Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 27, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Bilinear spatial control of the velocity term in a Kirchhoff plate equation *

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Abstract

We consider a bilinear optimal control problem with the state equation being a Kirchhoff plate equation. The control is a function of the spatial variables and acts as a multiplier of the velocity term. The unique optimal control, driving the state solution close to a desired evolution function, is characterized in terms of the solution of the optimality system.

1 Introduction

We consider the problem of controlling the solution of a Kirchhoff plate equation. The motion with appropriate boundary conditions describes the motion of a thin plate which is clamped along one portion of its boundary and has free vibrations on the other portion of the boundary. We consider bilinear optimal control, acting as a multiplier of a velocity term, is a function of the spatial variables x and y.

Given control

$$h \in U_M = \{h \in L^\infty(\Omega); -M \le h(x,y) \le M\},$$

the 'displacement' solution $w = w(h)$ of our state equation satisfies

$$\begin{aligned} w_{tt} + \Delta^2 w &= h(x,y) w_t \quad \text{on } Q = \Omega \times (0,T) \\ w(x,y,0) &= w_{01}(x,y), \quad w_t(x,y,0) = w_{02}(x,y) \\ w = \frac{\partial w}{\partial \nu} &= 0 \quad \text{on } \Sigma_0 = \Gamma_0 \times (0,T), \\ \Delta w + (1-\mu) B_1 w &= 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \\ \frac{\partial \Delta w}{\partial \nu} + (1-\mu) B_2 w &= 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,T), \end{aligned} \tag{1.1}$$

* *Mathematics Subject Classifications:* 49K20, 35F10.
Key words: Kirchhoff plate, optimal control, bilinear control.

Submitted July 26, 2000. Published May 1, 2001.

where $\Omega \subset \mathbb{R}^2$ with C^2 boundary, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \neq \emptyset$, $\nu = \langle n_1, n_2 \rangle$ is the outward unit normal vector on $\partial\Omega$, and

$$\begin{aligned} B_1 w &= 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx}, \\ B_2 w &= \frac{\partial}{\partial \tau}\left[(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})\right]. \end{aligned}$$

The direction τ in $B_2 w$ is the tangential direction along Γ_1. The plate is clamped along Γ_0 and has free vibrations along Γ_1. The constant μ, $0 < \mu < \frac{1}{2}$, represents Poisson's ratio.

We take as our objective functional

$$J(h) = \frac{1}{2}\left(\int_Q (w - z)^2 dQ + \beta \int_\Omega h^2(x, y)\, d\Omega\right)$$

where z is the desired evolution for the plate and the quadratic term in h represents the cost of implementing the control with weighting factor $\beta > 0$. For convenience, we assume that

$$\begin{aligned} z &\in C([0, T]; L^2(\Omega)), \\ z_t &\in C([0, T]; L^2(\Omega)). \end{aligned}$$

We seek to minimize the objective functional, i.e., characterize an optimal control $h^* \in U_M$ such that

$$J(h^*) = \min_{h \in U_M} J(h).$$

For background on plate models and control, see the books by Lagnese and Lions [15], Lagnese [13], Lagnese, Leugering, and Schmidt [14], Kormornik [11], Li and Yong [18], and Lions [19]. The bilinear control case treated here does not fit into the Riccati framework [17]; even though the objective functional is quadratic, the state equation has a bilinear term, hw_t. See [4, 6, 8, 9, 10, 12, 16] for control papers involving Kirchhoff plates. Bilinear control problems similar to the problem here were introduced in three papers by Ball, Marsden, and Slemrod [1-3], and in Bradley and Lenhart [5] (with control acting through the term hw). Note that in a recent paper by Bradley, Lenhart and Yong, the case of $h(t)w_t$ was treated [7].

