Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 24, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Existence results for a class of semi-linear evolution equations *

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Abstract

We prove the existence of regular solutions for the quasi-linear evolution

$$\frac{d}{dt}(x(t)+g(t,x(t))=Ax(t)+f(t,x(t)),$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators defined on a Banach space and the functions f,g are continuous.

1 Introduction

The class of equations considered in this paper have the form

$$\begin{gathered}\tfrac{d}{dt}(x(t)+g(t,x(t))=Ax(t)+f(t,x(t)),\quad t>0,\\ x(0)=x_0\,.\end{gathered}\tag{1.1}$$

We consider this system as a Cauchy problem on a Banach space X, where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$; $f,g:[0,T]\times\Omega\to X$ are appropriate continuous functions and Ω is an open subset of X. The case $g\equiv 0$ has an extensive literature. The books of Pazy [12], Krein [8], Goldstein [2] and the references contained therein, give a good account of important results.

Throughout this paper X will be a Banach space equipped with the norm $\|\cdot\|$ and the operator $A:D(A)\subset X\to X$ will be the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X. For the theory of strongly continuous semigroups, refer to [12] and [2]. We mention here only some notation and properties essential to our purpose. In particular, it is well known that there exist $\tilde{M}\geq 1$ and a real number w such that

$$\|T(t)\|\leq \tilde{M}e^{wt},\quad t\geq 0\,.$$

Mathematics Subject Classifications: 35A05, 34G20, 34A09.
Key words: Banach spaces, semigroup of linear operators, abstract differential equations, fractional powers of closed operators, regular solutions.

Submitted December 14, 2000. Published April 10, 2001.
Partially supported by grant 13394-5 from Fapesp Brazil

In what follows we assume that $\|T(t)\|$ is uniformly bounded by $\tilde{M}$ and that $0 \in \rho(A)$. In this case it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha < 1$, as a closed linear operator with domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|$$

defines a norm on $D((-A)^\alpha)$. Hereafter we represent by X_α the space $D((-A)^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$. The following properties are well known (see [12]).

Lemma 1 *Under the above conditions we have*

1. *If $0 < \alpha \leq 1$, then X_α is a Banach space.*
2. *If $0 < \beta \leq \alpha$, then $X_\alpha \to X_\beta$ is continuous and compact when the resolvent operator of A is compact.*
3. *For every constant $a > 0$, there exists $C_a > 0$ such that*
$$\|(-A)^\alpha T(t)\| \leq \frac{C_a}{t^\alpha}, \quad 0 < t \leq a.$$
4. *For every $a > 0$ there exists a positive constant C'_a such that*
$$\|(T(t) - I)(-A)^{-\alpha}\| \leq C'_a t^\alpha, \quad 0 < t \leq a.$$

By analogy with the abstract Cauchy problem

$$\dot{u}(t) = Au(t) + h(t) \tag{1.2}$$

we adopt the following definitions.

Definition 1 A function $x \in C([0, r) : X)$ is a mild solution of the abstract Cauchy problem (1.1) if the following holds: $x(0) = x_0$; for each $0 \leq t < r$ and $s \in [0, t)$, the function $AT(t-s)g(s, x(s))$ is integrable and

$$\begin{aligned} x(t) &= T(t)(x_0 + g(0, x_0)) - g(t, x(t)) - \int_0^t AT(t-s)g(s, x(s))ds \\ &\quad + \int_0^t T(t-s)f(s, x(s))ds\,. \end{aligned}$$

Definition 2 A function $x \in C([0, r) : X)$ is a classical solution of the abstract Cauchy problem (1.1) if $x(0) = x_0$, $x(t) \in D(A)$ for all $t \in (0, r)$, $\dot{x}$ is continuous on $(0, r)$, and $x(\cdot)$ satisfies (1.1) on $(0, r)$.

Definition 3 A function $x \in C([0, r) : X)$ is an S-classical (Semi-classical) solution of the abstract Cauchy problem (1.1) if $x(0) = x_0$, $\frac{d}{dt}(x(t) + g(t, x(t)))$ is continuous on $(0, r)$, $x(t) \in D(A)$ for all $t \in (0, r)$, and $x(\cdot)$ satisfies (1.1) on $(0, r)$.

This paper is organized as follows. In section 2 we discuss the existence of S-classical and classical solutions to the initial value problem (1.1). Our results are based on the properties of analytic semigroups and the ideas used in [12, chapter 5]. In section 3, some applications are considered.

Throughout this paper we assume that X is an abstract Banach space. The terminology and notations are those generally used in operator theory. In particular, if $X,\ Y$ are Banach spaces, we indicate by $\mathcal{L}\ (X:Y)$ the Banach space of the bounded linear operator of X into Y and we abbreviate to $\mathcal{L}\ (X)$ whenever $X=Y$. In addition $B_r(x:X)$ will denote the closed ball in space X with center at x and radius r.

