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Approximate equivalence transformations and invariant solutions of a perturbed nonlinear wave equation *

R. N. Ibragimov

Abstract

We discuss the properties of a perturbed nonlinear wave equation whose coefficients depend on the first-order spatial derivatives. In particular, we obtain a group of transformations which are stable with respect to the given perturbation, and derive the principal Lie algebra and its approximate equivalence transformation. The extension of the principal Lie algebra by one is obtained by means of a well-known classification of low dimensional Lie algebras. We also obtain some invariant solutions and classification of the perturbed equation.

1 Introduction

We consider the nonlinear wave equation

$$H(u) \equiv D\langle u\rangle - \varepsilon f(\frac{\partial u}{\partial x}) = 0, \tag{1}$$

where $D = \partial_{t^2}^2 - a\exp\{\partial_x u\}\,\partial_x^2$ is a nonlinear operator, a is constant and εf is an infinitesimal perturbation imposed on the principal part $D\langle u\rangle = 0$. For the sake of simplicity and without loss of generality, we put $a = 1$.

The classification problem of a family of equations involves the determination of the principal algebra $L_\wp$, the equivalence algebra $E_\wp$ and extension of $L_\wp$ by subalgebras of $E_\wp$ to divide the family into disjoint classes. In a recent paper [11], the nonlinear wave equation $u_{tt} = f(x, u_x)u_{xx} + g(x, u_x)$ was partially classified into thirty-three classes of equations and one of them is given there as the family of the form of (1).

The main goal of this paper is to find invariant solutions to (1). Since the perturbation destroys the group of transformations admitted by the principal part, the analysis is rather difficult. However we shall construct symmetries

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which are stable with respect to the perturbation. More specifically, we shall find a mapping $\mathbf{F} : (x, t; u) \to (x, t; u')$ which transforms (1) into

$$\frac{\partial^2 u'}{\partial t^2} - \frac{\partial^2 u'}{\partial x'^2} \exp\left\{\frac{\partial u'}{\partial x'}\right\} - \varepsilon f'(\frac{\partial u'}{\partial x'}) + o(\varepsilon) = 0, \tag{2}$$

i.e., the form of (1) is unchanged. Further we construct the principal Lie algebra which enables us to classify (1). As the result of Lie group classification, we determine the unknown function f and, thus, find invariant solutions admitted by (1).

The approximate method of group analysis, used in this paper, was developed first by Ovsyannikov [7] while the problem of group classification of partial differential equations according to their symmetries was first considered by Sophus Lie [4]. The general approach to finding the symmetry group of differential equations can be found, for example, in [2], [5], [8].

Recently, several papers, which are closely related to the present work, were published. To name a few, Ames et al [1] investigated the group properties of quasilinear hyperbolic equation of the form $u_{tt} = f(u_x)u_{xx}$. The investigation was later generalized by Torrisi at [9], [10] to equation of the form $u_{tt} = f(x, u_x)u_{xx}$.

2 Group Classification

We wish to find the approximate equivalence transformations for (1). In this case, a natural modification of equivalence transformation that involves approximate transformations (as in [3]) is used. Since an equivalence transform is a nongenerate change of variables x, t and u which transform (1) to the same form as (2) (generally with different function $f(u_x)$), we apply the Lie infinitesimal method to calculate the group of equivalence transformations of the system

$$\begin{gathered} D\langle u\rangle - \varepsilon f = o(\varepsilon), \\ f_x = f_t = f_u = f_{u_t} = \varepsilon^{-1} o(\varepsilon) \end{gathered} \tag{3}$$

and suppose that the operator for approximate transformation groups be given in the form

$$\mathbf{X} = (\xi^0 + \varepsilon\xi^1)\partial_x + (\tau^0 + \varepsilon\tau^1)\partial_t + (\eta^0 + \varepsilon\eta^1)\partial_u + \varphi\partial_f,$$

where $\xi^\nu, \tau^\nu, \eta^\nu$ $(\nu = 0, 1)$ are functions of t, x and u while φ depends on variables t, x, u, u_x, u_t and f. Thus we rewrite the generator of the group as $\mathbf{X} = \mathbf{X}^0 + \varepsilon\mathbf{X}^1$, where $\mathbf{X}^0$ is a stable symmetry if it is admitted by unperturbed equation $D\langle u\rangle = 0$. According to [3], we call the corresponding symmetry generator $\mathbf{X}$ a deformation of the operator $\mathbf{X}^0$ which generates the elements of the principal Lie algebra $L_\wp$ for $D\langle u\rangle = 0$.