In section 2, we show well-posedness of our state problem. In section 3, we show the existence of an optimal control by a minimizing sequence argument. In section 4, we derive a characterization for optimal controls, in terms of the solutions of an optimality system. The optimality system consists of the state equation coupled with an adjoint equation, and it is derived by differentiating the objective functional and the map $h \to w(h)$ with respect to the control. In section 5, we prove that the optimal control is unique for small time, T, provided that initial data are taken to be sufficiently smooth.

2 Well-posedness of the State Equation

We will begin by proving existence, uniqueness, and regularity results for the state equation. We first define our solution spaces:

$$H^2_{\Gamma_0}(\Omega) = \left\{ w \in H^2(\Omega) | w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}$$

and

$$\mathcal{H} = H^2_{\Gamma_0}(\Omega) \times L^2(\Omega).$$

Note that the bilinear form on $H^2_{\Gamma_0}(\Omega)$,

$$a(u,v) = \int_\Omega \{\Delta w \Delta v + (1-\mu)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}w_{xx}\} \, d\Omega$$

induces a norm on $H^2_{\Gamma_0}(\Omega)$ which is equivalent to the usual H^2 norm on $H^2_{\Gamma_0}(\Omega)$ (see [12]).

Definition Given $h \in U_M$, $\tilde{w} = \tilde{w}(h) = (w(h), w_t(h))$ is a weak solution to (1.1) if $\tilde{w} \in C([0,T]; \mathcal{H})$, $\tilde{w}(0) = (w_{01}, w_{02})$ and $\tilde{w}$ satisfies

$$\int_0^T \langle w_{tt}, \phi \rangle dt + \int_0^T a(w,\phi)(t) dt = \int_Q h w_t \phi \, d\Omega \, dt$$

for all $\phi \in H^2_{\Gamma_0}(\Omega)$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $[H^2_{\Gamma_0}(\Omega)]'$ and $H^2_{\Gamma_0}(\Omega)$.

Theorem 2.1 **(i)** *Let $\tilde{w}(0) = (w_{01}, w_{02}) \in \mathcal{H}$ and $h \in U_M$. Then the system (1.1) has a unique weak solution $\tilde{w} = \tilde{w}(h) = (w, w_t)$.*

(ii) *In addition, if $(w_{01}, w_{02}) \in D_0$ where*

$$D_0 = \Big\{ (w_{01}, w_{02}) \in (H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega) : \\ \Delta w_{01} + (1-\mu) B_1 w_{01} = 0 \text{ on } \Gamma_1, \\ \frac{\partial \Delta w_{01}}{\partial \nu} + (1-\mu) B_2 w_{01} = 0 \text{ on } \Gamma_1 \Big\}$$

for $h \in U_M$, then the weak solution satisfies

$$\tilde{w} \in C([0,T]; (H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega))$$

and $w_{tt} \in C([0,T]; L^2(\Omega))$.
Furthermore, (1.1) holds in the L^2 sense.

Proof. (i) To write the system in semigroup form, we define the operator $\mathcal{A}$: $\mathcal{A}w = \Delta^2 w$ with domain

$$\mathcal{D}(\mathcal{A}) = \Big\{ w \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) : \Delta w + (1-\mu)B_1 w = 0 \text{ on } \Gamma_1 \\ \text{and } \frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w = 0 \text{ on } \Gamma_1 \Big\}.$$

Then define operator $A : H^4(\Omega) \times H^2_{\Gamma_0}(\Omega) \to \mathcal{H}$ by

$$A\tilde{w} = \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \tilde{w} \quad \text{with } \mathcal{D}(A) = D_0 .$$

Then the stated equation (1.1) can be written as

$$\frac{d}{dt}\tilde{w}(t) = A\tilde{w}(t) + B\tilde{w}(t)$$
$$\tilde{w}(0) = \tilde{w}_0 = \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix}$$

with $B\tilde{w}(t) = \begin{bmatrix} 0 \\ hw_t(t) \end{bmatrix}$. Using skew-adjointness, the operator A generates a strongly continuous unitary group on $\mathcal{H}$. Since B is a bounded perturbation of A on $\mathcal{H}$, by standard semigroup theory [20], we have the conclusion of (i).