For a bounded function $\xi:[0,a]\to X$ and $0<t<a$ we will employ the notation

$$\|\xi(\cdot)\|_t=\sup\{\|\xi(s)\|:s\in[0,t]\}.$$

Finally for $x_0\in X$, we will use the notation $x(\cdot,x_0)$ for the mild solution of (1.1).

2 Regular Solutions

The existence of mild solutions for the abstract Cauchy problem (1.1) follows from [5, Theorems 2.1, 2.2]; for this reason we omit the proofs of the next two results.

Theorem 1 *Let $x_0\in X$ and assume that the following conditions hold*

a) *There exist $\beta\in(0,1)$ and $L\geq 0$ such that the function g is X_β-valued and satisfies the Lipschitz condition*

$$\|(-A)^\beta g(t,x)-(-A)^\beta g(s,y)\|\leq L(|t-s|+\|x-y\|)$$

for every $0\leq s,t\leq T$ and $x,y\in\Omega$, and $\|(-A)^{-\beta}\|L<1$.

b) *The function f is continuous and takes bounded sets into bounded sets.*

c) *The semigroup $(T(t))_{t\geq 0}$ is compact.*

Then there exists a mild solution $x(\cdot,x_0)$ of the abstract Cauchy problem (1.1) defined on $[0,r]$ for some $0<r<T$.

Theorem 2 *Let $x_0\in X$ and assume that the following conditions hold:*

a) *There exist $\beta\in(0,1)$ and $L\geq 0$ such that the function g is X_β-valued and satisfies the Lipschitz condition*

$$\|(-A)^\beta g(t,x)-(-A)^\beta g(s,y)\|\leq L(|t-s|+\|x-y\|)$$

for every $0\leq s,t\leq T$ and $x,y\in\Omega$ and $\|(-A)^{-\beta}\|L<1$.

b) The function f is continuous and there exists $N > 0$ such that

$$\|f(t,x) - f(s,y)\| \le N(|t-s| + \|x-y\|)$$

for every $0 \le s,t \le T$ and $x, y \in \Omega$.

Then there exists a unique mild solution $x(\cdot, x_0)$ of the abstract Cauchy problem (1.1) defined on $[0,r]$ for some $0 < r \le T$.

The existence of S-classical and classical solutions, requires some additional assumptions on the functions g, f. In particular, in the next result we assume that the following assumption hold.

Assumptions on f and g: There exist $0 < \alpha < \beta < 1$ and an open set $\Omega_\alpha \subset X_\alpha$ such that the functions f and $(-A)^\beta g$ are continuous on $[0,T] \times \Omega_\alpha$, and there exist $L > 0$ and $0 < \gamma_1, \gamma_2 < 1$ such that for every $(t, x_1), (s, x_2) \in [0,T] \times \Omega_\alpha$ we have

$$\begin{gathered}
\|(-A)^\beta g(t,x_1) - (-A)^\beta g(s,x_2)\| \le L\{|t-s|^{\gamma_1} + \|x_1 - x_2\|_\alpha\}, \\
\|f(t,x_1) - f(s,x_2)\| \le N\{|t-s|^{\gamma_2} + \|x_1 - x_2\|_\alpha\}, \\
L\|(-A)^{\alpha-\beta}\| < 1\,.
\end{gathered}$$

Theorem 3 *Let $x_0 \in \Omega_\alpha$ and assume that f and g satisfy the above assumptions, that g is $D(A)$-valued continuous and that $1 - \beta < \min\{\beta - \alpha, \gamma_1, \gamma_2\}$. Then there exists a unique S-classical solution $x(\cdot, x_0) \in C([0,r] : X)$ for some $0 < r < T$.*

Proof. Let $0 < r_1 < T$ and $\delta > 0$ such that

$$V = \{(t,x) \in [0,r_1] \times X_\alpha : \ \|(-A)^\alpha x - (-A)^\alpha x_0\| < \delta\} \subset [0,T) \times \Omega_\alpha.$$

Assuming that the functions f and $(-A)^\beta g$ are bounded on V by $C_1 > 0$, we choose $0 < r < r_1$ such that

$$\begin{gathered}
\|(T(\cdot) - I)(-A)^\alpha x_0\|_r \le \frac{(1-\mu)\delta}{6}, \\
\|(T(\cdot) - I)(-A)^\alpha g(0,x_0)\|_r \le \frac{(1-\mu)\delta}{6}, \\
\|(-A)^{\alpha-\beta}\| L r^{\gamma_1} + C_{1-\beta+\alpha} C_1 \frac{r^{\beta-\alpha}}{\beta-\alpha} + C_\alpha C_1 \frac{r^{1-\alpha}}{1-\alpha} < \frac{(1-\mu)\delta}{6}, \\
L C_{1-\beta+\alpha} \frac{r^{\beta-\alpha}}{\beta-\alpha} + N C_\alpha \frac{r^{1-\alpha}}{1-\alpha} < 1 - \mu,
\end{gathered}$$

where $\mu = \|(-A)^{\alpha-\beta}\| L$ and C_α, $C_{1-\beta+\alpha}$ are the constants in Lemma 1.