In the extended space with variables $(x, t, u, u_x, u_t, \dots)$, the second prolongation of operator $\mathbf{X}$ is

$$\mathbf{X}^{[2]} = \mathbf{X} + (\zeta_0^{(x)} + \zeta_1^{(x)})\partial_{u_x} + (\zeta_0^{(t)} + \zeta_1^{(t)})\partial_{u_t} + (\zeta_0^{(xx)} + \zeta_1^{(xx)})\partial_{u_{xx}} + (\zeta_0^{(tt)} + \zeta_1^{(tt)})\partial_{u_{tt}} + \psi_x\partial_{f_x} + \psi_t\partial_{f_t} + \psi_u\partial_{f_u} + \psi_{u_t}\partial_{f_{u_t}} + \dots$$

Here the following notation is used ($\nu = 0, 1$ and $\theta \in \{x, t, u, u_t\}$):

$$\begin{aligned} \zeta_\nu^{(x)} &= \eta_x^\nu + u_x(\eta_u^\nu - \partial_x\xi^\nu - u_x\partial_u\xi^\nu) - u_t(\partial_x\tau^\nu + u_x\partial_u\tau^\nu), \\ \zeta_\nu^{(t)} &= \eta_t^\nu + u_t(\eta_u^\nu - \partial_t\tau^\nu - u_t\partial_u\tau^\nu) - u_x(\partial_t\xi^\nu + u_t\partial_u\xi^\nu), \end{aligned} \tag{4}$$

$$\begin{aligned} \zeta_\nu^{(xx)} = \; & \eta_{xx}^\nu + (2\eta_{xu}^\nu - \xi_{xx}^\nu)u_x - \tau_{xx}^\nu u_t + (\eta_u^\nu - 2\xi_x^\nu)u_{xx} \\ & -2\tau_x^\nu u_{xt} + (\eta_{uu}^\nu - 2\xi_{xu}^\nu)u_x^2 - 2\tau_{xu}^\nu u_t u_x - \xi_{uu}^\nu u_x^3 - \tau_{uu}^\nu u_t u_x^2 \\ & -3\xi_u^\nu u_x u_{xx} - \tau_u^\nu u_t u_{xx} - 2\tau_u^\nu u_x u_{xt}, \end{aligned}$$

$$\begin{aligned} \zeta_\nu^{(tt)} = \; & \eta_{tt}^\nu + (2\eta_{tu}^\nu - \tau_{tt}^\nu)u_t - \xi_{tt}^\nu u_x + (\eta_u^\nu - 2\tau\xi_t^\nu)u_{tt} \\ & -2\xi_t^\nu u_{xt} + (\eta_{uu}^\nu - 2\tau_{tu}^\nu)u_t^2 - 2\xi_{tu}^\nu u_t u_x - \tau_{uu}^\nu u_t^3 - \xi_{uu}^\nu u_t^2 u_x \\ & -3\tau_u^\nu u_t u_{tt} - \xi_u^\nu u_x u_{tt} - 2\xi_u^\nu u_t u_{xt}, \end{aligned}$$

$$\begin{aligned} \psi_\theta = \; & d_\theta^* \langle\varphi\rangle - f_t d_\theta^* \langle\tau^0 + \varepsilon\tau^1\rangle - f_x d_\theta^* \langle\xi^0 + \varepsilon\xi^1\rangle - f_u d_\theta^* \langle\eta^0 + \varepsilon\eta^1\rangle \\ & + f_{u_t} d_\theta^* \left\langle\zeta_0^{(t)} + \varepsilon\zeta_1^{(t)}\right\rangle + f_{u_x} d_\theta^* \left\langle\zeta_0^{(x)} + \varepsilon\zeta_1^{(x)}\right\rangle, \end{aligned}$$

where

$$d_\theta^* = \partial_\theta + f_\theta\partial_f + f_{\theta t}\partial_{f_t} + f_{\theta x}\partial_{f_x} + f_{\theta u}\partial_{f_u} + f_{\theta u_t}\partial_{f_{u_t}} + \dots.$$

Since $f_\theta = 0 \; \forall\theta \in \{x, t, u, u_t\}$, i.e., $d_\theta^* = \partial_\theta$, the infinitesimal invariance criterion for system (3) becomes

$$\mathbf{X}^{[2]}\left[H(u)\right]|_{(M)} = o(\varepsilon), \tag{5}$$

$$\mathbf{X}^{[2]}(\varepsilon f_\theta) = o(\varepsilon), \tag{6}$$

where the symbol $|_{(M)}$means evaluated on the manifold M, defined by (1).

