Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 22, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Dynamics of polynomial systems at infinity *

Efthimios Kappos

Abstract

The behaviour of dynamics 'at infinity' has not received much attention, even though it was central to Poincaré's analysis of qualitative dynamics. Poincaré's 'sphere' is actually a projective plane and our treatment of dynamics at infinity in more than two dimensions requires the use of $\mathbb{R}P^n$. In control theory, 'strange' transients have been reported by Kokotović and Sussmann, where they go by the name of 'peaking behaviour'. These have a simple explanation when we consider the dynamics on the Poincaré compactification of state space. In this work, we propose to give an analysis of the issues arising in trying to examine the dynamics at infinite radius for dynamical systems in *arbitrary* dimension. Use is made of the Newton polytope and of recent results on principal parts of vector fields.

1 Introduction

The *'behaviour at infinity'* of a dynamical system is crucial to an understanding of its global dynamics. Before the development of the theory of dynamical systems, the qualitative approach of its main pioneer, Henri Poincaré, involved defining dynamics on a compact state space that is in fact the projective plane, see [13]. For a variety of reasons, the subsequent development of dynamical systems paid little attention to the question of interesting, or pathological dynamics *'far away'* (exceptions are references [3, 10, 16] and a few others.) Perhaps because many practical systems are 'dissipative,' attention has focussed on 'local' problems where the theory of normal forms plays a major role. Still, the subject is treated in a limited way in some of the main references, such as the book [1] of the Andronov school, and in Lefschetz [11]. Modern texts completely ignore this aspect, an exception being Perko [12].

Recently, in the context of nonlinear control systems having a certain diagonal structure, the phenomenon of *peaking* was observed which involves a family of trajectories originating arbitrarily close to one of the invariant manifolds of a stable equilibrium point that have arbitrarily large transients (see Section 4.)

* *Mathematics Subject Classifications:* 34C11, 34D23, 37B30, 52B12.
Key words: Dynamics on manifolds, Newton polytopes, dissipative systems, peaking, Poincaré and Bendixson spheres.

Submitted August 16, 2000. Published April 4, 2001.

It therefore seems appropriate to re-examine techniques for systematically analyzing trajectories far away and to re-visit the classical subject of the Poincaré and Bendixson spheres.

In this paper we set up a general method for obtaining dynamics on compact manifolds whose trajectories are almost everywhere in one-to-one correspondence with the trajectories of a flow in Euclidean space. We make an effort to update the classical treatments in [11] and [1] and to go beyond them in several respects.

We mainly consider dynamical systems arising from a vector field defined in euclidean n-dimensional space $\mathbb{R}^n$:

$$\dot{x} = F(x), \tag{1}$$

where $x = (x_1, \dots, x_n)^T$ and $F = (F_1, \dots, F_n)^T$. *We do not assume that the vector field is* **complete**. The main class of vector fields we shall consider is the finitely generated module of **polynomial vector fields** over $\mathbb{R}[x_1, \dots, x_n]$, the ring of polynomials in n variables. We denote by $\deg F_i$ the *total degree* of F_i and use the *multi-index notation*

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \text{so that} \quad \deg x^\alpha = |\alpha| = \sum_i \alpha_i.$$

The standard basis in $T_x\mathbb{R}^n$ will be denoted by $\mathbf{e}_i, i = 1, \dots, n$ (rather than $\frac{\partial}{\partial x_i}$.) We then use the notation

$$x^{\alpha + a\mathbf{e}_i} = x_1^{\alpha_1} \cdots x_i^{\alpha_i + a} \cdots x_n^{\alpha_n}.$$

The notation $(x_1, \dots, \widehat{x_i}, \dots, x_n)$ will denote the $(n-1)$-dimensional array with the ith element x_i omitted.

2 Bendixson one-point compactification

The obvious way to attempt to define dynamics on a compact state space is to use stereographic projection to define a vector field on the one-point compactification of $\mathbb{R}^n$, namely the sphere S^n.

Let us assume that the two n-dimensional manifolds $\mathbb{R}^n$ and S^n are embedded in $\mathbb{R}^{n+1}$ in such a way as to have $\mathbb{R}^n$ be the tangent plane to the sphere S^n at the *'north pole'* $\{Z = 1\}$; the case $n = 2$ helps in the visualization, see Figure 1. We use the coordinates $X_1, \dots, X_n, Z$ in $\mathbb{R}^{n+1}$ and hence for the sphere $S^n = \{\sum X_i^2 + Z^2 = 1\}$. Stereographic projection from the south pole sends points at infinity to the south pole. Now if we give the tangent plane $\{Z = -1\}$ the coordinates $\xi_1, \dots, \xi_n, \zeta$ and use projection from the north pole, we get the change of coordinates

$$\xi_i = \frac{4x_i}{\sum_{j=1}^n x_j^2} \quad \text{and conversely} \quad x_i = \frac{4\xi_i}{\sum_{j=1}^n \xi_j^2} \tag{2}$$

