^2(\Delta_i, L^2_\sigma(\Omega)))} \Big\}^{1/2},$$

we obtain a Banach space B'. The elements $u \in B$ [B', respectively] are continuous functions on I_i $(i = 1, \dots, n)$ with the values in $L^2(\Delta_i, L^2_\sigma(\Omega))$ $(i = 1, \dots, n)$. Hence, the mappings

$$\ell_i : B \ni u \to \ell_i u = u_{|t_i=0} \in L^2(\Delta_i, L^2_\sigma(\Omega)) \ (i = 1, \dots, n),$$

are defined and continuous on B [B', respectively] .

We denote by F the Hilbert space $L^2(I, B^1_{2,\sigma}(\Omega)) \times \prod_{i=1}^{n} L^2(\Delta_i, L^2_\sigma(\Omega))$. The elements of F are of the form $\{f,\ \varphi_1, \dots, \varphi_n\}$, where $f \in L^2(I, B^1_{2,\sigma}(\Omega))$, $\varphi_i \in L^2(\Delta_i, L^2_\sigma(\Omega))$ $\quad (i = 1, \dots, n)$, and the norm is defined by:

$$\|\{f,\ \varphi_1, \dots, \varphi_n\}\|_F = \Big\{ \|f\|^2_{L^2(I,B^1_{2,\sigma}(\Omega))} + \sum_{i=1}^{n} \|\varphi_i\|^2_{L^2(\Delta_i, L^2_\sigma(\Omega))} \Big\}^{1/2}.$$

If the operator L is closable then we denote by $\overline{L}$ the closure of L and by $D(\overline{L})$ its domain.

Definition 3.5 A solution of the abstract equation

$$\overline{L}u = \{f,\ \varphi_1, \dots, \varphi_n\}$$

is called a **strongly generalized solution** of problem (5)-(8) [(5)-(7) and (8'), respectively].

In concluding this section, we shall state the following lemma, which can be applied to obtain a priori estimates for the solutions.

Lemma 3.1 *If f_k $(k = 1, 2, 3)$ are nonnegative functions on I; f_1 and f_2 are integrable on I; $\tau \to f_3(\tau)$ is non-decreasing with respect to τ_i $(i = 1, \dots, n)$ and*

$$\int_{I^\tau} f_1(t)dt + f_2(\tau) \le c \sum_{i=1}^{n} \int_0^{\tau_i} f_2(\tau_1, \dots, \tau_{i-1}, t_i, \tau_{i+1}, \dots, \tau_n) + f_3(\tau),$$

where c is a positive constant, then

$$\int_{I^\tau} f_1(t)dt + f_2(\tau) \le \exp\left\{nc\sum_{i=1}^{n}\tau_i\right\}\cdot f_3(\tau),$$

where $I^\tau := \prod_{i=1}^{n}(0,\tau_i),\ 0<\tau_i<T_i\ (i=1,\dots,n)$.

The proof of this lemma is analogous to the proof of Lemma 1 in [8].

4 Uniqueness and continuous dependence

In this section we will establish a priori estimates. Thus we will deduce the uniqueness and continuous dependence of the solutions upon the data.

Theorem 4.1 *If Assumptions A1 and A2 are satisfied then there is a constant $c>0$, independent of u such that*

$$\|u\|_B \le c\,\|Lu\|_F \tag{10}$$

$$[\|u\|_{B'} \le c\,\|Lu\|_{F'}\,,\ \textit{respectively}] \tag{10'}$$

for $u \in D(L)$.