In zero-order approximation ($\varepsilon = 0$), system (5)–(6) yields the system of determining equations in the form

$$\zeta_0^{(tt)} - \exp\{u_x\}(\zeta_0^{(x)}u_{xx} + \zeta_0^{(xx)}) = 0, \quad \varepsilon\psi_\theta = 0 \tag{7}$$

which gives $\varphi_\theta = 0$ and $(\zeta_0^{(x)})_\theta = 0$ ($\forall\theta \in \{x, t, u, u_t\}$) since f is a differential variable which is algebraically independent from f_{u_x}. Thus φ is the function of u_x and f only. Consequently, differentiation of (4) with respect to x and splitting it into independent parts, yields

$$\eta_{xx}^0 = \tau_{xx}^0 = \tau_{xu}^0 = \eta_{xu}^0 - \xi_{xx}^0 = \xi_{xu}^0 = 0.$$

In a way similar to the above, we derive the following equations

$$\begin{gathered}\eta^0_{xt} = \tau^0_{xt} = \tau^0_{tu} = \eta^0_{tu} - \xi^0_{xt} = \xi^0_{tu} = 0,\\ \eta^0_{xu} = \eta^0_{uu} - \xi^0_{xu} = \tau^0_{xu} = \tau^0_{uu} = \xi^0_{uu} = 0,\\ \tau^0_x = \tau^0_u = 0\end{gathered}$$

from which we find

$$\begin{gathered}\xi^0 = a_1 u + \alpha_1(t)x + \alpha_2(t),\\ \tau^0 = \tau^0(t),\\ \eta^0 = \beta_1(t)u + a_2 x + \beta_2(t),\end{gathered}$$

where $a_1$and a_2 are constant coefficients. Similarly, splitting the first of (7) we obtain

$$\begin{gathered}\xi^0 = (c_5 + c_6)x + c_2,\\ \tau^0 = c_5 t + c_1,\\ \eta^0 = 2c_6 x + (c_5 + c_6)u + c_4 t + c_3\end{gathered}$$

with constants $c_1, \dots, c_6$. Thus the unperturbed equation (1) is stable with respect to the group G^0 of the transformations defined by the following generators:

$$\begin{gathered}\mathbf{X}^0_1 = \partial_t, \quad \mathbf{X}^0_2 = \partial_x, \quad \mathbf{X}^0_3 = \partial_u, \quad \mathbf{X}^0_4 = t\partial_u,\\ \mathbf{X}^0_5 = t\partial_t + x\partial_x + u\partial_u, \quad \mathbf{X}^0_6 = x\partial_x + (u + 2x)\partial_u.\end{gathered}$$

So equation $D\langle u\rangle = 0$ is invariant with respect to a group G^0, i.e., the unperturbed nonlinear wave equation admits G^0 whenever u solves that equation. Note that if one rewrites (5) in the form $(\xi \cdot \nabla H) \mid_{(M)}= 0$, it becomes evident that (5) is the condition for the vector field $\xi = (\xi, \tau)$ to be tangent to the manifold M.

In a way similar to the above, we write the invariance condition for (1) up to the first order where $M = \left\{u:\ \partial_t^2 u = \exp\{\partial_x u\}\, \partial_x^2 u + \varepsilon f\right\}$. After splitting of the determinant equation

$$Z(u) - \varepsilon\varphi = 0,$$

where we denote

$$Z(u) = (\zeta_0^{(tt)} + \varepsilon\zeta_1^{(tt)}) - (\zeta_0^{(x)} + \varepsilon\zeta_1^{(x)})u_{xx}\exp\{u_x\} - (\zeta_0^{(xx)} + \varepsilon\zeta_1^{(xx)})\exp\{u_x\}$$

