

# Uniform stability of displacement coupled second-order equations \*

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## Abstract

We prove that the uniform stability of semigroups associated to displacement coupled dissipator systems is equivalent to the uniform stability of velocity coupled dissipator systems. Using this equivalence, we give sufficient conditions for obtaining uniform stability and exact controllability of displacement coupled dissipator systems.

## 1 Introduction

We consider a linear oscillator in a Hilbert space  $H$  represented by

$$\partial_{tt}u(t) + \mathcal{A}u(t) = h, \quad (1)$$

where  $\mathcal{A}$  is a (generally unbounded) positive self-adjoint operator on  $H$ . Russell [12, p. 340] proposed to introduce

certain *indirect damping* mechanisms which arise, not from insertion of damping terms into the original equations describing the mechanical motion, but by coupling those equations to further equations describing other processes in the structure ...

He described, in the same work, two types of *indirect damping*: the velocity coupled dissipator and the displacement coupled dissipator. Works on *indirect damping* mechanisms of the first type leading to exponential decay of the total energy may be found in [2, 3, 4, 7, 12]. In this paper, our attention will be focused on the second type of indirect damping mechanisms. The description given in [12] is the following. Consider a system with displacement vector  $(w, z)$ , velocity  $(\partial_t w, \partial_t z)$  and energy form

$$E(w, z)(t) = \frac{1}{2} \left( \left\langle \begin{pmatrix} w \\ z \end{pmatrix}, S \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_{H \times G} + \|\partial_t w\|_H^2 + \|\partial_t z\|_G^2 \right). \quad (2)$$

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where  $G$  is a second Hilbert space and  $S$  is a positive self-adjoint operator on  $H \times G$  representable in operator matrix form as

$$S = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad (3)$$

The energy  $E(w, z)(t)$  is conserved for the second order system

$$\begin{pmatrix} \partial_{tt}w \\ \partial_{tt}z \end{pmatrix} + S \begin{pmatrix} w \\ z \end{pmatrix} = 0. \quad (4)$$

Damping is then introduced in the second equation of the system:

$$\begin{pmatrix} \partial_{tt}w \\ \partial_{tt}z \end{pmatrix} + S \begin{pmatrix} w \\ z \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ \partial_t z \end{pmatrix} = 0.$$

At this level, Russell [12] assumes the inertial forces in the  $z$  system are small in comparison with the damping and, then,  $\partial_{tt}z$  is discarded. In our work, we do not adopt this last assumption. Moreover, we replace the constant  $\gamma$  by an (eventually unbounded) positive self-adjoint operator  $D$  acting on  $G$

$$\begin{pmatrix} \partial_{tt}w \\ \partial_{tt}z \end{pmatrix} + S \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ D\partial_t z \end{pmatrix} = 0. \quad (5)$$

Our problem is then to find, among all these damping mechanisms, those for which the energy of the resulting system has an exponential decay to zero (Russell was interested in the analyticity of the associated semigroup).

This paper is organized as follows. In the second section, we give the relation between the velocity coupled dissipator and the displacement coupled dissipator. The third section is devoted to uniform stability of displacement coupled dissipator. In the fourth section we give some applications (two displacement coupled wave equations and Timoshenko beam). And in the last section, we give a result of exact controllability of the displacement coupled dissipator.

## 2 Relation between displacement coupling and velocity coupling

Under suitable assumptions, we will show the equivalence between coupling through displacements and coupling through velocities of two elastic systems.

Let  $H$  and  $G$  be Hilbert spaces. Let  $A$  and  $C$  be positive self-adjoint unbounded operators acting on  $H$  and  $G$  respectively, with compact resolvents. Let  $B$  be an unbounded operator from  $G$  to  $H$  such that  $D(A) \subset D(B^*)$  and  $D(C) \subset D(B)$ . Let  $D$  be a positive self-adjoint operator on  $G$ .

We start by giving conditions for the well-posedness of system (4). We denote by  $(\cdot, \cdot)_H$  and  $(\cdot, \cdot)_G$  the scalar products on  $H$  and  $G$  respectively and we set  $X = H \times G$ .