On the set

$$S = \{y \in C([0,r] : X) : y(0) = (-A)^\alpha x_0, \ \|y(t) - (-A)^\alpha x_0\| \le \delta, \ t \in [0,r]\}$$

we define the operator

$$\begin{aligned}\Psi(y)(t) &= T(t)(-A)^{\alpha}(x_0+g(0,x_0))-(-A)^{\alpha}g(t,(-A)^{-\alpha}y(t))\\ &\quad+\int_0^t(-A)^{1-\beta+\alpha}T(t-s)(-A)^{\beta}g(s,(-A)^{-\alpha}y(s))ds \qquad (2.1)\\ &\quad+\int_0^t(-A)^{\alpha}T(t-s)f(s,(-A)^{-\alpha}y(s))\,ds.\end{aligned}$$

For the mapping Ψ we consider the decomposition $\Psi=\Psi_1+\Psi_2$, where

$$\begin{aligned}\Psi_1(y)(t) &= T(t)(-A)^{\alpha}(x_0+g(0,x_0))-(-A)^{\alpha}g(t,(-A)^{-\alpha}y(t))\\ &\quad+\int_0^t(-A)^{1-\beta+\alpha}T(t-s)(-A)^{\beta}g(s,(-A)^{-\alpha}y(s))ds,\\ \Psi_2(y)(t) &= \int_0^t(-A)^{\alpha}T(t-s)f(s,(-A)^{-\alpha}y(s))ds.\end{aligned}$$

Next we prove that Ψ_1 and Ψ_2 are well defined, that Ψ satisfies a Lipschitz condition and that the ranges of Ψ is contained in S.

Since $T(\cdot)$ is analytic, the function $s\to AT(t-s)$ is continuous in the uniform operator topology on $[0,t)$, consequently the function $AT(t-s)g(s,(-A)^{-\alpha}y(s))$ is continuous on $[0,t)$. Moreover from lemma 1 we have

$$\|(-A)^{1-\beta+\alpha}T(t-s)(-A)^{\beta}g(s,(-A)^{-\alpha}y(s))\|\le\frac{C_{1-\beta+\alpha}C_1}{(t-s)^{1-\beta+\alpha}},$$

$s\in[0,t)$, which implies that $\|(-A)^{1-\beta+\alpha}T(t-s)g(s,(-A)^{-\alpha}y(s))\|$ is integrable on $[0,t)$. We thus conclude that Ψ_2 is well defined and with values in $C([0,r]:X)$. It's clear from the previous remark that Ψ_1 is also well defined and with values in $C([0,r]:X)$.

It remain to show that the operator Ψ is a contraction on S. Firstly we prove that the range of Ψ is contained in S. Let y be a function in S. Then for $t\in[0,r]$ we get

$$\begin{aligned}&\|\Psi(y)(t)-(-A)^{\alpha}x_0\|\\ &\le \|(T(t)-I)(-A)^{\alpha}(x_0+g(0,x_0))\|\\ &\quad+\|(-A)^{\alpha}g(0,x_0)-(-A)^{\alpha}g(t,(-A)^{-\alpha}y(t))\|\\ &\quad+\int_0^t\frac{C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}}\|(-A)^{\beta}g(s,(-A)^{-\alpha}y(s))\|ds\\ &\quad+\int_0^t\frac{C_{\alpha}}{(t-s)^{\alpha}}\|f(s,(-A)^{-\alpha}y(s))\|ds\\ &\le \frac{2(1-\mu)\delta}{6}+\|(-A)^{\alpha-\beta}\|L\{r^{\gamma_1}+\|(-A)^{\alpha}x_0-y(t)\|\}\\ &\quad+\int_0^t\left(\frac{C_{1-\beta+\alpha}C_1}{(t-s)^{1-\beta+\alpha}}+\frac{C_{\alpha}C_1}{(t-s)^{\alpha}}\right)ds\\ &\le \frac{2(1-\mu)\delta}{6}+\|(-A)^{\alpha-\beta}\|L\{r^{\gamma_1}+\delta\}+C_{1-\beta+\alpha}C_1\frac{r^{\beta-\alpha}}{\beta-\alpha}+C_{\alpha}C_1\frac{r^{1-\alpha}}{1-\alpha}.\end{aligned}$$

¿From the choice of r we conclude that

$$\|\Psi(y) - (-A)^{\alpha} x_0\|_r \le \delta$$

so that $\Psi(y) \in S$.