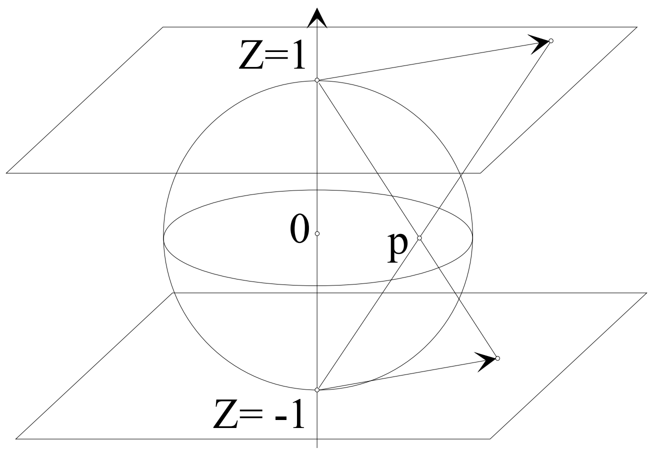

Figure 1: The Bendixson sphere compactification.

The elementary proof of this is given in the Appendix.

Taking the derivative with respect to time of equation 2 gives

$$\dot{\xi}_i = \frac{1}{4}(\sum_{j\neq i}\xi_j^2 - \xi_i^2)F_i - \frac{1}{2}\xi_i\sum_{j\neq i}\xi_j F_j \tag{3}$$

where

$$F_i = F_i(\frac{4\xi}{\sum_k \xi_k^2}).$$

This gives a vector field *away from the point* $\xi = 0$; we shall denote it by G.

We then have that the above transformation gives a one-to-one map between trajectories of the system 1 and the trajectories of the vector field in the complement of the origin in both spaces.

In order to obtain a well-defined system on the sphere, we need to scale the vector field in eq. 3 so that is is defined at the origin $\xi = 0$.

Definition 2.1. *The class* $\mathcal{N} \subset \mathcal{X}(\mathbb{R}^n)$ *of* **normalizable** *dynamics is the subset of the set of vector fields* F *in* $\mathbb{R}^n$ *for which a function* $\rho : \mathbb{R}^n \to \mathbb{R}$ *exists such that, for the transformed vector field* G*, the limit*

$$\lim_{\xi\to 0}\rho(\xi)G(\xi)$$

is defined and the direction fields of ρG *and* G *coincide, where* G *is defined.*

The class of *polynomial vector fields* is normalizable. The normalizing function can be taken to be $\rho(\xi) = R^{2N}$, where R is the norm of the vector ξ, $R^2 = \sum_k \xi_k^2$ and for some positive N, possibly smaller than M, where

$$M = \max_i\{\deg F_i\}.$$

Since the two coordinate patches $U_1 = \{Z > -1\}$ and $U_2 = \{Z < 1\}$ cover the sphere, we have shown that

Proposition 1. *For any normalizable vector field F in $\mathbb{R}^n$, there is defined a direction field in the sphere S^n topologically orbitally equivalent to F on the open subset $\{Z > -1\}$ of S^n.*

Example 1. *An elementary example is the non-complete vector field*

$$\dot{x} = x^2$$

in $\mathbb{R}^1$ with a degenerate equilibrium at the origin.

The dynamics on U_2 are given by

$$\dot{\xi} = -\xi^2(\frac{16}{\xi^2}) = -16$$

which means a vector field on the sphere S^1 with a single *equilibrium point.*

Dissipativeness and Lyapunov functions

The main class of dynamics in $\mathbb{R}^n$ of practical interest is the class of **dissipative dynamics**, i.e. those with a globally asymptotically stable compact attracting set.

Proposition 2. *The system (1) is dissipative iff the point at infinity on the sphere S^n is a repeller.*

It is sometimes (but certainly not always) easier to check the local stability of an equilibrium point rather than to come up with a global Lyapunov function. Thus, the above Proposition can be of practical use. Quite often, though, the point at infinity is a highly degenerate equilibrium, whose stability is hard to establish.

Dissipativeness can be defined by the existence of a global, compact attractor $\mathcal{A}$ or by the existence of a *proper Lyapunov function* strictly decreasing towards the value at the compact set. The quotient flow obtained by collapsing the attractor $\mathcal{A}$ to a point is a gradient-like flow with a single attracting equilibrium; the repeller at infinity is the *complementary repeller* of $\mathcal{A}$ in the terminology of the Conley index (see [8].)