**Proposition 1** *If there exists  $c \in [0, \frac{1}{2}[$  such that, for all  $(u, v) \in D(A) \times D(C)$*

$$|(u, Bv)| < c \left( \|A^{1/2}u\|_H^2 + \|C^{1/2}v\|_G^2 \right) \quad (6)$$

*then  $S = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$  is positive self-adjoint on  $X$  with domain  $D(S) = D(A) \times D(C)$ .*

**Proof**  $S$  is clearly symmetric and condition (6) implies its invertibility. In fact, for  $(h, g) \in H \times G$ , we look for a solution  $(u, v) \in D(S)$  of the system

$$\begin{aligned} Au + Bv &= h \\ B^*u + Cv &= g. \end{aligned} \quad (7)$$

Solving for  $u$  in the first equation of (7) and replacing in the second, we obtain

$$\begin{aligned} u &= -A^{-1}Bv + A^{-1}h, \\ (C - B^*A^{-1}B)v &= g - B^*A^{-1}h. \end{aligned}$$

Thus, (7) has a solution in  $D(S)$  if and only if  $C - B^*A^{-1}B$  is boundedly invertible. But, actually, one has

$$C - B^*A^{-1}B = C^{1/2}(I - C^{-1/2}B^*A^{-1}BC^{-1/2})C^{1/2}.$$

Now, condition (6) may be rewritten as

$$\left| (A^{1/2}u, (A^{-1/2}BC^{-1/2})C^{1/2}v) \right| < c \left( \|A^{1/2}u\|_H^2 + \|C^{1/2}v\|_G^2 \right)$$

for all  $(u, v) \in D(S)$ . Set  $x = A^{1/2}u \in D(A^{1/2})$ ,  $y = C^{1/2}v \in D(C^{1/2})$  and  $T = A^{-1/2}BC^{-1/2}$ . Then

$$|(x, Ty)| < c (\|x\|_H^2 + \|y\|_G^2).$$

Let  $x = Ty$  in this last inequality. Then

$$\|Ty\|_H^2 \leq \frac{c}{1-c} \|y\|_G^2 \quad y \in D(C^{1/2}).$$

Then, since  $C^{-1/2}B^*A^{-1}BC^{-1/2} = T^*T$  and  $\frac{c}{1-c} < 1$ , it follows that  $I - C^{-1/2}B^*A^{-1}BC^{-1/2}$  is boundedly invertible and, moreover maps  $D(C^{1/2})$  into itself. So, the conclusion follows since

$$v = C^{-1/2}(I - C^{-1/2}B^*A^{-1}BC^{-1/2})^{-1}C^{-1/2}(g - B^*A^{-1}h) \in D(C).$$

This implies that  $S$  is self-adjoint. The positivity follows from (6).  $\diamond$

As a consequence of this proposition, it follows that system (4) is well-posed in the energy space  $V = D(S^{\frac{1}{2}}) \times X$ .

**Remark 2** Using (6) we have

$$\begin{aligned} & (1-2c) \left( \left\langle \begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_{H \times G} + \|\partial_t w\|_H^2 + \|\partial_t z\|_G^2 \right) \\ & \leq \left( \left\langle \begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_{H \times G} + \|\partial_t w\|_H^2 + \|\partial_t z\|_G^2 \right) \\ & \leq (1+2c) \left( \left\langle \begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_{H \times G} + \|\partial_t w\|_H^2 + \|\partial_t z\|_G^2 \right) \end{aligned}$$

**Theorem 3** If  $(w, z)$  is a solution of system (5), then  $(u, v)$  defined by

$$\begin{aligned} u &= -A^{-1/2} \partial_t w \\ v &= z \end{aligned} \tag{8}$$

is a solution of the system

$$\begin{aligned} \partial_{tt} u &= -Au + A^{-1/2} B \partial_t v \\ \partial_{tt} v &= -B^* A^{-1/2} \partial_t u - (C - B^* A^{-1} B) v - D \partial_t v \end{aligned} \tag{9}$$

Moreover, under the assumptions of Proposition 1,  $C - B^* A^{-1} B$  is self-adjoint positive. Conversely, if  $(u, v)$  is a solution of (9), then  $(w, z)$  defined by

$$\begin{aligned} w &= A^{-1/2} u_t - A^{-1} B v \\ z &= v \end{aligned} \tag{10}$$

is a solution of (5).

**Proof** We use direct computations. From (8), one has

$$\partial_t u = -A^{-1/2} \partial_{tt} w = -A^{-1/2} (-Aw - Bz) = A^{1/2} w + A^{-1/2} Bz \tag{11}$$

and thus

$$\partial_{tt} u = A^{1/2} \partial_t w + A^{-1/2} B \partial_t z = -Au + A^{-1/2} B \partial_t v.$$

For the second equation in  $v$ , using (11) one has

$$\begin{aligned} \partial_{tt} v &= -B^* w - Cv - D \partial_t v \\ &= -B^* A^{-1/2} \partial_t u - (C - B^* A^{-1} B) v - D \partial_t v. \end{aligned}$$

and this gives system (9). The converse is obtained by the same computations.