into independent parts and solving the resulting equations, we obtain

$$\begin{gathered}\tau^1 = a_1 t + a_2, \quad \xi^1 = a_3 x + a_4,\\ \eta^1 = a_3 u + 2(a_1 - a_3)x + \frac{a_5 t^2}{2} + a_6 t + a_7,\\ \varphi = (c_1 - 2c_3)f + a_5\end{gathered}$$

with constants $a_1, \dots, a_7$. The last equations together with the group G^0 generate 13 dimensional approximate Lie algebra of approximate equivalence transformations G^1spanned by generators

$$\begin{gathered}
\mathbf{X}_1^1 = \partial_t, \quad \mathbf{X}_2^1 = \partial_x, \quad \mathbf{X}_3^1 = \partial_u, \quad \mathbf{X}_4^1 = t\partial_u, \\
\mathbf{X}_5^1 = t\partial_t + x\partial_x + u\partial_u - f\partial_f, \quad \mathbf{X}_6^1 = x\partial_x + (u+2x)\partial_u + f\partial_f, \\
\mathbf{X}_7^1 = \varepsilon\partial_t, \quad \mathbf{X}_8^1 = \varepsilon\partial_x, \quad \mathbf{X}_9^1 = \varepsilon\partial_u, \quad \mathbf{X}_{10}^1 = \varepsilon t\partial_u, \\
\mathbf{X}_{11}^1 = \varepsilon(t\partial_t + 2x\partial_x), \quad \mathbf{X}_{12}^1 = \varepsilon(x\partial_x + [u-2x]\,\partial_u), \quad \mathbf{X}_{13}^1 = \varepsilon(t^2\partial_u + 2\partial_f).
\end{gathered}$$

It is sufficient, for group classification, to consider the point approximate equivalence transformations corresponding to nontrivial generators $\mathbf{X}_5^1, \mathbf{X}_6^1$ and $\mathbf{X}_{13}^1$. These transformations are given by

$$\begin{gathered}
x' = a_1a_2x, \quad t' = a_1t, \quad f' = \varepsilon(2a_1a_2a_3 + a_1a_2f), \\
u' = \varepsilon(a_1a_2a_3t^2 - 2a_1(a_2-1)x + a_1a_2u).
\end{gathered}$$

To find the principal Lie algebra $L_\wp$ for (1) and to find those functions f for which $L_\wp$ is extended, we seek the admitted operator in the form $\mathbf{Y} = \mathbf{X} - \varphi\partial_f$. The invariance condition for (1)

$$\mathbf{Y}^{(2)}\,[H(u)]\,|_{(M)} = 0$$

yields the determining equation as

$$Z(u) - \varepsilon f'(\zeta_0^{(x)} + \varepsilon\zeta_1^{(x)}) = 0.$$

For the zero order approximation we obtain a similar result as above whereas for the first order approximation the determining equation takes the form

$$(\eta_u^0 - 2\tau_t^0)f + \zeta_1^{(tt)} - \exp\{u_x\}\,(u_{xx_1}\zeta_1^{(xx)} + \zeta_1^{(x)}) - f'\zeta_x^0 = 0. \tag{8}$$

Substituting $u_{tt} = \exp\{u_x\}\,u_{xx}$ and considering arbitrary f we split (8) into independent parts to obtain

$$\begin{gathered}
\xi^1 = a_1x + a_2, \quad \tau^1 = a_3t + a_4, \\
\eta^1 = a_1u + 2(a_1 - a_2)x + a_5t + a_6.
\end{gathered}$$

Thus the principal 10 dimensional Lie algebra $L_\wp$ has the basis

$$\begin{gathered}
\mathbf{Y}_1 = \partial_u, \quad \mathbf{Y}_2 = \partial_x, \quad \mathbf{Y}_3 = \partial_t, \quad \mathbf{Y}_4 = t\partial_u, \\
\mathbf{Y}_5 = \varepsilon\mathbf{Y}_1, \quad \mathbf{Y}_6 = \varepsilon\mathbf{Y}_2, \quad \mathbf{Y}_7 = \varepsilon\mathbf{Y}_3, \quad \mathbf{Y}_8 = \varepsilon\mathbf{Y}_4, \\
\mathbf{Y}_9 = \varepsilon(t\partial_t - 2x\partial_u), \quad \mathbf{Y}_{10} = \varepsilon(x\partial_x + [2x+u]\,\partial_u).
\end{gathered} \tag{9}$$

We show that these symmetries are admitted by (1) in Appendix 1. If we consider the function f not arbitrary, then (8) reduces to

$$(\eta_u^0 - 2\tau_t^0)f + \eta_{tt}^1 - f'\eta_x^0 = 0$$

which is equivalent to relation

$$\delta + (c_6 - c_5)f - 2c_6 f' = 0, \tag{10}$$

where δ is a constant.