Proof. Taking the square of the norm in the space $L^2(I^\tau, B^1_{2,\sigma}(\Omega))$ of $\mathcal{L}u$, we get

$$\begin{aligned}\int_{I^\tau}\Big\|\sum_{i=1}^{n}\frac{\partial u}{\partial t_i}\Big\|^2_{B^1_{2,\sigma}(\Omega)}dt + \int_{I^\tau}\|P(t)u\|^2_{B^1_2(\Omega)}\,dt&\\ -2\sum_{i=1}^{n}\int_{\Omega\times I^\tau}\Im_x P(t)u\cdot\Im_x(\sigma\frac{\partial u}{\partial t_i})dx &= \int_{I^\tau}\|f\|^2_{B^1_{2,\sigma}(\Omega)}\,dt\,;\end{aligned}$$

respectively

$$\begin{aligned}\int_{I^\tau}\left\|\sum_{i=1}^{n}\frac{\partial u}{\partial t_i}\right\|^2_{B^1_{2,\sigma}(\Omega)}dt + \int_{\Omega\times I^\tau}p^2(J_x\sigma,t)(\frac{\partial u}{\partial x})^2dx&\\ -2\sum_{i=1}^{n}\int_{\Omega\times I^\tau}p(J_x\sigma,t)\frac{\partial u}{\partial x}\cdot\Im_x(\sigma\frac{\partial u}{\partial t_i})dx &= \int_{I^\tau}\|f\|^2_{B^1_{2,\sigma}(\Omega)}\,dt\,.\end{aligned}$$

Observe that

$$-2\sum_{i=1}^{n}\int_{\Omega\times I^\tau}\Im_x P(t)u\cdot\Im_x(\sigma\frac{\partial u}{\partial t_i})dx$$

$$\left[-2\sum_{i=1}^{n}\int_{\Omega\times I^\tau}p(J_x\sigma,t)\frac{\partial u}{\partial x}\cdot\Im_x(\sigma\frac{\partial u}{\partial t_i})dx\right]$$

$$
\begin{aligned}
= \quad & \sum_{i=1}^{n} \int_{Q_i} \sigma(x) p(J_x\sigma, t^{i,\tau}) u^2(x, t^{i,\tau}) dx dt^i \\
& - \sum_{i=1}^{n} \int_{\Omega \times I^\tau} \sigma(x) \frac{\partial p(J_x\sigma, t)}{\partial t_i} u^2 dx dt \\
& + 2 \sum_{i=1}^{n} \int_{\Omega \times I^\tau} \sigma(x) \frac{\partial p(J_x\sigma, t)}{\partial J_x\sigma} u \cdot \Im_x(\sigma \frac{\partial u}{\partial t_i}) dx dt,
\end{aligned}
$$

it yields

$$
\begin{aligned}
& \int_{I^\tau} \Big\| \sum_{i=1}^{n} \frac{\partial u}{\partial t_i} \Big\|^2_{B^1_{2,\sigma}(\Omega)} dt + \int_{I^\tau} \| P(t) u \|^2_{B^1_2(\Omega)} dt \qquad (11) \\
& \sum_{i=1}^{n} \int_{Q_i} \sigma(x) p(J_x\sigma, t^{i,\tau}) u^2(x, t^{i,\tau}) dx dt^i \\
& \Big[\int_{I^\tau} \Big\| \sum_{i=1}^{n} \frac{\partial u}{\partial t_i} \Big\|^2_{B^1_{2,\sigma}(\Omega)} dt + \int_{\Omega \times I^\tau} p^2(J_x\sigma, t) (\frac{\partial u}{\partial x})^2 dx \\
& \sum_{i=1}^{n} \int_{Q_i} \sigma(x) p(J_x\sigma, t^{i,\tau}) u^2(x, t^{i,\tau}) dx dt^i \Big] \\
= \quad & \int_{I^\tau} \| f \|^2_{B^1_{2,\sigma}(\Omega)} dt + \sum_{i=1}^{n} \int_{Q_i} \sigma(x) p(J_x\sigma, t^{i,0}) \varphi^2(x, t^i) dx dt^i \\
& \sum_{i=1}^{n} \int_{\Omega \times I^\tau} \sigma(x) \frac{\partial p(J_x\sigma, t)}{\partial t_i} u^2 dx dt \\
& + 2 \sum_{i=1}^{n} \int_{\Omega \times I^\tau} \sigma(x) \frac{\partial p(J_x\sigma, t)}{\partial J_x\sigma} u \cdot \Im_x(\sigma \frac{\partial u}{\partial t_i}) dx dt.
\end{aligned}
$$