Proof. By basic Conley index theory, the complementary attractor of a repelling equilibrium at the North pole on the sphere is a compact set. Thus the *'if'* direction follows.

Next note that, in the complement of the two poles, the change of coordinates of equation (2) is a diffeomorphism. Thus, the derivative of a Lyapunov function V along the trajectory is the result of evaluating an exact one-form along a vector field, $\frac{dV}{dt} = dV(F(x))$, which is clearly independent of the coordinates chosen. (Note that we are considering the *unscaled* version of the vector field in the patch U_2.) Since away from the compact attractor, we have

$$\frac{dV}{dt} < 0$$

and since V is proper, we get that the south pole is a repeller. $\square$

Let us look at a familiar example.

Example 2 (The Lorenz dynamics).

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= -\beta z + xy\end{aligned} \tag{4}$$

where the parameters σ, ρ, β. *Since the divergence* $\operatorname{div} F = -\sigma - 1 - \beta < 0$, *the attracting set cannot be of dimension three.*

Proposition 3. *There is an increasing sequence of compact sets* K_i *(so* $K_{i+1} \supset K_i$*) such that* $\lim K_i = \mathbb{R}^n$ *and each* K_i *is positively invariant for the flow of the Lorenz system.*

Proof. Lorenz, see [17], Appendix C, uses the function

$$V(x,y,z) = \frac{1}{2}(\rho x^2 + \sigma y^2 + \sigma(z-2\rho)^2)$$

which gives

$$\begin{aligned}&\frac{dV}{dt} = \sigma(-\rho x^2 - y^2 - \beta z^2 + 2\rho\beta z)\\ &= \sigma(-\rho x^2 - y^2 - \beta(z-\rho)^2 + \beta\rho^2)\end{aligned}$$

which is negative as soon as the sum of squares dominates the constant term. The function V is thus a *Lyapunov function* outside a compact set and its *sublevel sets* $\{V(x) \leq k_i\}$ supply the desired compact sets, for appropriate k_i. $\square$

The Lorenz equations are thus dissipative for all *(positive) parameter values and thus can be defined as dynamics on the sphere* S^3. *Also note that the levels of* V *are clearly spheres far away.*

State spaces other than euclidean ones

The natural state space of a dynamical systems is often a manifold. In cases where this manifold is a product of some euclidean space with a compact manifold, the compactification procedure still works, by only compactifying the euclidean summand. The simple pendulum equations,

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\gamma y - \sin x,\end{aligned} \tag{5}$$

for example, live in the space $\mathbb{R} \times S^1$. Here, the one-point compactification gives a two-sphere, S^2. Care must be exercised to take a Lyapunov function that is also x-periodic.

3 Poincaré compactification

The key to the Poincaré compactification is to consider the state space $\mathbb{R}^n$ as the affine plane $\{Z = 1\}$ in $\mathbb{R}^{n+1}$ and to extend the vector field on $\mathbb{R}^n$ to a **direction field** in $\mathbb{R}P^n$ (see Figure 2.) Since a whole $(n-1)$-dimensional space

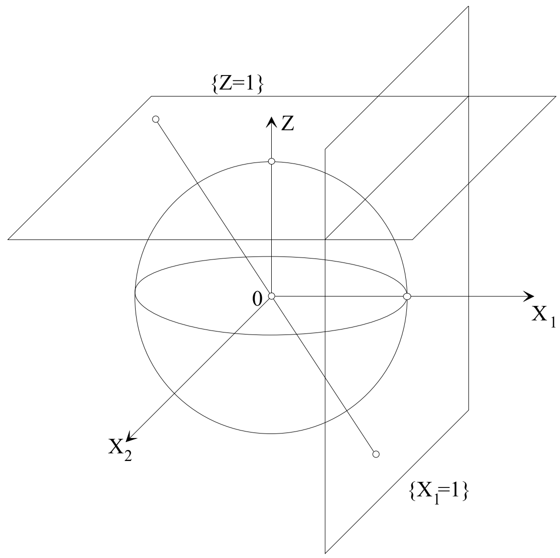

Figure 2: The Poincaré compactification.

of infinities is used, the dynamics at infinity tend to be considerably simpler that for the Bendixson one-point compactification.

Compactifying Dynamics to the Projective space

The affine space $\mathbb{R}^n$ gives a coordinate patch

$$\{Z = 1\}$$

of the projective space $\mathbb{R}P^n$, whose homogeneous coordinates will be written

$$[X_1; \ldots ; X_n; Z].$$

The space

$$\{Z = 0\}$$

provides a collection of *'lines at infinity'* equivalent to $\mathbb{R}P^{n-1}$.