(ii) Assume that $\tilde{w}_0 \in D_0$ and $h \in U_M$. From variation of parameters [20] and (i),

$$\tilde{w}(t) = e^{At}\tilde{w}_0 + \int_0^t e^{A(t-\tau)} B(\tilde{w})(\tau)d\tau, \tag{2.1}$$

where e^{At} represents the semigroup generated by A. Proceeding to formally differentiate (2.1) in the t variable and defining a new variable $\tilde{v} = (v_1, v_2) = \frac{d\tilde{w}}{dt}$, we seek a solution of the form:

$$\tilde{v}(t) = Ae^{At}\tilde{w}_0 + B\tilde{w}(t) + \int_0^t Ae^{A(t-\tau)} B\tilde{w}(\tau)d\tau.$$

Setting

$$F\tilde{v} = Ae^{At}\tilde{w}_0 + B\tilde{w}(t) + \int_0^t Ae^{A(t-\tau)} B\tilde{w}(\tau)d\tau, \tag{2.2}$$

we seek a fixed point of F, i.e. we seek a unique point $\tilde{v} \in C([0,T];\mathcal{H})$ such that

$$F\tilde{v} = \tilde{v}.$$

Note that

$$\begin{aligned}\int_0^t Ae^{A(t-\tau)} B\tilde{w}(t)dt &= -\int_0^t \frac{d}{d\tau}\Big(e^{A(t-\tau)} B\tilde{w}(\tau)\Big)d\tau + \int_0^t e^{A(t-\tau)} \frac{d}{d\tau} B\tilde{w}(\tau)d\tau \\ &= -B\tilde{w}(t) + e^{At}\begin{bmatrix} 0 \\ hw_{02} \end{bmatrix} + \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ hw_{\tau\tau} \end{bmatrix} d\tau,\end{aligned}$$

where $w_{02} = w_t(x, y, 0)$. Thus from (2.2), F can be rewritten as

$$F(\tilde{v}) = Ae^{At}\tilde{w}_0 + e^{At}\begin{bmatrix} 0 \\ hw_{02} \end{bmatrix} + \int_0^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ hv_2 \end{bmatrix} d\tau.$$

Since $\tilde{w}_0 \in \mathcal{D}(A)$ and $h = h(x, y) \in U_M \subset L^\infty(\Omega)$, $F : C([0, T]; \mathcal{H}) \to C([0, T]; \mathcal{H})$ is bounded. We now verify that F is a contraction on $C([0, T]; \mathcal{H})$ for small T_0 for $0 \le t \le T_0$,

$$\begin{aligned}\|F\tilde{v}_1 - F\tilde{v}_2\|_{C([0,T_0];\mathcal{H})} &\le \left\| \int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ h(v_{12} - v_{22}) \end{bmatrix} d\tau \right\|_{C([0,T_0];\mathcal{H})} \\ &\le \sup_{0\le t\le T_0} \int_0^t \|h(v_{12} - v_{22})(\tau)\|_{L^2(\Omega)} d\tau \\ &\le T_0 M \|\tilde{v}_1 - \tilde{v}_2\|_{C([0,T_0];\mathcal{H})}.\end{aligned}$$

Taking $T_0 < \frac{1}{M}$, we have the F is a contractive mapping on $C([0, T_0], \mathcal{H})$. To complete the proof, we set $\tilde{v}(T_0)$ (with $\tilde{v}$ being the fixed point) as the new initial data and repeat the argument to obtain F as a contraction on $C([T_0, 2T_0], \mathcal{H})$. Repeating this procedure yields the result on $[0, T]$.

We observe first that

$$(w_t, w_{tt}) \in C([0, T]; \mathcal{H}),$$

and then $hw_t \in L^2(Q)$ with equation (1.1), gives

$$\Delta^2 w \in C([0, T]; L^2(\Omega)).$$

By standard elliptic theory, $w \in C([0, T]; H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega))$. □

We now present an *a priori* estimate needed for the existence of an optimal control.