In light of the Cauchy inequality, the last sum of integrals on the right-hand side of (11) are dominated as follows

$$
\begin{aligned}
& 2 \sum_{i=1}^{n} \int_{\Omega \times I^\tau} \sigma(x) \frac{\partial p(J_x\sigma, t)}{\partial J_x\sigma} u \cdot \Im_x(\sigma \frac{\partial u}{\partial t_i}) dx dt \\
\leq \quad & \int_{\Omega \times I^\tau} \sigma^2(x) (\frac{\partial p(J_x\sigma, t)}{\partial J_x\sigma})^2 u^2 dx dt + \int_{I^\tau} \Big\| \sum_{i=1}^{n} \frac{\partial u}{\partial t_i} \Big\|^2_{B^1_{2,\sigma}(\Omega)} dt,
\end{aligned}
$$

where the second term will be absorbed in the left-hand side of (11).

By virtue of Assumptions A1-A2 and the Poincarré inequality [26]:

$$
\int_a^b u^2(x, .) dx \leq \frac{(b-a)^2}{2} \int_a^b (\frac{\partial u(x, .)}{\partial x})^2 dx + \frac{1}{b-a} \left\{ \int_a^b u(x, .) dx \right\}^2
$$

we obtain

$$\int_{I^\tau} \|P(t)u\|^2_{B^1_2(\Omega)}\,dt + \sum_{i=1}^n \int_{\Delta_i^\tau} \left\|u(.,t^{i,\tau})\right\|^2_{L^2_\sigma(\Omega)}\,dt^i \tag{12}$$
$$\leq \quad c_5 \left\{ \int_{I^\tau} \|f\|^2_{B^1_{2,\sigma}(\Omega)}\,dt + \sum_{i=1}^n \int_{\Delta_i^\tau} \left\|\varphi_i(.,t^i)\right\|^2_{L^2_\sigma(\Omega)}\,dt^i \right\} + c_6 \int_{I^\tau} \|u\|^2_{L^2_\sigma(\Omega)}\,dt$$

respectively

$$\int_{I^\tau} \|u\|^2_{H^1(\Omega)}\,dt + \sum_{i=1}^n \int_{\Delta_i^\tau} \left\|u(.,t^{i,\tau})\right\|^2_{L^2_\sigma(\Omega)}\,dt^i \tag{12'}$$
$$\leq c_5' \left\{ \int_{I^\tau} \|f\|^2_{B^1_{2,\sigma}(\Omega)}\,dt + \sum_{i=1}^n \int_{\Delta_i^\tau} \left\|\varphi_i(.,t^i)\right\|^2_{L^2_\sigma(\Omega)}\,dt^i \right\} + c_6' \int_{I^\tau} \|u\|^2_{L^2_\sigma(\Omega)}\,dt,$$

where

$$c_5 := \frac{\max(1,c_1)}{\min(1,c_0)} \quad \left[c_5' := \frac{\max(1,c_1)}{\min(c_0, \frac{c_0^2}{2}, (\frac{c_0}{b-a})^2)}, \text{ respectively} \right],$$

$$c_6 := \frac{c_4 c_3^2 + nc_2}{\min(1,c_0)} \quad \left[c_6' := \frac{c_4 c_3^2 + nc_2}{\min(c_0, \frac{c_0^2}{2}, (\frac{c_0}{b-a})^2)}, \text{ respectively} \right]$$

and

$$\Delta_i^\tau = (0,\tau_1) \times \ldots \times (0,\tau_{i-1}) \times (0,\tau_{i+1}) \times \ldots \times (0,\tau_n) \ (i = 1,\ldots,n).$$