The other n coordinate patches correspond to $\{X_i \neq 0\}$. Let us present the case of the ith coordinate patch, $\{X_i = 1\}$. From the equality of homogeneous coordinates in the overlap,

$$[x_1; \ldots ; x_n; 1] = [X_1; \ldots ; X_{i-1}; Z; X_{i+1}; \ldots ; X_n]$$

we obtain by differentiation the vector field

$$\begin{array}{lll} \dot{X}_1 & = & Z(F_1 - X_1 F_i) \\ \dots & & \dots \\ \dot{X}_{i-1} & = & Z(F_{i-1} - X_{i-1} F_i) \\ \dot{Z} & = & -Z^2 F_i \\ \dot{X}_{i+1} & = & Z(F_{i+1} - X_{i+1} F_i) \\ \dots & & \dots \\ \dot{X}_n & = & Z(F_n - X_n F_i) \end{array} \tag{6}$$

where each vector field component is expressed in the new coordinates

$$\tilde{F}_i(X_1, \dots, Z, \dots X_n) = F_i(\frac{X_1}{Z}, \dots, \frac{X_{i-1}}{Z}, \frac{1}{Z}, \frac{X_{i+1}}{Z}, \dots \frac{X_n}{Z})$$

and is hence a Laurent polynomial. The above equations establish the equivalence of the dynamical systems on the overlap $\{X_i \neq 0, Z \neq 0\}$ of the two coordinate patches in $\mathbb{R}P^n$. As it stands, the dynamical system above is not defined for $Z = 0$. The next step is thus to obtain if possible a well-defined *vector field* in $\mathbb{R}P^n$ from F by some kind of scaling or normalization. In the case of polynomial vector fields, the obvious (and familiar) solution is to multiply the right-hand sides of equation 6 by an appropriate power of Z to obtain a polynomial vector field, call it G_i (see, for example, [12] or [1].) If we scale by an *even* power, Z^{2k}, we say that *the scaling is even* and we can define a vector field in $\mathbb{R}P^n$ by patching the vector fields defined in the $(n+1)$ patches along small neighbourhoods of the codimension-two sets $\{x_i = 1, Z = 1\}$, where the vector fields coincide.

Let us examine the process of transforming the vector field in more detail, with the aim of obtaining information about the *global dynamics* on $\mathbb{R}P^n$ and to point out an important modelling issue motivated by the notion of *genericity* in dynamical systems.

Newton Polytopes and Normalization

We assume $F_i \in \mathbb{R}[x_1, \dots, x_n]$, $1 \leq i \leq n$. We work in the ith coordinate patch $\{X_i = 1\}$.

Let us define the following map for monomials:

$$c_\alpha x^\alpha \mapsto (\alpha, c_\alpha) \in \mathbb{Z}^n \times \mathbb{R}^n. \tag{7}$$

We shall think of the image as a point α on the integer lattice of the first quadrant of $\mathbb{R}^n$, with the coefficient c_α as a label affixed at the point. The map $c_\alpha x^\alpha \mapsto \alpha \in \mathbb{Z}^n$ is the **exponent map**.

Now the change of coordinates between the different affine charts gives an **involution** (a linear transformation A such that $A^2 = I$) in the exponent map domain, given by the matrices

$$A^i = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & \cdots & & \cdots & \\ -1 & -1 & -1 & -1 & -1 \\ & \cdots & & \cdots & \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \tag{8}$$

where $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots x_n)$ is meant to transform to

$$X = (X_1, \dots, X_{i-1}, Z, X_{i+1} \dots, X_n).$$

Thus the monomial $c_\alpha x^\alpha$ is mapped to $c_\alpha X^{A\alpha}$. For example, $3x_1^2 x_2^4 x_3$ maps, in the $\{X_2 = 1\}$ patch, to $3X_1^2 Z^{-7} X_3$. It is clear that only Z appears with a non-positive exponent, namely $-|\alpha|$.

Applying the exponent map to each of the monomials of

$$x_1 \cdots \widehat{x}_j \cdots x_n F_j(x_1, \dots, x_n)$$

(F_j polynomial) we get the *support* of F_j, $\operatorname{supp} F_j$, of the non-zero alphas. The **Newton polytope Γ of the polynomial vector field** F is the convex hull of $\cup_j \operatorname{supp} F_j$. Clearly, Γ is a compact convex subset of the *first quadrant* $\{x_i \geq 0\ ;\ \forall i\}$. The shifting involved in this definition (see [4]) is special to vector fields; for a polynomial, one uses the exponent map directly; Koushnirenko [9] has given definitions of Newton polytopes for power series and for Laurent polynomials as well; these are not needed here. Even though it clearly depends on the chosen coordinates, the Newton polytope of a polynomial p contains a surprising amount of information about the singularities of p (see Arnol'd et.al. [2].)