On the other hand for $x(\cdot)$, $y(\cdot) \in S$ and $t \in [0, r]$,

$$\begin{aligned}
&\|\Psi(y)(t) - \Psi(x)(t)\| \\
&\quad\le \|(-A)^{\alpha} g(t, (-A)^{-\alpha} y(t)) - (-A)^{\alpha} g(t, (-A)^{-\alpha} x(t))\| \\
&\quad+ \int_0^t \frac{C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}} \|(-A)^{\beta} g(s, (-A)^{-\alpha} y(s)) - (-A)^{\beta} g(s, (-A)^{-\alpha} x(s))\| ds \\
&\quad+ \int_0^t \frac{C_{\alpha}}{(t-s)^{\alpha}} \|f(s, (-A)^{-\alpha} y(s)) - f(s, (-A)^{-\alpha} x(s))\| ds \\
&\quad\le \|(-A)^{\alpha-\beta}\| L \|y(t) - x(t)\| + \int_0^t \{\frac{L C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}} + \frac{N C_{\alpha}}{(t-s)^{\alpha}}\} \|y - x\|_r ds,
\end{aligned}$$

thus

$$\|\Psi(y) - \Psi(x)\|_r \le \left(L\|(-A)^{\alpha-\beta}\| + L C_{1-\beta+\alpha} \frac{r^{\beta-\alpha}}{\beta-\alpha} + N C_{\alpha} \frac{r^{1-\alpha}}{1-\alpha} \right) \|y - x\|_r.$$

The last estimate and the choice of r imply that Ψ is a contraction mapping on S. Let $y(\cdot)$ be the unique fixed point of the operator Ψ in S. We affirm that $y(\cdot)$ is locally Hölder continuous. In fact, let ϑ be a real number with $0 < \vartheta < \min\{1 - \alpha, \beta - \alpha\}$ and $\vartheta + \beta > 1$, and let $\tilde{C} > 0$ be the constant guaranteed in Lemma 1, such that for all $0 \le s \le t \le T$ and $0 < h < 1$

$$\|(T(h) - I)(-A)^{\alpha} T(t-s)\| \le \frac{\tilde{C} h^{\vartheta}}{(t-s)^{\vartheta+\alpha}}, \quad 0 \le s < t$$

and

$$\|(T(h) - I)(-A)^{1-\beta+\alpha} T(t-s)\| \le \frac{\tilde{C} h^{\vartheta}}{(t-s)^{1-\beta+\alpha+\vartheta}}, \quad 0 \le s < t\,.$$

For $t \in [0, r)$ and $h > 0$ sufficiently small,

$$\begin{aligned}
&\|y(t+h) - y(t)\| \\
&\quad\le \|(T(h) - I)(-A)^{\alpha} T(t)(x_0 - g(0, x_0))\| \\
&\qquad + \|(-A)^{\alpha-\beta}\| L \{h^{\gamma_1} + \|y(t+h) - y(t)\|\} \\
&\qquad + \int_0^t \|(T(h) - I)(-A)^{1-\beta+\alpha} T(t-s)(-A)^{\beta} g(s, (-A)^{-\alpha} y(s))\| ds \\
&\qquad + \int_t^{t+h} \|(-A)^{1-\beta+\alpha} T(t+h-s)(-A)^{\beta} g(s, (-A)^{-\alpha} y(s))\| ds \\
&\qquad + \int_0^t \|(T(h) - I)(-A)^{\alpha} T(t-s) f(s, (-A)^{-\alpha} y(s))\| ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^{t+h}\|(-A)^\alpha T(t+h-s)f(s,(-A)^{-\alpha}y(s))\|ds\\
\le\;&\frac{\tilde C}{t^{(\alpha+\vartheta)}}\|x_0-g(0,x_0)\|h^\vartheta+L\|(-A)^{\alpha-\beta}\|\{h^{\gamma_1}+\|y(t+h)-y(t)\|\}\\
&+\int_0^t\frac{\tilde C h^\vartheta C_1}{(t-s)^{1-\beta+\alpha+\vartheta}}ds+\int_t^{t+h}\frac{C_{1-\beta+\alpha}C_1}{(t+h-s)^{1-\beta+\alpha}}ds\\
&+\int_0^t\frac{\tilde C h^\vartheta C_1}{(t-s)^{\alpha+\vartheta}}ds+\int_t^{t+h}\frac{C_\alpha C_1}{(t+h-s)^{\alpha}}ds\\
\le\;&\frac{C(x_0)h^\vartheta}{t^{\vartheta+\alpha}}+C_2h^{\gamma_1}+L\|(-A)^{\alpha-\beta}\|\|y(t+h)-y(t)\|+C_3h^\vartheta\\
&+C_4h^{\beta-\alpha}+C_5h^{1-\alpha}
\end{aligned}
$$