We further analyze the classifying relation (10) to obtain non-equivalent forms of f. To this end we consider two different cases (see Appendix 2 for more details).

Case 1 . If $\delta = 0$, then for $\gamma = \frac{c_6 - c_5}{2c_6}$ $(c_6 \neq 0)$ we obtain the eleventh symmetry, namely

$$\mathbf{Y}_{11} = 2(1-\gamma)x\partial_x + (1-2\gamma)t\partial_t + (2x_1 + 2\,[1-\gamma]\,u)\partial_u.$$

In other words, the equation

$$\frac{\partial^2 u}{\partial t^2} - \exp\left\{\frac{\partial u}{\partial x}\right\}\frac{\partial^2 u}{\partial x^2} - \varepsilon A \exp\left\{\frac{\partial u}{\partial x}\right\} = 0, \quad A > 0,\ \gamma \in \mathbb{R} \tag{11}$$

admits 11 dimensional Lie algebra.

In particular, for $\gamma = \frac{1}{2}$, we have $Y_{11} = \frac{1}{\varepsilon}\mathbf{Y}_{10}$.

Case 2 . If $\delta \neq 0$, the eleventh symmetry is given by

$$\mathbf{Y}_{11} = 2(1-\gamma)x\partial_x + (1-2\gamma)t\partial_t + (2x + 2\,[1-\gamma]\,u - \frac{\delta}{2c_6}t^2)\partial_u.$$

3 The adjoint group and invariant solutions

Now we are ready to construct the adjoint group of the algebra L_{10} and thus find some approximate invariant solutions. We start by giving the definition of *inner automorphism*. See [3] or [7] for more details.

Definition 1 Let $X_1, \ldots, X_r$ be the selected basis of the vector space L_r. Accordingly, the structure constants $c^{\lambda}_{\mu\nu}$ are known and any $X \in L$ is written as $X = e^{\mu}X_{\mu}$. Hence, the elements of L_r are represented by vectors $e = (e^1, \ldots, e^r)$. Let L^A_r be a Lie algebra spanned by the following operators

$$E_{\mu} = c^{\lambda}_{\mu\nu}e^{\nu}\partial_{e^{\lambda}}, \quad \mu = 1, \ldots, r. \tag{12}$$

with the Lie Bracket defined by formula $[X_1, X_2] = X_1X_2 - X_2X_1$. The algebra L^A_r generates the group G^A of linear transformations of $\{e^{\mu}\}$. These transformations determine the automorphism of the algebra L_r known as inner automorphism. The group G^A is called group of automorphism of L_r, or the adjoint group of L_r.

We now consider the commutators of L_{10} given in the follwiong table

$[X_iX_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-\varepsilon(2X_6-X_2)$	0	0	εX_5
X_2	$\varepsilon(2X_6-X_2)$	0	$2\varepsilon X_6$	0	0
X_3	0	$-2\varepsilon X_6$	0	εX_4	0
X_4	0	0	$-\varepsilon X_4$	0	$-\varepsilon X_6$
X_5	$-\varepsilon X_5$	0	0	εX_6	0
X_6	$-\varepsilon X_6$	0	0	0	0
X_7	$2X_6-X_2$	0	$2X_6$	0	0
X_8	0	0	$-X_4$	0	$-X_6$
X_9	$-X_5$	0	X_5	X_6	0
X_{10}	$-X_6$	0	0	0	0

$[X_iX_j]$	X_6	X_7	X_8	X_9	X_{10}
X_1	εX_6	$-(2X_6-X_2)$	0	X_5	X_6
X_2	0	0	0	0	0
X_3	0	$-2X_6$	X_4	$-X_5$	0
X_4	0	0	0	$-X_6$	0
X_5	0	0	X_6	0	0
X_6	0	0	0	0	0
X_7	0	0	0	0	0
X_8	0	0	0	$-X_{10}$	0
X_9	0	0	X_{10}	0	0
X_{10}	0	0	0	0	0