A *support hyperplane* of Γ is a hyperplane maximizing the value of some one-form β on Γ.

The *facets* γ of the boundary of the Newton polytope of a vector field F are intersections of Γ with a supporting hyperplane; they are compact, convex polytopes of dimension at most $n-1$. The union of the facets whose support hyperplane co-vectors have negative entries form the **Newton diagram** $\mathcal{N}$ of the vector field F. The restrictions $F_\gamma = \sum_{\alpha \in \gamma} c_\alpha x^\alpha$ are called the **quasi-homogeneous components** of F. Lastly, we use the fact that a linear transformation A of vectors gives a transformation by the inverse A^{-1} for *co-vectors* and the fact that the matrices A above are involutions to obtain the transformation βA for the co-vectors of the supporting hyperplanes.

Proposition 4. *Let Γ be the Newton polytope of the vector field F.*

In the ith patch, the Newton polytope of the transformed vector field of equation 6 is exactly equal to the affine transformation of Γ by A^i, followed by a shift in the ith direction (found from the maximal degree of the monomials in F.)

Hence, the **Newton diagram** *of the transformed vector field is the transform of the union of facets of F with support covectors β such that $\beta A^i < 0$.*

Note the convenience of the above Proposition in being able to check the single Newton polytope of F, instead of computing all the transformed ones.

The notion of a **principal part** of a vector field at an equilibrium (the terms of the vector field mapping to the Newton diagram) is crucial to the generalization of the classical Grobman-Hartman Theorem by Brunella and Miari [4]. Vector fields with the same principal parts have locally equivalent dynamics. A condition that makes the principal part concept useful is the absence of dynamics of the centre-focus type (roughly, in the plane, we need a trajectory tending to the equilibrium at a well-defined angle.) We are interested in finding the *principal parts at infinity* of the vector field F. We assume the origin is an equilibrium of the transformed vector field.

Corollary 1. *In dimension two, assume that, in the ith patch, the origin is a nondegenerate equilibrium, in the sense of [4]. Then the vector field G_i is topologically equivalent to its A^i-transformed (and appropriately shifted) principal part modulo centre-focus.*

Remark 1. *Computing convex hulls is a classical problem in Computational Geometry ([14], [6].) In dimension two, it is even implemented in software such as* `maple` *and* `matlab`*.*

Proof. Let $\mathbf{1}$ be a vector of ones and use (α, k) for the exponent of the monomial $x^\alpha \mathbf{e}_k$ in the kth component of a vector field. Let MNP stand for the modified Newton polytope map which, using this notation, is defined by

$$x^\alpha \mathbf{e}_k \mapsto x^{\alpha+\mathbf{1}-\mathbf{e}_k} e_k$$

or

$$(\alpha, k) \mapsto (\alpha + \mathbf{1} - \mathbf{e}_k, k).$$

Now the vector field in the ith patch defined in equation 6 maps (α, k) to $(A\alpha, k)$; to get the MNP, we distinguish the two cases: $k = i$ and $k \neq i$. Since we shall later normalize by a power of Z, we ignore the factor Z common to all components.

In the former case, we get

$$(\alpha, k) \mapsto (A\alpha + \mathbf{1} - \mathbf{e}_k, k).$$

For $k = i$, we get the monomial (α, i) contributing to both the ith and the kth component of the vector field, in the first case giving

$$(\alpha, i) \mapsto (A\alpha + \mathbf{1} - \mathbf{e}_i + \mathbf{e}_i, i) = (A\alpha + \mathbf{1}, i)$$

(because we multiply by Z) and, in the second case

$$(\alpha, i) \mapsto (A\alpha + \mathbf{1} - \mathbf{e}_k + \mathbf{e}_k, k) = (A\alpha + \mathbf{1}, k)$$

(because of the X_k multiplying F_i in equation 6.)

It is easy to check that $A\mathbf{1} = \mathbf{1} - (n+1)\mathbf{e}_i$, $A\mathbf{e}_k = \mathbf{e}_k - \mathbf{e}_i$ for $k \neq i$ and $A\mathbf{e}_i = -\mathbf{e}_i$. We now have that

$$A(\alpha + \mathbf{1} - \mathbf{e}_k) = A\alpha + \mathbf{1} - \mathbf{e}_k - n\mathbf{e}_i, k$$

and

$$A(\alpha + \mathbf{1}) = A\alpha + \mathbf{1} - n\mathbf{e}_i$$

and hence the involution A maps the MNP of F to the MNP of the transformed vector field, except for the shift by $n\mathbf{e}_i$, which is immaterial, since we are going to scale anyway by a power of Z.