where the constants C_i are independent of t. If $\bar\rho=\min\{\vartheta,\gamma_1\}$, the last inequality can be rewritten in the form

$$\|y(t+h)-y(t)\|\le\frac{C(\alpha,\beta,\vartheta,t,x_0)}{1-\mu}h^{\bar\rho}$$

since $\mu=L\|(-A)^{\alpha-\beta}\|<1$. Therefore the function $y(\cdot)$ is locally $\bar\rho$-Hölder continuous on $(0,r)$, moreover, we can to assume that $\bar\rho+\beta>1$. Now it is easy to show that $s\to(-A)^\beta g(s,(-A)^{-\alpha}y(s))$ and $s\to f(s,(-A)^{-\alpha}y(s))$ are ρ-Hölder continuous on $(0,r)$, where $\rho=\min\{\bar\rho,\gamma_2\}$ and $\rho+\beta>1$. From this remark, in [2, Theorem 2.4.1] and Lemma 2 below, we infer that the function

$$
\begin{aligned}
x(t)\;=\;&T(t)(x_0+g(0,x_0))-g(t,(-A)^{-\alpha}y(t))\\
&+\int_0^t(-A)^{1-\beta}T(t-s)(-A)^\beta g(s,(-A)^{-\alpha}y(s))ds \qquad (2.2)\\
&+\int_0^tT(t-s)f(s,(-A)^{-\alpha}y(s))ds
\end{aligned}
$$

is X_α-valued, that the integral terms in (2.2) are functions in $C^1([0,r]:X)$ and that $x(t)\in D(A)$ for all $t\in(0,r)$. Operating on $x(\cdot)$ with $(-A)^\alpha$, we conclude that $(-A)^{-\alpha}y=x$ and hence that $x(t)+g(t,x(t))$ is a C^1 function on $(0,b)$. The proof is completed. ♢

The proof of the next Lemma is analogous to the proof in [2, Theorem 2.4.1]. However there are some differences that require special attention and we include the principal ideas of this proof for completeness.

Lemma 2 *Let $0<\beta<1$ and $g\in C([0,T]:X_{1-\beta})$. Assume that $g:[0,T]\to X$ is θ-Hölder continuous in $(0,T)$ with $\beta+\theta>1$. If $y:[0,T]\to X$ is defined by*

$$y(t)=\int_0^t(-A)^{1-\beta}T(t-s)g(s)ds,$$

then $y(t)\in D(A)$ for every $t\in[0,T)$ and $\dot y\in C([0,T):X)$.

Proof. For $t \in [0,T)$ we rewrite $y(t)$ in the form

$$\int_0^t (-A)^{1-\beta}T(t-s)(g(s)-g(t))ds + \int_0^t (-A)^{1-\beta}T(t-s)g(t)ds = v(t)+w(t). \tag{2.3}$$

Clearly, $Aw(t) = T(t)(-A)^{1-\beta}g(t) - (-A)^{1-\beta}g(t) \in C([0,T]:X)$. For $\epsilon > 0$, sufficiently small we define the function

$$v_\epsilon(t) := \begin{cases} \int_0^{t-\epsilon}(-A)^{1-\beta}T(t-s)(g(s)-g(t))ds\,, & \text{for } t \in [\epsilon, T), \\ 0 & \text{for } t \in [0,\epsilon). \end{cases}$$

It is clear that $v_\epsilon(t) \in D(A)$. Moreover for $0 < \delta_1 < \delta_2$

$$\begin{aligned} \|Av_{\delta_2}(t) - Av_{\delta_1}(t)\| &\leq \int_{t-\delta_2}^{t-\delta_1} \|(-A)^{2-\beta}T(t-s)(g(s)-g(t))\|ds \\ &\leq C_{2-\beta}(\delta_2^{\beta+\theta-1} - \delta_1^{\beta+\theta-1}). \end{aligned}$$

The last inequality proves that Av_δ is convergent, $\beta+\theta > 1$, and therefore

$$A(v(t)) = \int_0^t A^{2-\beta}T(t-s)(g(s)-g(t))ds \tag{2.4}$$

since A is a closed operator. From the previous remark it follows that $y(t) \in D(A)$ for $t \in [0,T]$. The continuity of $\partial_t y$ follows as in [2, Theorem 2.4.1]. $\diamondsuit$

In the rest of this paper for a function $j : [0,b] \times X \to X$ and $h \in \mathbb{R}$ we denote by $\partial_h j$ to the function

$$\partial_h j(t) = \frac{j(t+h)-j(t)}{h}.$$

Moreover, if j is differentiable we will employ the decomposition:

$$j(t+s,y) - j(t,y) = D_1 j(t,y)s + W_1(j,t,t+s,y) \tag{2.5}$$

and

$$j(t,y+y_1) - j(t,y) = D_2 j(t,y)\cdot y_1 + W_2(j,t,y,y+y_1) \tag{2.6}$$

where

$$\begin{aligned} \frac{W_1(j,t,t+s,y)}{|s|} &\to 0 \qquad \text{as} \quad s \to 0 \\ \frac{W_2(j,t,y,y+y_1)}{\|y_1\|} &\to 0 \qquad \text{as} \quad y_1 \to 0\,. \end{aligned}$$

To prove the next theorem, we need a preliminary result which is interesting in its own right.