To eliminate the last term on the right-hand side of the above inequality, we apply Lemma 3.1 by denoting the first integral of the left-hand side by $f_1(\tau)$, the sum of the last integrals on the same side by $f_2(\tau)$, and the first integral and the sum of integrals on Δ_i^τ by $f_3(\tau)$. Consequently, we get

$$\|P(t)u\|^2_{L^2(I^\tau, B^1_2(\Omega))} + \sum_{i=1}^n \|u(\ldots,\tau_i,\ldots)\|^2_{L^2(\Delta_i^\tau, L^2_\sigma(\Omega))} \tag{13}$$
$$\leq \quad c_7 \Big\{ \|f\|^2_{L^2(I, B^1_{2,\sigma}(\Omega))} + \sum_{i=1}^n \|\varphi_i\|^2_{L^2(\Delta_i, L^2_\sigma(\Omega))} \Big\};$$

respectively

$$\|u\|^2_{L^2(I^\tau, H^1(\Omega))} + \sum_{i=1}^n \|u(\ldots,\tau_i,\ldots)\|^2_{L^2(\Delta_i^\tau, L^2_\sigma(\Omega))} \tag{13'}$$
$$\leq c_7' \Big\{ \|f\|^2_{L^2(I, B^1_{2,\sigma}(\Omega))} + \sum_{i=1}^n \|\varphi_i\|^2_{L^2(\Delta_i, L^2_\sigma(\Omega))} \Big\}$$

where $c_7 := c_5 \exp\left\{ nc_6 \sum_{i=1}^n T_i \right\}$ $[c_7' := c_5' \exp\left\{ nc_6' \sum_{i=1}^n T_i \right\}$, respectively].

The right-hand side of (13) [(13'), respectively] is independent of τ_i ($i = 1, \dots, n$). Hence replacing the left-hand side by its upper bound with respect to τ_i from 0 to T_i ($i = 1, \dots, n$), we obtain (10) [(10'), respectively] with $c = c_7^{\frac{1}{2}}$ [$c = {c'_7}^{\frac{1}{2}}$, respectively]. The proof of Theorem 4.1 is complete. $\diamondsuit$

Remark 4.1 We can obtain such estimate for a special case of problem (1)-(3) and (4'), for which $P(t)$ is the Bessel operator, *i.e.*, for which $\sigma(x) := x$, $p(J_x\sigma, t) := x$, $\Omega := (0, b)$, if we take the inner product in $L^2(\Omega \times I)$ of the considered equation with the following integro-differential operator: $Mu = x\sum_{i=1}^n \Im_x \frac{\partial u}{\partial t_i} + x^2 \sum_{i=1}^n \frac{\partial u}{\partial t_i} - x\frac{\partial u}{\partial x}$. $\diamondsuit$

It follows from estimation (10) [(10'), respectively] that there is a bounded inverse operator L^{-1} on the range $R(L)$ of L. However, since we have no information concerning $R(L)$, except that $R(L) \subset F$, we must extend L so that the estimation (10) [(10′), respectively] holds for the extension and its range is the whole space. We first show that $L : B$ [B', respectively]$\to F$, with the domain $D(L)$, has a closure, i.e., the closure of the graph $\Gamma(L) \subset B \times F$ [$B' \times F$, respectively] of L is the graph $\Gamma(\overline{L}) = \overline{\Gamma(L)}$ of the new linear operator $\overline{L}$.

Proposition 4.1 *If the assumptions of Theorem 4.1 are satisfied, then the operator L with domain $D(L)$ is closable.*

Proof. The proof of this proposition is similar to the proof of Proposition 1 in Bouziani [8]. $\diamondsuit$

The following corollaries are immediate consequences of Theorem 4.1 and of Proposition 4.1.