To find the transformations that give rise to the adjoint group of L_{10}, we seek the generators of the adjoint algebra L_{10}^A in the form (12), i.e., $E_\mu = c_{\mu\nu}^\lambda e^\nu \partial_{e^\lambda}$, $\mu = 1, \dots, 10$,where the structure constants are given by $[X_\mu, X_\nu] = c_{\mu\nu}^\lambda X_\lambda$. We find (see Appendix 3)

$$
\begin{gathered}
\mathbf{E}_1 = (\varepsilon e^2 + e^7)\partial_{e^2} + (\varepsilon e^5 + e^9)\partial_{e^5} + (\varepsilon e^6 - 2e^7 - 2\varepsilon e^2 + e^{10})\partial_{e^6}, \\
\mathbf{E}_2 = 2\varepsilon e^1\partial_{e^6} - \varepsilon e^1\partial_{e^2} + 2\varepsilon e^3\partial_{e^6}, \\
\mathbf{E}_3 = -\varepsilon e^2\partial_{e^6} + \varepsilon e^4\partial_{e^4} - 2e^7\partial_{e^6} + e^8\partial_{e^4} - e^9\partial_{e^5}, \\
\mathbf{E}_4 = \varepsilon e^3\partial_{e^4} - \varepsilon e^5\partial_{e^6} - e^9\partial_{e^6}, \\
\mathbf{E}_5 = -\varepsilon e^1\partial_{e^5} + \varepsilon e^4\partial_{e^6} + e^8\partial_{e^6}, \\
\mathbf{E}_6 = -\varepsilon e^1\partial_{e^6}, \\
\mathbf{E}_7 = 2e^1\partial_{e^6} - e^1\partial_{e^2} + 2e^3\partial_{e^6}, \\
\mathbf{E}_8 = -e^3\partial_{e^4} - e^5\partial_{e^6} + e^9\partial_{e^{10}}, \\
\mathbf{E}_9 = -e^1\partial_{e^5} + e^3\partial_{e^5} + e^4\partial_{e^6} + e^8\partial_{e^{10}}, \\
\mathbf{E}_{10} = -e^1\partial_{e^6}.
\end{gathered}
$$

We further solve Lie equations for these operators to obtain the following adjoint transformations which give rise to the adjoint group elements of the algebra L_{10}:

$$\begin{gathered}
e'^1 = e^1, \\
e'^2 = 2(a_2 - a_1^\varepsilon a_3 a_7)e^1 - 2\varepsilon a_1^\varepsilon a_3 e^2 + (\frac{a_1^\varepsilon}{\varepsilon} - 1)e^7, \\
e'^3 = e^3, \\
e'^4 = (\varepsilon + 1)a_3 e^4 - ([\varepsilon + 1]\, a_3)(a_8 + a_4\varepsilon)e^3, \\
e'^5 = -\varepsilon a_5 e^1 + a_1^\varepsilon e^5 + (\frac{a_1^\varepsilon}{\varepsilon} - 1)e^9, \\
e'^6 = a_1^\varepsilon(2\varepsilon\left[e^1 + e^3\right] a_2 - 2a_3\left[\varepsilon e^2 + e^7\right] + a_4\left[\varepsilon e^5 + e^{10}\right]) + a_5(\varepsilon e^4 + e^8) - \\
\varepsilon a_6 e^1 + 2a_7(e^1 + e^3) - a_8 e^5 + a_9 e^4 - a_{10}e^1 - 2e^7 + e^{10} + e^6, \\
e'^7 = e^7, \\
e'^8 = e^8, \\
e'^9 = e^9, \\
e'^1 0 = a_9 e^8 - a_8 e^9 + e^{10}.
\end{gathered}$$

We now construct some regular invariant approximate solutions for (11). To this end we seek the approximate invariants for operator $\mathbf{X}$ in the form

$$J(x, t, u, \varepsilon) = J_0(x, t, u) + \varepsilon J_1(x, t, u)$$

which are determined by equation

$$\mathbf{X}J = o(\varepsilon). \tag{13}$$

Thus (13) splits into

$$\mathbf{X}_0 J_0 \ = 0 \quad \text{and} \quad \mathbf{X}_0 J_1 = -X_1 J_0.$$

Among other generators, (11) admits the generators

$$\mathbf{Z}_1 = \varepsilon(t\partial_t - 2x\partial_u), \tag{14}$$

$$\mathbf{Z}_2 = (\partial_t + t\partial_u) + \mathbf{Z}_1, \tag{15}$$

$$\mathbf{Z}_3 = (\partial_x + t\partial_u) + \varepsilon(x\partial_x + (u + 2x)\partial_u). \tag{16}$$

Operators (14)–(16) are linear combination of generators $\mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_7, \mathbf{Y}_8$ and $\mathbf{Y}_9$ given in (9).