Lemma 3 *Under the assumptions in Theorem 2, if $x_0 \in D(A)$ and $g(0,x_0) \in D(A)$, then $x(\cdot) = x(\cdot, x_0)$ is Lipschitz on closed intervals.*

Proof. Initially we prove that $x(\cdot)$ is β-Hölder continuous on a closed interval $[0,b]$. Using the continuity of $(-A)^\beta g$ and f we can to assert that $(-A)^\beta g(s,x(s))$ and $f(s,x(s))$ are bounded by $C_1>0$ on $[0,b]$. Employing that $x_0\in D(A)$ and that $g(0,x_0)\in D(A)$; for $t\in[0,b)$ and $h>0$ we have

$$\begin{aligned}
&\|x(t+h)-x(t)\|\\
&\le\ C_2h+\|g(t+h,x(t+h))-g(t,x(t))\|\\
&\quad+\int_0^t\frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|(-A)^\beta g(s+h,x(s+h))-(-A)^\beta g(s,x(s))\|ds\\
&\quad+\int_0^h\|(-A)^{1-\beta}T(t+h-s)(-A)^\beta g(s,x(s))\|ds\\
&\quad+\tilde M\int_0^t\|f(s+h,x(s+h))-f(s,x(s))\|ds+\tilde M\int_0^h\|f(s,x(s))\|ds
\end{aligned}$$

thus

$$\begin{aligned}
\|x(t+h)-x(t)\|\ &\le\ C_3h^\beta+\|(-A)^{-\beta}\|L\|x(t+h)-x(t)\|\\
&\quad+\int_0^t\{\frac{C_{1-\beta}L}{(t-s)^{1-\beta}}+N\tilde M\}\|x(s+h)-x(s)\|ds.
\end{aligned}$$

Since $\|(-A)^{-\beta}\|L<1$, the Gronwall-Bellman inequality [12, Lemma 5.6.7] implies that $x(\cdot)$ is β-Hölder continuous. Reiterating the previous estimates and using that $x(\cdot)$ is β-Hölder; if $t\in[0,T)$ and $h>0$ we get

$$\begin{aligned}
&\|x(t+h)-x(t)\|\\
&\le C_4h+\|(-A)^{-\beta}\|L\|x(t+h)-x(t)\|\\
&\quad+\int_0^t\{\frac{C_{1-\beta}L}{(t-s)^{1-\beta}}+N\tilde M\}\|x(s+h)-x(s)\|ds\\
&\quad+\int_0^h\frac{C_{1-\beta}L}{(t+h-s)^{1-\beta}}\|(-A)^\beta g(s,x(s))-(-A)^\beta g(0,x_0)\|ds\\
&\quad+\int_0^h\|T(t+h-s)(-A)g(0,x_0)\|ds
\end{aligned}$$

then

$$\begin{aligned}
\|x(t+h)-x(t)\|\ &\le\ C_5h^{2\beta}+\|(-A)^{-\beta}\|L\|x(t+h)-x(t)\|\\
&\quad+\int_0^t\{\frac{C_{1-\beta}L}{(t-s)^{1-\beta}}+N\tilde M\}\|x(s+h)-x(s)\|ds.
\end{aligned}$$

The assumption $\|(-A)^{-\beta}\|L<1$ and Gronwall Bellman inequality, implies that $x(\cdot)$ is 2β-Hölder continuous. Clearly the previous routine permit to infer that $x(\cdot)$ is Lipschitz continuous, this completes the proof. ♢

In the next theorem we establish the existence of classical solutions using some usual regularity assumptions on the functions f and $(-A)^\beta g$.

Theorem 4 *Assume that* $(-A)^{1-\beta}g(\cdot)$ *and* $f(\cdot)$ *are continuously differentiable functions on* $[0,T]\times\Omega$. *If* x_0, $g(0,x_0)\in D(A)$ *and* $\|D_2g(0,x_0)\|_{\mathcal{L}(X)}<1$ *then* $\dot{x}(\cdot,x_0)\in C([0,b]:X)$ *for some* $0<b<T$.