Lemma 2.1 (*A priori* estimate) *Given $\tilde{w}(0) \in \mathcal{H}$ and $h \in U_M$, the weak solution to (1.1) satisfies*

$$\|\tilde{w}\|_{C([0,T];\mathcal{H})} \le (1 + MT)^{1/2} e^{C_2 MT} \|\tilde{w}(0)\|_{\mathcal{H}}. \tag{2.3}$$

Proof. Since D_0 is dense in $\mathcal{H}$, there exists a sequence, $\{\tilde{w}(0)^n\}$ in D_0, such that

$$\tilde{w}(0)^n \to \tilde{w}(0) \quad \text{strongly in } \mathcal{H}.$$

Denoting by $\tilde{w}^n$ the solution of (1.1) with initial data $\tilde{w}(0)^n$ and control h, then $\tilde{w}^n$ has the additional regularity from Theorem 2.1(ii). Using w^n_t as a multiplier in (1.1), we obtain

$$\begin{aligned}0 &= \int_0^s \int_\Omega (w^n_{tt} w^n_t + \Delta^2 w^n w^n_t - h(w^n_t)^2)\, d\Omega\, dt \\ &= \int_0^s \int_\Omega \frac{1}{2}\frac{d}{dt}(w^n_t)^2\, d\Omega\, dt + \int_0^s \frac{1}{2}\frac{d}{dt} a(w^n, w^n) dt - \int_0^s \int_\Omega h(w^n_t)^2\, d\Omega\, dt\,.\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}(w_t^n)^2(x,y,s)\,d\Omega+\frac{1}{2}a(w^n,w^n)(s)\\
&\quad=\frac{1}{2}\|w_{02}^n\|^2_{L^2(\Omega)}+\frac{1}{2}a(w_{01}^n,w_{01}^n)+\int_0^s\int_{\Omega}h(w_t^n)^2\,d\Omega\,dt\\
&\quad\le\frac{1}{2}\|\tilde{w}(0)^n\|^2_{\mathcal{H}}+M\int_0^s\|\tilde{w}^n(t)\|^2_{\mathcal{H}}dt.
\end{aligned}$$

Gronwall's Inequality implies

$$\sup_{0\le s\le T}\left\{\int_{\Omega}(w_t^n)^2(x,y,t)\,d\Omega+a(w^n,w^n)(s)\right\}\le\|\tilde{w}(0)^n\|^2_{\mathcal{H}}(1+2MT)e^{\tilde{C}MT},\tag{2.4}$$

which gives the desired result for smooth approximations. Now we can pass to the limit and obtain (2.3) for $\tilde{w}$.

3 Existence of Optimal Controls

We now prove the existence of an optimal control by a minimizing sequence argument.

Theorem 3.1 *There exists an optimal control $h^*\in U_M$, which minimizes the objective functional $J(h)$ over h in U_M.*

Proof. Let $\{h^n\}$ be a minimizing sequence in U_M, i.e.,

$$\lim_{n\to\infty}J(h^n)=\inf_{h\in U_M}J(h).$$

By Lemma 2.1, for $\tilde{w}^n=\tilde{w}(h^n)$,

$$\|\tilde{w}^n\|_{C([0,T],\mathcal{H})}\le C_1e^{C_2MT}.$$

On a subsequence, we have

$$\begin{aligned}
&w^n\rightharpoonup w^*\quad\text{weakly* in } L^\infty([0,T];H^2_{\Gamma_0}(\Omega)),\\
&w_t^n\rightharpoonup w_t^*\quad\text{weakly* in } L^\infty([0,T];L^2(\Omega)),\\
&w_{tt}^n\rightharpoonup w_{tt}^*\quad\text{weakly* in } L^\infty([0,T];(H^2_{\Gamma_0}(\Omega))'),\\
&h^n\rightharpoonup h^*\quad\text{weakly in } L^2(\Omega).
\end{aligned}$$