Corollary 4.1 *Suppose that the assumptions of Theorem 4.1 are satisfied. Then there is a constant $c > 0$ such that*

$$\|u\|_B \le c \left\|\overline{L}u\right\|_F \quad \text{for } u \in D(\overline{L}) \tag{14}$$

respectively

$$\|u\|_{B'} \le c \left\|\overline{L}u\right\|_{F'} \quad \text{for } u \in D(\overline{L}). \tag{14'}$$

Corollary 4.2 *If the assumptions of Theorem 4.1 are fulfilled and problem (5)-(8) [(5)-(7) and (8') respectively] has a strongly generalized solution then this solution is unique and depends continuously on $\{f, \varphi_1, \dots, \varphi_n\}$.*

Corollary 4.3 *If the assumptions of Theorem 4.1 are satisfied then $R(\overline{L}) = \overline{R(L)}$ and $\overline{L}^{-1} = \overline{L^{-1}}$, where $R(L)$ and $R(\overline{L})$ denote the set of values taken by L and $\overline{L}$, respectively, and $\overline{L^{-1}}$ is the continuous extension of L^{-1} from $R(L)$ to $\overline{R(L)}$.*

5 Existence of a solution

We are, now, in a position to state and to prove the main result of this paper:

Theorem 5.1 *If the assumptions of Theorem 4.1 are fulfilled then, for any* $f \in L^2(I, B^1_{2,\sigma}(\Omega))$ *and* $\varphi_i \in L^2(\Delta_i, L^2_\sigma(\Omega))$ $(i = 1, \dots, n)$, *problem (5)-(8) [(5)-(7) and (8'), respectively] admits a unique strongly generalized solution* $u = \overline{L}^{-1}\{f, \varphi_1, \dots, \varphi_n\}$ *such that*

$$u \in \prod_{i=1}^{n} C(\overline{I}_i, L^2(\Delta_i, L^2_\sigma(\Omega))), \quad P(t) \in L^2(I, B^1_2(\Omega))$$

$$\left[u \in L^2(I, H^1(\Omega)) \cap \prod_{i=1}^{n} C(\overline{I}_i, L^2(\Delta_i, L^2_\sigma(\Omega))), \ \textit{respectively} \right].$$

Proof. Corollary 4.3 states that to prove the existence of the solution, in the sense of Definition 3.5, for any $\{f, \varphi_1, \dots, \varphi_n\}$, it is sufficient to prove that $R(L)$ does not have an orthogonal complement in F. For this purpose we need:

Proposition 5.1 *If, for some function* $\omega \in L^2(I, B^1_{2,\sigma}(\Omega))$ *and for all* $u \in D(L)$ *with* $\ell_i u = 0$ $(i = 1, \dots, n)$, *we have*

$$(\mathcal{L}u, \omega)_{L^2(I, B^1_{2,\sigma}(\Omega))} = 0 \tag{15}$$

then $\omega \equiv 0$ *almost everywhere in* $\Omega \times I$.

Assume for the moment that Proposition 5.1 has been proved and return to the proof of Theorem 5.1.

Let $W = (\omega, \omega_1, \dots, \omega_n) \in F$ be orthogonal to the set $R(L)$. Consequently,

$$(\mathcal{L}u, \omega)_{L^2(I, B^1_{2,\sigma}(\Omega))} + \sum_{i=1}^{n} (\ell_i u, \omega_i)_{L^2(\Delta_i, L^2_\sigma(\Omega))} = 0, \quad \forall u \in D(L). \tag{16}$$

Assume in (16) that u is any element of $D(L)$ such that $\ell_i u = 0$ $(i = 1, \dots, n)$. We then conclude, by Proposition 5.1, that $\omega \equiv 0$ almost everywhere in $\Omega \times I$. Thus, (16) implies that

$$\sum_{i=1}^{n} (\ell_i u, \omega_i)_{L^2(\Delta_i, L^2_\sigma(\Omega))} = 0, \quad \forall u \in D(L).$$

Since $\ell_i u$ $(i = 1, \dots, n)$ are independent and the ranges of the operators ℓ_i $(i = 1, \dots, n)$ are dense in $L^2(\Delta_i, L^2_\sigma(\Omega))(i = 1, \dots, n)$, it follows that $\omega_i \equiv 0$ $(i = 1, \dots, n)$ almost everywhere in $\Omega \times \Delta_i$ $(i = 1, \dots, n)$. ♢

To complete the proof of Theorem 5.1, it remains to prove Proposition 5.1.