### 3 Uniform stability

Our main result is the uniform stability of (5). Before giving the assumptions on the operators  $A, B, C$  and  $D$  to obtain the uniform stability we define the

operators

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ -A & 0 & -B & 0 \\ 0 & 0 & 0 & I \\ -B^* & 0 & -C & -D \end{pmatrix},$$

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & I & 0 & 0 \\ -A & 0 & 0 & A^{-1/2}B \\ 0 & 0 & 0 & I \\ 0 & -B^*A^{-1/2} & -C + B^*A^{-1}B & -D \end{pmatrix}.$$

The operator  $\mathcal{A}_1$  has dense domain,

$$D(\mathcal{A}_1) = D(A) \times D(A^{1/2}) \times D(C) \times (D(C^{1/2}) \cap D(D)).$$

The system (5) can be written as

$$\partial_t Y = \mathcal{A}_1 Y. \quad (12)$$

and (9) as

$$\partial_t Z = \tilde{\mathcal{A}} Z. \quad (13)$$

**Theorem 4** *The energy of system (12) decays exponentially if and only if the energy of system (13) decays exponentially in the same energy space.*

**Proof** Using the result of the Theorem 2, we have a bounded invertible transformation  $P \in \mathcal{L}(D(A^{1/2}) \times H \times D(C^{1/2}) \times G)$ , such that

$$\mathcal{A}_1 P = P \tilde{\mathcal{A}} \quad (14)$$

where

$$P = \begin{pmatrix} 0 & A^{-1/2} & -A^{-1}B & 0 \\ -A^{1/2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} 0 & -A^{-1/2} & 0 & 0 \\ A^{1/2} & 0 & A^{-1/2}B & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Applying  $P$  to (13) leads to

$$\begin{aligned} \partial_t PZ &= P \tilde{\mathcal{A}} Z \\ &= \mathcal{A}_1 PZ, \end{aligned}$$

Then  $PZ := W$  is a solution of (12). If we suppose the uniform stability of (13), that is: there exist positive constants  $M, \omega$  such that

$$\|Z(t)\|_{D(A^{1/2}) \times H \times D(C^{1/2}) \times G}^2 \leq M \|Z(0)\|_{D(A^{1/2}) \times H \times D(C^{1/2}) \times G}^2 \exp(-\omega t), \quad t \geq 0,$$

then for  $t \geq 0$ , we have

$$\begin{aligned} & \|P^{-1}W(t)\|_{D(A^{1/2}) \times H \times D(C^{1/2}) \times G}^2 \\ & \leq M \|P^{-1}W(0)\|_{D(A^{1/2}) \times H \times D(C^{1/2}) \times G}^2 \exp(-\omega t). \end{aligned}$$

Using the boundedness of  $P^{-1}$ , we obtain the uniform stability of (12).  $\diamond$

**Remark 5** 1)  $\mathcal{A}_1$  (respectively  $\tilde{\mathcal{A}}$ ) generates a  $C_0$ -semigroup on  $D(A^{1/2}) \times H \times D(C^{1/2}) \times G$ , denoted by  $S_{\mathcal{A}_1}(t)$  (respectively  $S_{\tilde{\mathcal{A}}}(t)$ ). And

$$S_{\mathcal{A}_1}(t) = PS_{\tilde{\mathcal{A}}}(t)P^{-1}, \quad t \geq 0$$

2) The semigroups  $S_{\mathcal{A}_1}(t)$  and  $S_{\tilde{\mathcal{A}}}(t)$  have the same type.

Denote by  $B_1 := A^{-1/2}B$  and  $C_1 := C - B^*A^{-1}B$ . To give a sufficient conditions for the uniform stabilization of the system (5) we will need the following assumption: (H)  $B_1$  is boundedly invertible and the operators  $D^{-1/2}B_1^{-1}A^{1/2}$ ,  $D^{-1/2}C_1B_1^{-1}A^{-1/2}$ ,  $D^{1/2}B_1^{-1}$  and  $A^{-1/2}B_1^{-1}D^{-1/2}$  all extend to bounded operators on  $X$ .