The convergence of the w_{tt}^n sequence follows from the PDE (1.1) and the estimate from Lemma 2.1

In weak form, w^n satisfies

$$\int_0^T[\langle w_{tt}^n,\phi\rangle+a(w^n,\phi)(t)]dt=\int_Q h^nw_t^n\phi dQ,\tag{3.1}$$

where we now allow that $\phi = \phi(x,y,t)$ in $L^2([0,T],\mathcal{H})$ and ϕ_t in $L^2(Q)$. In the convergence as $n \to \infty$, the only difficult term is on the RHS of (3.1). Examining the RHS we see that

$$\begin{aligned}\text{RHS } &= \int_0^T \int_\Omega h^n(x,y) w_t^n(x,y,t)\phi(x,y,t)\, d\Omega\, dt \\ &= -\int_0^T \int_\Omega h^n(x,y) w^n(x,y,t)\phi_t(x,y,t)\, d\Omega\, dt \\ &\quad + \int_\Omega h^n(x,y)\left(w^n(x,y,T)\phi(x,y,T) - w_{01}\phi(x,y,0)\right)\, d\Omega\,,\end{aligned}$$

where we have used the fact that $w_{01}^n = w_{01}$ for all n. Now by a standard result from semigroup theory (see, for example [20]), we know that $w^n(x,y,t) \in C([0,T],H^2_{\Gamma_0})$, which implies that $w^n(x,y,T) \in H^2_{\Gamma_0}(\Omega) \subset C(\Omega)$, since $\Omega \subset \mathbb{R}^2$. As a consequence, we may pass with a limit on this equation to obtain

$$\begin{aligned}\text{RHS } \to &-\int_0^T \int_\Omega h^*(x,y) w^*(x,y,t)\phi_t(x,y,t)\, d\Omega\, dt \\ &+ \int_\Omega h^*(x,y)\left(w^*(x,y,T)\phi(x,y,T) - w_{01}\phi(x,y,0)\right)\, d\Omega \quad \text{as } n \to \infty.\end{aligned}$$

From this we obtain $\tilde{w}^* = \tilde{w}(h^*)$, which is the weak solution of (1.1) with control h^*. Since the objective functional is lower semi-continuous with respect to weak convergence, we obtain

$$J(h^*) = \inf_{h \in U_M} J(h)$$

and h^* is an optimal control.

4 Necessary Conditions

We now derive necessary conditions that any optimal control must satisfy. To derive these necessary conditions, we must differentiate our functional $J(h)$ and $w = w(h)$ with respect to h. The differentiation of J results in a characterization of optimal controls in terms of the optimality system.

Lemma 4.1 *The mapping*

$$h \in U_M \to \tilde{w}(h) \in C([0,T];\mathcal{H})$$

is differentiable in the following sense:

$$\frac{\tilde{w}(h + \varepsilon\ell) - \tilde{w}(h)}{\varepsilon} \rightharpoonup \tilde{\psi} \quad \textit{weakly* in } L^\infty([0,T];\mathcal{H}),$$

as $\varepsilon \to 0$, for any $h, h+\varepsilon\ell \in U_M$. Moreover, the limit $\tilde{\psi} = (\psi, \psi_t)$ is a weak solution to the following system

$$\begin{aligned}
\psi_{tt} + \Delta^2\psi - h\psi_t = \ell w_t \quad \textit{in } Q\\
\psi(x,y,0) = \psi_t(x,y,0) = 0 \quad \textit{in } \Omega\\
\psi = \frac{\partial\psi}{\partial\nu} = 0 \quad \textit{on } \Sigma_0\\
\Delta\psi + (1-\mu)B_1\psi = 0 \quad \textit{on } \Sigma_1\\
\frac{\partial\Delta\psi}{\partial\nu} + (1-\mu)B_2\psi = 0 \quad \textit{on } \Sigma_1\,.
\end{aligned} \tag{4.1}$$