Proof of Proposition 5.1. Equality (15) can be written in the form

$$\sum_{i=1}^{n} \int_I (\frac{\partial u}{\partial t_i}, \omega)_{B^1_{2,\sigma}(\Omega)} dt = \int_I (P(t)u, \sigma\omega)_{B^1_2(\Omega)} dt.$$

In the above formula, we put

$$u = \sum_{i=1}^{n} \Im_{t_i}(e^{k\tau_i}\theta) = \sum_{i=1}^{n} \int_0^{t_i} e^{k\tau_i}\theta d\tau_i \,,$$

where k is a constant such that

$$k \geq \frac{1}{c_0}(c_2 + c_3^2 c_4), \tag{17}$$

θ satisfies conditions

$$\begin{gathered} \theta \in L^2(I, B^1_{2,\sigma}(\Omega)), \\ \sum_{i=1}^{n} \Im_{t_i}(e^{k\tau_i}\theta) \in L^2(I, B^1_{2,\sigma}(\Omega)), \\ p(J_x\sigma, t)(\frac{\partial \Im_{t_i}(e^{k\tau_i}\theta)}{\partial x}) \in L^2(I, L^2(\Omega)), \\ P(t)\Im_{t_i}(e^{k\tau_i}\theta) \in L^2(I, B^1_2(\Omega)), \end{gathered}$$

and u verifies conditions (7), (8) [(7), (8'), respectively]. Consequently, we get

$$\sum_{i=1}^{n} \int_I e^{kt_i}(\theta, \omega)_{B^1_{2,\sigma}(\Omega)} dt = \sum_{i=1}^{n} \int_I (P(t)\Im_{t_i}(e^{k\tau_i}\theta), \sigma\omega)_{B^1_2(\Omega)} dt \tag{18}$$

The left-hand side of (18) shows that the mapping

$$L^2(I, B^1_{2,\sigma}(\Omega)) \ni \theta \to \sum_{i=1}^{n} \int_I (P(t)\Im_{t_i}(e^{k\tau_i}\theta), \sigma\omega)_{B^1_2(\Omega)} dt$$

is a linear continuous functional if the function ω, on the right-hand side of (22), satisfies the following properties:

$$\omega \in L^2(I, B^1_{2,\sigma}(\Omega)), \tag{19}$$

$$\Im_x^2(\sigma\omega) \in L^2(I, L^2(\Omega)), \tag{20}$$

$$\frac{\partial \Im^*_{t_i}(p(J_x\sigma, t)\Im_x(\sigma\omega))}{\partial x} \in L^2(I, L^2(\Omega)), \tag{21}$$

$$\frac{\partial}{\partial x}(\frac{e^{kt_i}}{\sigma(x)} \frac{\partial \Im^*_{t_i}(p(J_x\sigma, t)\Im_x(\sigma\omega))}{\partial x}) \in L^2(I, L^2(\Omega)), \tag{22}$$

and

$$\Im_x^2(\sigma\omega)_{|x=b} = 0, \tag{23}$$

$$\Im_x(\sigma\omega)_{|x=b} = 0, \tag{24}$$

where

$$\Im^*_{t_i}\eta = \int_{t_i}^{T_i} \eta(J_x\sigma, t^{i,\tau}) d\tau_i \ (i = 1, \dots, n)$$

[Here properties (20) and (23) are specific to problem (5)-(8)].