Proof: Let $x(\cdot)=x(\cdot,x_0)$ and $z(\cdot)$ be a solution of the integral equation

$$\begin{aligned} z(t) &= T(t)(Ax_0+Ag(0,x_0)+f(0,x_0))+h(t)-D_2g(t,x(t))(z(t)) \\ &\quad +\int_0^t(-A)^{1-\beta}T(t-s)D_2(-A)^{\beta}g(s,x(s))(z(s))ds \\ &\quad +\int_0^t T(t-s)D_2f(s,x(s))(z(s))ds \end{aligned} \tag{2.7}$$

where

$$z(0)=Ax_0+Ag(0,x_0)+f(0,x_0)-D_1g(0,x_0)-D_2g(0,x_0)(z(0))$$

and

$$\begin{aligned} h(t) &= -D_1g(t,x(t))+\int_0^t(-A)^{1-\beta}T(t-s)D_1(-A)^{\beta}g(s,x(s))ds \\ &\quad +\int_0^t T(t-s)D_1f(s,x(s))ds\,. \end{aligned}$$

The existence and uniqueness of local solution for (2.7), is consequence of the contraction mapping principle and the condition $\|D_2g(0,x_0)\|_{\mathcal{L}(X)}<1$; we omit details. In what follows we assume that $x(\cdot)$ and $z(\cdot)$ are defined on $[0,2b]$ where $0<2b<T$ and $\|D_2g(\theta,x_\theta)\|_{2b}<\eta<1$. Using the notations introduced in (2.5)-(2.6), for t in $[0,b]$ and $h>0$ sufficiently small, we have

$$\begin{aligned} &\|\xi(t,h)\| \\ &= \|\frac{x(t+h)-x(t)}{h}-z(t)\| \\ &\le \|T(t)(\frac{T(h)-I}{h}x_0-A(x_0))\| \\ &\quad +\|\frac{1}{h}\int_0^h T(t+h-s)f(s,x(s))ds-T(t)f(0,x_0)\| \\ &\quad +\|T(t)(\frac{T(h)-I}{h})g(0,x_0)+\frac{1}{h}\int_0^h(-A)T(t+h-s)g(s,x(s))ds\| \\ &\quad +\|D_1g(t,x(t+h))-D_1g(t,x(t))\|+\|D_2g(t,x(t))(\xi(t,h))\| \\ &\quad +\|\frac{W_1(g,t,t+h,x(t+h))}{h}\|+\|\frac{W_2(g,t,x(t),x(t+h))}{h}\| \\ &\quad +\int_0^t\frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|D_1(-A)^{\beta}g(s,x(s+h))-D_1(-A)^{\beta}g(s,x(s))\|ds \\ &\quad +\int_0^t\frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|D_2(-A)^{\beta}g(s,x(s))(\xi(s,h))\|ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|\frac{W_1((-A)^\beta g, s, s+h, x(s+h))}{h}\|ds \\
&+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|\frac{W_2((-A)^\beta g, s, x(s), x(s+h))}{h}\|ds \\
&+\int_0^t \tilde{M}\|D_1 f(s, x(s+h)) - D_1 f(s, x(s))\|ds \\
&+\int_0^t \tilde{M}\|D_2 f(s, x(s))\|\|\xi(s,h)\|ds + \int_0^t \tilde{M}\|\frac{W_1(f, s, s+h, x(s+h))}{h}\|ds \\
&+\tilde{M}\int_0^t \|\frac{W_2(f, s, x(s), x(s+h))}{h}\|ds.
\end{aligned}
$$

On the other hand, from lemma 2 we know that $x(\cdot)$ is Lipschitz continuous; therefore,

$$
\frac{W_2((-A)^\beta g, s, x(s), x(s+h))}{\|x(s+h) - x(s)\|} \cdot \frac{\|x(s+h) - x(s)\|}{h} \to 0 \quad \text{as} \quad h \to 0
$$

and

$$
\frac{W_2(f, s, x(s), x(s+h))}{\|x(s+h) - x(s)\|} \cdot \frac{\|x(s+h) - x(s)\|}{h} \to 0 \quad \text{as} \quad h \to 0
$$

uniformly for $s \in [0, b]$. This enables us to rewrite the last inequality in the form

$$
\begin{aligned}
&\|\xi(t,h)\| \\
&\quad = \quad \|\frac{x(t+h) - x(t)}{h} - z(t)\| \\
&\quad \le \quad \rho(t,h) + \frac{1}{h}\int_0^h \frac{C_{1-\beta}}{(t+h-s)^{1-\beta}}\|(-A)^\beta g(0, x_0) - (-A)^\beta g(s, x(s))\|ds \\
&\qquad +\|D_2 g(t, x(t))(\xi(t,h))\| + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|D_2(-A)^\beta g(s, x(s))\|\|\xi(s,h)\|ds \\
&\qquad +\tilde{M}\int_0^t \|D_2 f(s, x(s)\|\|\xi(s,h)\|ds
\end{aligned}
$$