Proof. Denote by $\tilde{w}^\varepsilon = \tilde{w}(h+\varepsilon\ell)$ and $\tilde{w} = \tilde{w}(h)$. By (1.1), $(\tilde{w}^\varepsilon - \tilde{w})/\varepsilon$ is a weak solution of

$$\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_{tt} + \Delta^2\left(\frac{w^\varepsilon - w}{\varepsilon}\right) = h\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t + \ell w_t^\varepsilon \text{ in } Q$$

with homogeneous initial and boundary conditions. Using the proof of Lemma 2.1 with source term ℓw_t^ε, we obtain

$$\left\|\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon}\right\|_{C([0,T];\mathcal{H})} \le \|\ell w_t^\varepsilon\|_{L^2(Q)}\, e^{CMT}.$$

But we have *a priori* estimates on w_t^ε,

$$\|\ell w_t^\varepsilon\|_{L^2(Q)} \le T\|\ell\|_\infty \|\tilde{w}^\varepsilon\|_{C(0,T;\mathcal{H})} \le (1+MT)^{1/2} e^{C_2MT}\|\tilde{w}(0)\|_{\mathcal{H}},$$

using Lemma 2.1 on $\tilde{w}^\varepsilon$. Hence on a subsequence, as $\varepsilon \to 0$,

$$\frac{\tilde{w}^\varepsilon - \tilde{w}}{\varepsilon} \rightharpoonup \tilde{\psi} \quad \text{weakly* in } L^\infty([0,T];\mathcal{H}).$$

Similar to the proof of Theorem 3.1, we obtain that $\tilde{\psi}$ is a weak solution of (4.1). □

We obtain the existence of an adjoint solution and use it in the differentiation of the map $h \to J(h)$ to obtain our characterization of an optimal control.

Theorem 4.1 *Given an optimal control h^* in U_M and corresponding state solution $\tilde{w}^* = \tilde{w}(h^*)$ to (1.1), there exists a unique weak solution*

$$\tilde{p} = (p, p_t) \in C([0,T];\mathcal{H})$$

to the adjoint problem:

$$\begin{aligned}
p_{tt} + \Delta^2 p + hp_t = w^* - z \quad \textit{in } Q\\
p = \frac{\partial p}{\partial\nu} = 0 \quad \textit{on } \Sigma_0\\
\Delta p + (1-\mu)B_1 p = 0 \quad \textit{on } \Sigma_1\\
\frac{\partial\Delta p}{\partial\nu} + (1-\mu)B_2 p = 0 \quad \textit{on } \Sigma_1\\
p(x,y,T) = p_t(x,y,T) = 0 \quad \textit{(transversality condition).}
\end{aligned} \tag{4.2}$$

Furthermore

$$h^*(x,y) = \max\Big(-M, \min\Big(-\frac{1}{\beta}\int_0^T w_t^* p(x,y,t)dt, M\Big)\Big). \tag{4.3}$$

Proof. The proof of existence of the solution to the adjoint equation is similar to the proof of existence of solution of the state equation since the source term $(w^* - z) \in C([0,T], L^2(\Omega))$. However, since $p(x,y,T) = 0$, there is a difference in the constant in the *a priori* estimate:

$$\begin{aligned}\sup_{0\le s\le T}&\Big\{\int_\Omega (p_t^n)^2(x,y,t)\,d\Omega + a(p^n,p^n)(s)\Big\}\\ &\le \|w^* - z\|^2_{C([0,T],L^2(\Omega))}(1+2MT)e^{\tilde{C}MT}.\end{aligned}$$