By replacing ω by θ in equality (18), the standard integration by parts of the right-hand side of the obtained equality leads to

$$\begin{aligned}
&\sum_{i=1}^{n}\int_I e^{kt_i}\,\|\theta\|^2_{B^1_{2,\sigma}(\Omega)}\,dt \qquad (25)\\
&= -\frac{1}{2}\sum_{i=1}^{n}\int_{\Omega\times\Delta_i}\sigma(x)e^{-kT_i}p(J_x\sigma,t^{i,T})(\Im_{T_i}(e^{kt_i}\theta))^2dxdt^i\\
&\quad -\frac{1}{2}\sum_{i=1}^{n}\int_{\Omega\times I}\sigma(x)e^{-kt_i}(kp(J_x\sigma,t)-\frac{\partial p(J_x\sigma,t)}{\partial t_i})(\Im_{t_i}(e^{k\tau_i}\theta))^2dxdt\\
&\quad -\sum_{i=1}^{n}\int_{\Omega\times I}\sigma(x)\frac{\partial p(J_x\sigma,t)}{\partial J_x\sigma}\Im_{t_i}(e^{k\tau_i}\theta)\Im_x(\sigma\theta)dxdt.
\end{aligned}$$

Applying the Cauchy inequality to the last integral on the right-hand side of (25), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\int_I e^{kt_i}\,\|\theta\|^2_{B^1_{2,\sigma}(\Omega)}\,dt\\
&= -\frac{1}{2}\sum_{i=1}^{n}\int_{\Omega\times\Delta_i}\sigma(x)e^{-kT_i}p(J_x\sigma,t^{i,T})(\Im_{T_i}(e^{kt_i}\theta))^2dxdt^i\\
&\quad -\frac{1}{2}\sum_{i=1}^{n}\int_{\Omega\times I}\sigma(x)e^{-kt_i}\left\{kp-\frac{\partial p}{\partial t_i}-2\sigma(x)(\frac{\partial p}{\partial J_x\sigma})^2\right\}(\Im_{t_i}(e^{k\tau_i}\theta))^2dxdt.
\end{aligned}$$

Since the integrals on $\Omega\times\Delta_i$ $(i,\dots,n)$ are negative then it follows, by Assumptions A1-A2, that

$$\begin{aligned}
&\sum_{i=1}^{n}\int_I e^{kt_i}\,\|\theta\|^2_{B^1_{2,\sigma}(\Omega)}\,dt\\
&\le -(kc_0-c_2-2c_3^2c_4)\sum_{i=1}^{n}\int_{\Omega\times I}\sigma(x)e^{-kt_i}(\Im_{t_i}(e^{k\tau_i}\theta))^2dx\,dt.
\end{aligned}$$

The above inequality and (17) imply that

$$\sum_{i=1}^{n}\int_I e^{kt_i}\,\|\theta\|^2_{B^1_{2,\sigma}(\Omega)}\,dt\le 0,$$

and thus $\omega\equiv 0$ almost everywhere in $\Omega\times I$. The proof of Theorem 5.1 is complete. ♢

Remark 5.1 Applying a similar argumentation to those given above, our results can be generalized to the following nonlocal pluriparabolic problem:

$$(\mathcal{L}v)(x,t):=\sum_{i=1}^{n}\frac{\partial v}{\partial t_i}-\frac{1}{\sigma(x)}\operatorname{sign}\prod_{i=1}^{n}(1-|\lambda_i|^2)P(t)v=g(x,t)$$

$$
\begin{gathered}
\text{for } (x,t) \in \Omega \times I, \\
(\ell_i v)(x,t) := v(x,t^{i,0}) - \lambda_i v(x,t^{i,T}) = \Phi_i(x,t^i) \\
\text{for } (x,t) \in Q_{i,0}\ (i = 1,\dots,n), \\
\int_\Omega \sigma(x) v(x,t)dx = E(t) \quad \text{for } \ t \in I, \\
\int_\Omega x\sigma(x) v(x,t)dx = G(t) \quad \text{for } t \in I, \\
\left[\frac{\partial v(a,t)}{\partial x} = H(t) \text{ for } t \in I, \text{ respectively} \right].
\end{gathered}
$$

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