The proposition now follows from the transformation rule for covectors, under the stated conditions. $\square$

The Corollary is immediate from the results of Brunella and Miari.

Just as it has now become common to expect *local dynamics* to be of *low codimension*, we can require the dynamics **at infinity** to be of low codimension as well. The results of Brunella et.al. can be combined with the above setting to examine when the principal parts of vector fields at infinity are generic. The details are left to an extended version of this work.

4 Examples

Gradient dynamics with two finite minima

The examination of relations between properties of a polynomial, such as its degree, and the number and nature of its critical points is an interesting and non-trivial problem. It turns out that to do the counting properly, one needs a definition of **critical points at infinity** for functions $f : \mathbb{R}^n \to \mathbb{R}$ ([7].) Durfee gives *five* different definitions, which he then shows to be equivalent.

Through our dynamical viewpoint, we approach this question via the *gradient vector field* obtained from the given function.

Let us take a concrete example (adapted from [5].) It is the polynomial

$$f(x, y) = (x^2y - x - 1)^2 + (x^2 - 1)^2$$

which is easily seen to have just two (local) **minima**, at $(-1, 0)$ and $(1, 2)$, and **no other (finite) critical points**! In terms of the gradient flow

$$-\nabla f(x, y),$$

the gradient dynamics has two attractors and no other equilibria. We shall examine the **global phase portrait** of this system obtained from the Poincaré compactification we have described. Clearly, on the compact state space $\mathbb{R}P^2$, we must have more equilibrium points, by basic Morse theory.

The phase portrait of the system dynamics $\dot{x} = -\nabla f(x)$ is shown in Figure 3. The Newton polytope of the gradient vector field is shown in Figure 4.

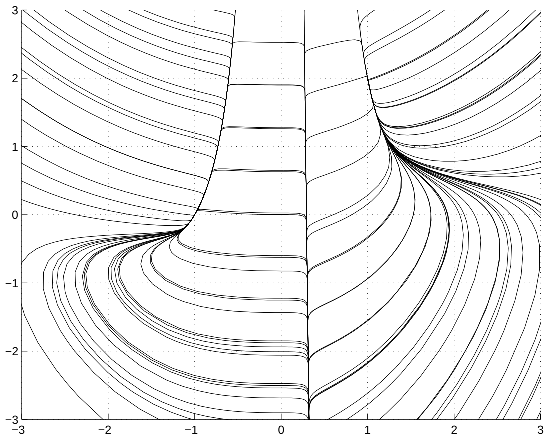

Figure 3: Phase portrait of the two-minimum system.

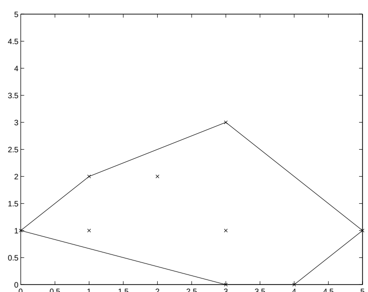

Figure 4: Newton polygon of the two-minimum example.

In the y-patch, $(x, y) \mapsto (\frac{X}{Z}, \frac{1}{Z})$, we find that we have three additional equilibria, at $(\pm\sqrt{2}, 0)$ and $(0, 0)$. The pair of equilibria at $(\pm\sqrt{2}, 0)$ are repellers, while the one at the origin is degenerate (so that the Corollary is not applicable to it.) The instability can be checked by effecting the shift $(X, Z) \mapsto (X \pm 1, Z)$ on the Newton polygon, obtaining the polygon shown in Figure 5, and checking that the equilibrium is nondegenerate with unstable linear principal part. The three 'asymptotic' curves visible in the phase portrait of Figure 3 become *unstable* (for the two outer curves) and *stable* (for the middle one) *manifolds* of the degenerate equilibrium (on the positive-Z side.) On the other side (as $y \to -\infty$) there is a single unstable curve. It appears from the simulations that there is then a connecting orbit (homoclinic in projective space) from the origin to itself (in the $\{y = 1\}$-patch.) Its existence has not been shown here, however.

The X-axis is invariant, with dynamics

$$\dot{X} = X^2(X^2 - 1).$$

Notice that the two systems can be 'sewn together' along the line $\{y = 1\} =$

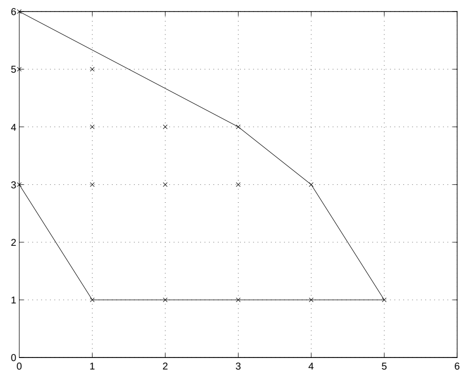

Figure 5: Transformed Newton polygon centred at $X = 1$.