We now proceed to characterize the optimal control in terms of the state $\tilde{w} = (w, w_t)$ and and adjoint $\tilde{p} = (p, p_t)$. Let $h^* + \varepsilon\ell$ be another control in U_M and $\tilde{w}^\varepsilon = \tilde{w}(h^* + \varepsilon\ell)$ be the corresponding solution to the state equation. Then since J achieves its minimum at h^*, we have

$$\begin{aligned}0 &\le \lim_{\varepsilon\to0^+}\frac{J(h^*+\varepsilon\ell) - J(h^*)}{\varepsilon}\\ &= \lim_{\varepsilon\to0^+}\int_Q \Big(\frac{w^\varepsilon - w^*}{\varepsilon}\Big)\Big(\frac{w^\varepsilon + w^* - 2z}{2}\Big)\,dQ + \frac{\beta}{2}\int_\Omega (2\ell h^* + \varepsilon\ell^2)\,d\Omega\\ &= \int_Q \psi(w^* - z)dQ + \beta\int_\Omega h^*\ell\,d\Omega\,.\end{aligned}$$

Substituting in from the adjoint equation (4.2) for $w^* - z$ and then using ψ PDE (4.1), we obtain

$$\begin{aligned}0 &\le \int_0^T \langle\psi, p_{tt}\rangle dt + \int_0^T a(\psi,p)dt + \int_Q \psi h^* p_t\,dQ + \beta\int_\Omega h^*\ell\,d\Omega\\ &= \int_0^T \langle\psi_{tt}, p\rangle dt + \int_0^T a(\psi,p)dt - \int_Q \psi_t h^* p dQ + \beta\int_\Omega h^*\ell\,d\Omega\\ &= \int_\Omega \ell\Big(\beta h^* + \int_0^T (w_t^* p)dt\Big)\,d\Omega.\end{aligned}$$

Using a standard control argument based on the choices for the variation $\ell(x,y)$, we obtain the desired characterization for h^*:

$$h^*(x,y) = \max\Big(-M, \min\Big(-\frac{1}{\beta}\int_0^T w_t^* p(x,y,t)dt, M\Big)\Big).$$

5 Uniqueness of the Optimal Control

We now characterize the optimal control as the unique solution to the optimality system

$$
\begin{aligned}
w_{tt} + \Delta^2 w &= \max\Big(-M, \min\Big(-\frac{1}{\beta}\int_0^T w_t^* p(x,y,s)ds, M\Big)\Big) w_t \\
&\qquad\qquad \text{in } Q = \Omega \times (0,T) \\
p_{tt} + \Delta^2 p &= -\max\Big(-M, \min\Big(-\frac{1}{\beta}\int_0^T w_t^* p(x,y,s)ds, M\Big)\Big) p_t + w^* - z \text{ in } Q \\
w = p = \frac{\partial w}{\partial \nu} &= \frac{\partial p}{\partial \nu} = 0 \quad \text{on} \Sigma_0 = \Gamma_0 \times (0,T), \qquad \text{(OS)} \\
\Delta w + (1-\mu)B_1 w &= \Delta p + (1-\mu)B_1 p = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,T) \\
\frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w &= \frac{\partial \Delta p}{\partial \nu} + (1-\mu)B_2 p = 0 \quad \text{on } \Sigma_1 = \Gamma_1 \times (0,T) \\
w(x,y,0) = w_{01}(x,y), &\quad w_t(x,y,0) = w_{02}(x,y) \quad \text{on } \Omega, \\
p(x,y,T) &= p_t(x,y,T) = 0 \quad \text{on } \Omega \times T\,.
\end{aligned}
$$

The existence of solutions to (OS) is given by Theorems 2.1 and 4.1. We now prove uniqueness, provided that time T is sufficiently small and that the conditions on initial data are as in Theorem 2.1(ii).

Theorem 5.1 *The solution to the optimality system, (OS), is unique for T sufficiently small, optimal control $h \in U_M$ and initial data such that $(w_{01}, w_{02}) \in D_0$, as in Theorem 2.1(ii).*

Proof. Since $z_t \in C([0,T]; L^2(\Omega))$, an extension of Theorem 2.1(ii) gives $p_t \in C([0,T]; H^2(\Omega))$. Also, Theorem 2.1(ii) directly implies

$$w_t \in C([0,T]; H^2_{\Gamma_0}(\Omega)).$$

Consequently, we have that $w,\ p,\ w_t$, and p_t are all bounded functions over $\overline{Q}$.