**Proposition 6** Under condition (6) and assumption (H) we have the uniform stability of (5).

**Proof** The result of this proposition follows from the relation between displacement coupling and velocity coupling, using Theorem 3, then the uniform stability of the system (5) is equivalent to the uniform stability of the system

$$\begin{aligned} \partial_{tt}u &= -Au + A^{-1/2}B\partial_tv \\ \partial_{tt}v &= -B^*A^{-1/2}\partial_tu - (C - B^*A^{-1}B)v - D\partial_tv. \end{aligned}$$

Using a result in [1] we construct a Lyapunov function associated with the system above. Let  $U = \begin{pmatrix} u \\ v \end{pmatrix}$  and

$$\begin{aligned} \chi_\varepsilon \left( \begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix} \right) &:= \frac{1}{2} (\|U(t)\|_{D(S^{1/2})}^2 + \|\partial_t U(t)\|_X^2) \\ &+ \varepsilon \left( \frac{1}{2} (u, \partial_t u) + \frac{1}{4} \|D^{1/2}v\|^2 + \frac{1}{2} (v, \partial_t v) \right) \\ &+ \varepsilon \left( \frac{1}{2} (u, B_1 v) + (\partial_t v, B_1^{-1} \partial_t u) + (C_1 v, B_1^{-1} u) \right). \end{aligned}$$

Next, we prove (see [1]) that for  $\varepsilon > 0$  sufficiently small there exist  $a_\varepsilon, b_\varepsilon$  and  $c_\varepsilon$  positive constants such that

$$\begin{aligned} a_\varepsilon(\|U(t)\|_{D(S^{1/2})}^2 + \|\partial_t U(t)\|_X^2) &\leq \chi_\varepsilon\left(\begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix}\right) \\ &\leq b_\varepsilon(\|U(t)\|_{D(S^{1/2})}^2 + \|\partial_t U(t)\|_X^2) \end{aligned} \quad (15)$$

and

$$\frac{d}{dt}\chi_\varepsilon\left(\begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix}\right) \leq -c_\varepsilon \cdot \chi_\varepsilon\left(\begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix}\right). \quad (16)$$

Then the proof of this proposition is derived from (15) and (16). In fact, from (16) it follows that

$$\chi_\varepsilon\left(\begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix}\right) \leq \exp(-c_\varepsilon t)\chi_\varepsilon\left(\begin{pmatrix} U(0) \\ \partial_t U(0) \end{pmatrix}\right)$$

and from (15) we have

$$\|U(t)\|_{D(S^{1/2})}^2 + \|\partial_t U(t)\|_X^2 \leq \frac{b_\varepsilon}{a_\varepsilon}(\|U(0)\|_{D(S^{1/2})}^2 + \|\partial_t U(0)\|_X^2) \exp(-c_\varepsilon t).$$

## 4 Applications

### Particular cases

In this subsection we set  $H = G$  and assume that  $B, C$  and  $D$  are powers of the positive self-adjoint operator  $A$ , in this case we consider the system

$$\begin{aligned} \partial_{tt}u + Au + a.A^\alpha v &= 0 \\ \partial_{tt}v + A^\beta v + a.A^\alpha u + A^\gamma \partial_t v &= 0, \end{aligned} \quad (17)$$

where  $a \neq 0$  is a real constant such that

$$|a| \left\| A^{\alpha - (\frac{\beta+1}{2})} \right\|_H < 1.$$

and  $\alpha, \beta, \gamma$  are real constants. Our objective is to find conditions on  $\alpha, \beta, \gamma$  in order to obtain the uniform stability of the above system. By using our result (Theorem 3), the uniform stability of system (17) is equivalent to the uniform stability of the system

$$\begin{aligned} \partial_{tt}u + Au - a.A^{\alpha - \frac{1}{2}} \partial_t v &= 0 \\ \partial_{tt}v + A^\beta (I - a^2 A^{2\alpha - \beta - 1})v + a.A^{\alpha - \frac{1}{2}} \partial_t u + A^\gamma \partial_t v &= 0. \end{aligned} \quad (18)$$

We remark that the operator  $A^{2\alpha - \beta - 1}$  is a compact perturbation from  $D(A^{1/2})$  to  $H$ , then the uniform stability of system (18) is equivalent to the uniform stability of the system

$$\begin{aligned} \partial_{tt}u + Au - a.A^{\alpha - \frac{1}{2}} \partial_t v &= 0 \\ \partial_{tt}v + A^\beta v + a.A^{\alpha - \frac{1}{2}} \partial_t u + A^\gamma \partial_t v &= 0. \end{aligned} \quad (19)$$