$\{Z = 1\}$. The scaling we use is even, so the phase portrait at the y-infinity is patched to the finite phase portrait without a sign change.

In the x-patch, $(x, y) \mapsto (\frac{1}{Z}, \frac{Y}{Z})$, there are three additional equilibrium points, one at the origin and two at $(0, \pm\frac{1}{\sqrt{2}})$.

Peaking behaviour

The following example demonstrates **peaking behaviour** in a so-called upper triangular system, where the diagonal systems are both linear and asymptotically stable.

$$\begin{aligned} \dot{x} &= -x + x^2 y \\ \dot{y} &= -ky, \; k > 1 \end{aligned} \tag{9}$$

The origin is linearly stable and hence locally stable. The problem is that the quadratic term x^2y prevents some trajectories from converging to zero fast enough. In fact, for any bound K, there is a trajectory whose ω-limit set is 0 and whose distance from the y axis tends to zero as $t \to -\infty$, but such that its x coordinate exceeds K for some intermediate time.

Now the dynamics on the $\{X = 1\}$ plane, after scaling by Z^3, are given by

$$\begin{aligned} \dot{Z} &= Z(Z^2 - Y) \\ \dot{Y} &= -Y((k-1)Z^2 + Y), \end{aligned} \tag{10}$$

giving a degenerate equilibrium at zero. It is easily checked that the Z-axis is invariant and unstable and that the Y-axis is also invariant, with dynamics $\dot{Y} = -Y^2$.

In fact, the parabola

$$Y = (k+1)Z^2$$

is also invariant and stable and thus the equilibrium point exhibits a mixed saddle-stable-unstable dynamical behaviour. This parabola is of course the image of an invariant hyperbola in the original plane $\{Z = 1\}$, see Figure 6.

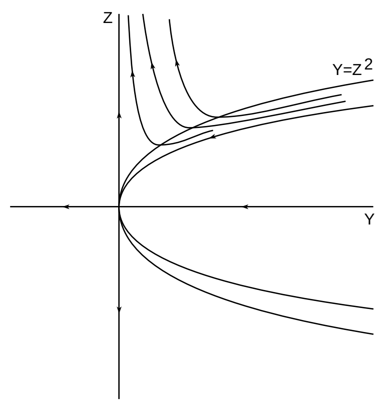

Figure 6: Phase portrait in the $\{x = 1\}$ plane

The saddle-like *'sector'* is thus responsible for the peaking behaviour. The full details of the phase portrait on $\mathbb{R}P^2$ are not difficult to obtain, but we omit them here. It is also possible to generalize this peaking example by taking *cross-terms* more general than x^2y. Details will be given elsewhere.

5 Conclusion

We have presented but the bare elements of a theory of global (polynomial) dynamics, combining a generalization of the classical Poincaré compactification with the powerful Newton polytope method, so useful in singularity theory and algebraic geometry. We have not touched on the topological information provided by the Whitney-Morse theory of relations between the topology of the state manifold and the indices of the equilibria of the vector field on it.

As the peaking example shows, a study of the compactified dynamics is sometimes necessary to clarify apparently strange transient dynamical behaviour. The two-minimum example shows that, even within the class of polynomial systems, expectations on the dynamics based on the intuition derived from compact state manifolds are occasionally wrong (two minima and no saddles). Compactification can resolve these ambiguities. It is clear that more examples need to be studied and that the genericity aspects must be more extensively addressed.

6 Appendix

Here we derive Equation (2). In $\mathbb{R}^{n+1}$, write $\mathbf{v} = (\mathbf{x}, Z)$, with $\mathbf{x} \in \mathbb{R}^n$ and $Z \in \mathbb{R}$. The unit sphere is

$$S^n = \{\mathbf{v}; |\mathbf{v}| = \sqrt{|\mathbf{x}| + Z^2} = 1\}$$

and the hyperplanes tangent to the North and South poles are

$$\mathcal{P}_N = \{\mathbf{v}; Z = 1\} \text{ and } \mathcal{P}_S = \{\mathbf{v}; Z = -1\}.$$