The operator (14) has the following functionally independent invariants:

$$h_1 = x \quad \text{quad} \quad h_2 = t\exp\left\{\frac{u}{2x}\right\}$$

and the corresponding approximate invariant solution is given by

$$u = 2x\ln(\frac{y}{t}),$$

where y satisfies the equation

$$y'' + \frac{2}{x}y' - \frac{(y')^2}{y} = \frac{\exp\left\{-2x\frac{y'}{y}\right\}}{y} + \frac{\varepsilon A y}{2x}.$$

The functionally independent invariants

$$h_1 = x + \varepsilon y_1(x, \frac{t^2}{2} - u) \quad \text{and} \quad h_2 = (\frac{t^2}{2} - u) + \varepsilon y_2(x, \frac{t^2}{2} - u)$$

are determined by operator (15). Consequently, assuming that y_1 and y_2 are equal to zero, the corresponding approximate invariant solution is given by

$$u = \frac{t^2}{2} - y(x),$$

where y satisfies the equation

$$y''(x) = \exp\{y'(x)\} - \varepsilon A.$$

Similarly, we find functionally independent invariants

$$h_1 = t + \varepsilon g_1(t, tx - u),$$
$$h_2 = (xt - u) + \varepsilon([xt - u + 2x] + g_2\,[t, xt - u])$$

for the last operator (16) and thus, assuming the function g_1 and g_2 to be zero, find the corresponding approximate invariant solution

$$u = \frac{2\varepsilon}{1+\varepsilon} - xt + 4\varepsilon A \exp\left\{-\frac{t}{2}\right\} + t + c,$$

where A and c are arbitrary constants.

4 Conclusion

In this paper a nonlinear wave equation with an infinitesimal perturbation has been considered. The construction of the principal Lie algebra, the equivalence transformation, the approximate principal Lie algebra, the approximate equivalence transformation and the approximate invariant solutions have been obtained. We have determined the function f from which the approximate principal Lie algebra extends by one and also we constructed some approximate invariant solutions for (1).

The problem of finding the optimal system of one-dimensional subalgebras of L_{10} and the invariant solutions still remain open questions as well as finding the Lagrangians and conservation laws for (1). We hope to return to these questions in a forthcoming paper.

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Appendix 1

As an example, we show that approximate symmetries (9) leave (1) invariant. Let us consider the last two generators

$$\mathbf{Y}_9 = \varepsilon(t\partial_t - 2x\partial_u) \quad \text{quad} \quad \mathbf{Y}_{10} = \varepsilon(x\partial_x + [2x+u]\,\partial_u)$$

We have

$$\begin{aligned}\mathbf{Y}_9^{[2]}&(u_{tt} - \exp\{u_x\}\,u_{xx} - \varepsilon f(u_x) = 0)\,|_{(u_{tt}=\exp\{u_x\}u_{xx})}\\ &= \zeta_1^{(tt)} - \exp\{u_x\}\,(\zeta_1^{(x)}u_{xx} + \zeta_1^{(xx)}),\end{aligned} \tag{A 1}$$

where we compute

$$\zeta_1^{(tt)} = u_{tt}, \quad \zeta_1^{(x)} = 2, \quad \zeta_1^{(xx)} = -u_{xx}.$$

Hence the right hand side of (A1) becomes

$$u_{tt} - 2\exp\{u_x\}\,u_{xx} + \exp\{u_x\}\,u_{xx}\,|_{(u_{tt}=\exp\{u_x\}u_{xx})} = 0.$$