**Proposition 7 ([1])** *When  $\beta \neq 1$ , system (19) is uniformly stable if*

$$\gamma \in [\max(2 - 2\alpha, 2\alpha - 2, 2\beta - 2\alpha), 2\alpha - 1].$$

*When  $\beta = 1$ , system (19) is uniformly stable if and only if*

$$\gamma \in [\max(0, 2\alpha - 2), 2\alpha - 1].$$

### Timoshenko beam

We consider in this example a model of Timoshenko beam [13, 14]. The equations of motion for this system are given by

$$\begin{aligned} \rho \partial_{tt} u &= K \partial_{xx} u - K \partial_x v \quad \text{in } ]0, l[ \times \mathbb{R}^+ \\ I_\rho \partial_{tt} v &= EI \partial_{xx} v + K(\partial_x u - v) - b(x) \partial_t v \quad \text{in } ]0, l[ \times \mathbb{R}^+ \\ u(0, t) &= u(l, t) = v(0, t) = v(l, t) = 0. \end{aligned} \quad (20)$$

This system is coupled with the initial conditions

$$\begin{aligned} u(x; 0) &= u_0(x) & \partial_t u(x; 0) &= u_1(x) \\ v(x; 0) &= v_0(x) & \partial_t v(x; 0) &= v_1(x) \end{aligned} \quad (21)$$

Here,  $t$  is the time variable and  $x$  is the space coordinate along the beam in its equilibrium position. The functions  $u(x, t)$  is the transverse displacement of the beam and  $v(x, t)$  is the rotation angle of a filament of the beam. The coefficients  $\rho, I_\rho, E$  and  $I$  are the mass per unit length, the mass moment of inertia of the cross section, Young's modulus and the moment of inertia of the cross section, respectively. The coefficient  $K$  is the shear modulus and  $b(x)$  is a positive function on  $[0, l]$ . The energy of the beam is given by

$$E(t) = \frac{1}{2} \int_0^l (\rho (\partial_t u)^2 + I_\rho (\partial_t v)^2 + EI (\partial_x v)^2 + K (\partial_x u - v)^2) dx$$

**Remark 8** In this example  $B = -\frac{K}{\rho} \partial_x$  is not a power of the operator  $A = \frac{K}{\rho} \partial_{xx}$ .

**Theorem 9 ([13, 14])** *If  $b(x) > 0$  on  $[0, l]$ , then*

$$E(t) \leq M.e^{-at} E(0) \quad \text{if and only if} \quad \frac{K}{\rho} = \frac{EI}{I_\rho}.$$

## 5 Exact Controllability

In this section we obtain an exact controllability result for displacement coupled dissipator systems. We suppose that  $D$  is **bounded** on  $G$ . Our result is as follows.

**Theorem 10** *There exists  $T > 0$  and  $c_1 \geq 0$  such that the solution of the system*

$$\begin{aligned} \begin{pmatrix} \partial_{tt}\varphi(t) \\ \partial_{tt}\phi(t) \end{pmatrix} &= \begin{pmatrix} -A & -B \\ -B^* & -C \end{pmatrix} \begin{pmatrix} \varphi(t) \\ \phi(t) \end{pmatrix} \\ \begin{pmatrix} \varphi(0) \\ \phi(0) \end{pmatrix} &= \begin{pmatrix} \varphi_0 \\ \phi_0 \end{pmatrix}, \quad \begin{pmatrix} \partial_t\varphi(0) \\ \partial_t\phi(0) \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \phi_1 \end{pmatrix} \end{aligned} \quad (22)$$

*satisfies*

$$\begin{aligned} &\left| \begin{pmatrix} \partial_t\varphi(0) \\ \partial_t\phi(0) \end{pmatrix} \right|_{H \times G}^2 + \left\langle \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \phi(0) \end{pmatrix}, \begin{pmatrix} \varphi(0) \\ \phi(0) \end{pmatrix} \right\rangle_{H \times G} \\ &\leq c_1 \int_0^T \left| D^{1/2} \partial_t \phi(t) \right|_G^2 dt \end{aligned}$$

*if and only if, the system*

$$\begin{aligned} \partial_{tt}u &= -Au + A^{-1/2}B\partial_tv \\ \partial_{tt}v &= -B^*A^{-1/2}\partial_tu - (C - B^*A^{-1}B)v - D\partial_tv \\ (u(0), v(0)) &= (u_0, v_0) \quad (\partial_tu(0), \partial_tv(0)) = (u_1, v_1) \end{aligned} \quad (23)$$

*is exponentially stable.*

**Proof** By Theorem 3, the uniform stability of (23) is equivalent to the uniform stability of (5). We write the system (5) as

$$\begin{aligned} \partial_{tt} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} &= 0 \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad \begin{pmatrix} \partial_tu(0) \\ \partial_tv(0) \end{pmatrix} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}. \end{aligned}$$

Using Proposition 1 we deduce that  $S$  is a self-adjoint, coercive operator on  $X$ , and  $\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$  is a bounded positive operator on  $X$ . Applying a result of Haraux [6], we conclude the present proof.

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