where $\rho(t,h) \to 0$ as $h \to 0$, uniformly for $t \in [0, b]$. Since $x(\cdot)$ is Lipschitz and $\|D_2 g(\cdot, x(\cdot))\|_b < \eta$, follow that

$$
\begin{aligned}
\|\xi(t,h)\| \quad \le \quad & \frac{1}{1-\eta}\rho(t,h) + \frac{C_{1-\beta} L C h^\beta}{\beta} \\
& + \frac{1}{1-\eta}\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\|D_2(-A)^\beta g(s, x(s))\|\|\xi(s,h)\|ds \\
& + \frac{1}{1-\eta}\tilde{M}\int_0^t \|D_2 f(s, x(s)\|\|\xi(s,h)\|ds.
\end{aligned}
$$

Finally, the Gronwall's inequality [12, Lemma 5.6.7] shows that $\xi(t,h) \to 0$ as $h \to 0$. Therefore, $\dot{x}(\cdot, x_0) = z(\cdot)$. This completes the proof. ♢

Corollary 1 *If g is a $D(A)$-valued continuous function then there exits a unique classical solutions of (1.1) defined on $[0, b]$ for some $0 < b < T$.*

Proof: From Theorem 4 we know that $x(\cdot) = x(\cdot, x_0) \in C^1([0, b] : X)$ for some $0 < b < T$. Since $x(\cdot, x_0)$ is Lipschtiz continuous in $[0, b]$, from [2, Theorem 2.4.1] and Lemma 2 we infer that $x(t) + g(t, x(t)) \in D(A)$ for $t \in [0, b]$ and therefore that $x(t) \in D(A)$ for $t \in [0, b]$. The proof is complete. ♢

3 Examples

In this section we sketch briefly some applications.

Functional Differential Equations with Unbounded Delay

The regularity results obtained in this work are used to prove the existence of regular solutions, "Classical" and "N-Classical" solutions, for a class of quasi-linear neutral functional differential equations with unbounded delay that can be modeled in the form

$$\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t), \quad t \geq \sigma, \tag{3.1}$$

$$x_\sigma = \varphi \in \mathcal{B}, \tag{3.2}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, as in Hale and Kato [3], and where the axioms are establish employing the terminology and notations used in Hino-Murakami-Naito [7]. A complete reference including results of existence of mild, strong and periodical solutions for (3.1)-(3.2) are the papers [4], [5]. The existence of "N-Classical " and " Classical" solutions is studied in [6], actually in preparation.

Partial Differential Equations of Sobolev Type

There is a extensive literature on semi-linear Sobolev evolution equations modeled in the form

$$\tfrac{d}{dt}(Bu(t)) = Au(t) + f(t, u(t)) \quad t > 0, \tag{3.3}$$

$$u(0) = u_0, \quad u_0 \in D(B)\,, \tag{3.4}$$

where A, B are closed linear operators on a Banach space X. The literature includes different and complete results concerning to existence, uniqueness and qualitative properties of mild, strong and classical solutions for (3.3)-(3.4) (see [1, 10, 13, 14]). Some usual assumptions on the operators A, B (see for example [1, 10]) are

- A, B are closed linear operators.

- $D(B) \subset D(A)$ and B has a continuous inverse.

¿From these assumptions and the Closed Graph Theorem it follows that AB^{-1} is a bounded linear operator on X. In this case the approach is to consider the related integral equation

$$x(t) = T(t)Bx_0 + \int_0^t T(t-s)f(s, B^{-1}x(s))ds, \tag{3.5}$$

where $T(t)$ with $t \geq 0$ is the semigroup generated by AB^{-1}.

We shall consider the abstract Cauchy problem

$$\tfrac{d}{dt}(u(t) + Bu(t)) = Au(t) + f(t, u(t)), \quad t > 0, \tag{3.6}$$

$$u(0) = u_0, \quad u_0 \in D(B), \tag{3.7}$$

where A, B are closed linear operators on a Banach space X and

- $D(A) \subset D(B)$ and B has a continuous inverse
- AB^{-1} is the infinitesimal generator of an analytic semigroup of bounded linear operators on X.

Under these conditions, we consider the associated system

$$\tfrac{d}{dt}(u + B^{-1}u(t)) = AB^{-1}u(t) + f(t, B^{-1}u(t)), \quad t > 0, \tag{3.8}$$

$$u(0) = B^{-1}u_0, \quad u_0 \in D(B). \tag{3.9}$$

If in addition B^{-1} is $D(AB^{-1})$-valued continuous and f is continuously differentiable, the existence of classical solutions for (3.8)-(3.9) and consequently for (3.6)-(3.7), follows from Corollary 1.

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