Similarly,

$$\begin{aligned}\mathbf{Y}_{10}^{[2]}&(u_{tt} - \exp\{u_x\}\,u_{xx} - \varepsilon f(u_x) = 0)\,|_{(u_{tt}=\exp\{u_x\}u_{xx})}\\ &= \zeta_1^{(tt)} - \exp\{u_x\}\,(\zeta_1^{(x)}u_{xx} + \zeta_1^{(xx)}),\end{aligned} \tag{A 2}$$

where

$$\zeta_1^{(tt)} = -2u_{tt}, \quad \zeta_1^{(x)} = -2, \quad \zeta_1^{(xx)} = 0.$$

Hence the right hand side of (A2) becomes

$$-2u_{tt} + 2\exp\{u_x\}\,u_{xx}\,|_{(u_{tt}=\exp\{u_x\}u_{xx})} = 0.$$

Thus the generators $\mathbf{Y}_9$ and $\mathbf{Y}_{10}$ leave (1) invariant.

Appendix 2

We use the relation (10) to determine non-equivalent forms of f.
Case 1. If $\delta = 0$, solution of (10) is given by

$$f = A\exp\{\gamma\}\,u_x,$$

where $\gamma = \frac{c_6 - c_5}{2c_6}$. Since $(1-2\gamma)c_6 = c_5$, we obtain

$$\begin{gathered}\tau^0 = (1-2\gamma)c_6 t + c_1, \quad \xi^0 = 2(1-\gamma)c_6 x + c_2, \\ \eta^0 = 2c_6 x + 2(1-\gamma)c_6 u + c_4 t + c_3, \\ \xi^1 = a_1 x + a_2, \quad \tau^1 = a_3 t + a_4, \\ \eta^1 = a_1 u + 2(a_1 - a_3) + a_5 t + a_6.\end{gathered}$$

Thus for any $\gamma \in \mathbb{R}$ ($c_6 \neq 0$) the extended symmetry is given by

$$\mathbf{Y}_{11} = 2(1-\gamma)x\partial_x + (1-2\gamma)t\partial_t + (2x + 2\left[1-\gamma\right]u)\partial_u.$$

Case 2. If $\delta \neq 0$, (10) yields

$$f = \frac{\delta}{2c_6\gamma} + 2c_6 \exp\left\{\gamma\right\} u_x.$$

Then

$$\begin{gathered}\eta^0 = 2c_6 x + 2c_6(1-\gamma)u - \gamma c_6 t^2 + c_4 t + c_3, \\ \xi^0 = 2(1-\gamma)c_6 x + c_2, \quad \tau^0 = (1-2\gamma)c_6 t + c_1.\end{gathered}$$

Thus the eleventh symmetry is

$$\mathbf{Y}_{11} = 2(1-\gamma)x\partial_x + (1-2\gamma)t\partial_t + (2x + 2\left[1-\gamma\right]u - \frac{\delta}{2c_6}t^2)\partial_u.$$

Appendix 3

As an example, we determine the generator E_1. The rest follow in the similar manner. Let $\mu = 1$ and $\lambda, \nu = 1, \ldots, 10$. We write the Lie brackets as

$$[X_1, X_\nu] = c^\lambda_{1\nu} X_\lambda.$$

For $\nu = 2$, we have

$$[X_1, X_2] = c^\lambda_{12} X_\lambda = c^1_{12} X_1 + c^2_{12} X_2 + \cdots + c^{10}_{12} X_{10}$$

and so we obtain

$$c^5_{12} = \varepsilon \quad \text{quad} \quad c^6_{12} = -2\varepsilon.$$

Further we find

$$\begin{gathered}c^5_{15} = \varepsilon \quad (\text{for } \nu = 5), \\ c^6_{16} = \varepsilon \quad (\text{for } \nu = 6), \\ c^6_{17} = -2 \quad \text{quad} \quad c^2_{17} = 1 \quad (\text{for } \nu = 7), \\ c^5_{19} = 1 \quad (\text{for } \nu = 9),\end{gathered}$$

and finally $c^6_{1,10} = 1$ (for $\nu = 10$) . Thus generator (12) has the form

$$\mathbf{E}_1 = (\varepsilon e^2 + e^7)\partial_{e^2} + (\varepsilon e^5 + e^9)\partial_{e^5} + (\varepsilon e^6 - 2e^7 - 2\varepsilon e^2 + e^{10})\partial_{e^6}.$$

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Ranis N. Ibragimov
Department of Applied Mathematics
University of Waterloo
Waterloo, ON, N2L 3G1, Canada
e-mail: ranis@nbnet.nb.ca