The stereographic projection from the South pole sends the point $\mathbf{v}$ to the point, $\mathbf{p}_N$, of intersection of S^n with the line

$$\ell_N = \{t\mathbf{v} + (1-t)(-\mathbf{e}_{n+1}), t \in \mathbb{R}\} = \{t(\mathbf{x}, 0) + (2t-1)\mathbf{e}_{n+1}, t \in \mathbb{R}\},$$

where $\mathbf{e}_{n+1}$ is the unit vector in the Z-direction. We thus have

$$|t(\mathbf{x}, 0) + (2t-1)\mathbf{e}_{n+1}|^2 = t^2|\mathbf{x}|^2 + (2t-1)^2|\mathbf{e}_{n+1}|^2 = 1,$$

which has the non-trivial solution

$$t = \frac{4}{4+\mathbf{r}^2},\ \mathbf{r} = |\mathbf{x}|.$$

Repeating for the projection from the North pole, we get a point $\mathbf{p}_S$ satisfying

$$|s(\xi, 0) - (2s-1)\mathbf{e}_{n+1}|^2 = s^2|\xi|^2 + (2s-1)^2|\mathbf{e}_{n+1}|^2 = 1,$$

giving

$$s = \frac{4}{4+\mathbf{r}'^2},\ \mathbf{r}' = |\xi|.$$

The change of coordinates means that

$$\begin{aligned} \mathbf{p}_n = \frac{4}{4+\mathbf{r}^2}(\mathbf{x}, 0) + (\frac{8}{4+\mathbf{r}^2} - 1)\mathbf{e}_{n+1} = \\ = \frac{4}{4+\mathbf{r}'^2}(\xi, 0) - (\frac{8}{4+\mathbf{r}'^2} - 1)\mathbf{e}_{n+1} = \mathbf{p}_S \end{aligned}$$

and, equating the Z-components, we check that $\mathbf{r}\mathbf{r}' = 4$ and therefore

$$\mathbf{x} = \frac{4+\mathbf{r}^2}{4+\mathbf{r}'^2}\xi = \frac{4}{\mathbf{r}'^2}\xi.$$

References

[1] A. A. Andronov et. al., *Qualitative Theory of Second-Order Dynamical Systems*, Israel Program for Scientific Translations (1973).

[2] V.A. Arnol'd et. al., *Dynamical Systems V: Singularity Theory*, Enc. of Mathematics, Springer (1996).

[3] V. H. Baširov, The Poincaré sphere, (in Russian) *Collection of articles dedicated to the memory of Mazit Ifatovič Al'muhamedov. Mat. Sb.* **16** 45–51 (1973).

[4] M. Brunella and M. Miari, Topological equivalence of a plane vector field with its principal part defined through Newton polyhedra, *J. Differential Equations*, **85**(2) 338–366 (1990)

[5] R. Davies, Problems, *Mathematics Magazine*, **61**, 1988, p.59.

[6] M. de Berg et.al., *Computational Geometry: Algorithms and Applications*, Springer, 1997.

[7] A. Durfee et.al., Counting Critical Points of Real Polynomials in Two Variables, *Am. Math Mo.*, **100**-3, 1993, pp.255–271.

[8] E. Kappos, The Conley index and global bifurcations, Parts I and II, *Int.J.Bif. and Chaos*, **5**–4, 1995 and **6**–12B, 1996.

[9] A.G. Koushnirenko, Polyhèdres de Newton et nombres de Milnor, *Inv. Math.* **32** 1–31 (1976).

[10] H. R. Latipov and Š. R. Šaripov, A study of the characteristics of the equation $\frac{dy}{dx} = \frac{b_{10}x+b_{01}y+Q_3(x,y)}{a_{10}x+a_{01}y+P_3(x,y)}$ on the Poincaré sphere, (in Russian) *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* **3** 13–17 (1963).

[11] S. Lefschetz, *Differential Equations: Geometric Theory*, 2nd ed., Interscience (1963).

[12] L. Perko, *Differential Equations and Dynamical Systems*, Springer (1991).

[13] H. Poincaré, Mémoire sur les courbes definis par des équations différentielles. In *Œvres*, tom I, 3-84.

[14] F. P. Preparata and M. I. Shamos, *Computational Geometry*, Springer, 1985.

[15] H. Röhrl and S. Walcher, Projections of polynomial vector fields and the Poincaré sphere, *J. Differential Equations* **139**(1) 22–40 (1997).

[16] Š. R. Šaripov, The distribution of singular points on the equator of the Poincaré sphere, (in Russian) *Trudy Samarkand. Gos. Univ. (N.S.) Vyp.* **144** 89–92 (1964).

[17] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors*, Springer (1982).

Efthimios Kappos
Department of Applied Mathematics
University of Sheffield, Sheffield, S3 7RH, U.K.
e-mail: E.Kappos@shef.ac.uk