Suppose we have two weak solutions corresponding to two optimal controls, h and $\overline{h}$:

$$\tilde{w} = (w, w_t), \tilde{p} = (p, p_t), \hat{w} = (\overline{w}, \overline{w}_t), \hat{p} = (\overline{p}, \overline{p}_t).$$

We then have that $(\tilde{w} - \hat{w})$ and $(\tilde{p} - \hat{p})$ are weak solutions to the following system of equations

$$
\begin{aligned}
(w - \overline{w})_{tt} + \Delta^2(w - \overline{w}) &= h w_t - \overline{h}\overline{w}_t \quad \text{in } Q \\
-(p - \overline{p})_{tt} - \Delta^2(p - \overline{p}) &= h p_t - \overline{h}\overline{p}_t + (\overline{w} - w) \quad \text{in } Q \\
(w - \overline{w})(t=0) = 0; &\quad (w - \overline{w})_t(t=0) = 0 \quad \text{in } \Omega \times \{0\} \\
(p - \overline{p})(t=T) = 0; &\quad (p - \overline{p})_t(t=T) = 0 \quad \text{in } \Omega \times \{T\},
\end{aligned}
$$

where we denote

$$h = \max\Big(-M, \min\Big(\int_0^T w_t p dt, M\Big)\Big), \tag{5.1.a}$$

$$\overline{h} = \max\Big(-M, \min\Big(\int_0^T \overline{w}_t \overline{p} dt, M\Big)\Big). \tag{5.1.b}$$

Also, we have homogeneous boundary conditions as in (OS). (Here, we have multiplied through the p-equation by -1.) Multiplying the w-equation by $(w - \overline{w})_t$ (resp. the p-equation by $(p - \overline{p})_t$) and integrating by parts over $\Omega \times [0, t]$ (resp. over $\Omega \times [t, T]$) we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big[((w - \overline{w})_t)^2(t) + ((p - \overline{p})_t)^2(t)\big]\, d\Omega \\ &+\frac{1}{2}a(w - \overline{w}, w - \overline{w})(t) + \frac{1}{2}a(p - \overline{p}, p - \overline{p})(t) \\ &\qquad = \int_\Omega \int_0^t (hw_t - \overline{h}\overline{w}_t)(w - \overline{w})_t d\tau d\Omega \\ &\qquad + \int_\Omega \int_t^T ((hp_t - \overline{h}\overline{p}_t)(p - \overline{p})_t + (\overline{w} - w)(p - \overline{p})_t) d\tau d\Omega. \end{aligned} \tag{5.2}$$

To estimate the RHS of equation (5.2), we note that

$$\begin{aligned} hw_t - \overline{h}\overline{w}_t &= h(w - \overline{w})_t + \overline{w}_t(h - \overline{h}), \\ hp_t - \overline{h}\overline{p}_t &= h(p - \overline{p})_t + \overline{p}_t(h - \overline{h}). \end{aligned}$$

Consequently, we have

$$\begin{aligned} &\int_\Omega \int_0^t |(hw_t - \overline{h}\overline{w}_t)(w - \overline{w})_t|\, d\tau\, d\Omega \\ &\le \int_\Omega \int_0^t \big[|h|((w - \overline{w})_t)^2 + |h - \overline{h}||\overline{w}_t||(w - \overline{w})_t|\big]\, d\tau\, d\Omega \\ &\le C\int_\Omega \int_0^t \big[((w - \overline{w})_t)^2 + |h - \overline{h}||(w - \overline{w})_t|\big]\, d\tau\, d\Omega \quad (h \text{ and } w_t \text{ are bounded}) \\ &\le C\int_\Omega \int_0^t ((w - \overline{w})_t)^2\, d\tau\, d\Omega \\ &\quad + C\Big(\int_\Omega |h - \overline{h}|^2\, d\Omega\Big)^{1/2}\Big(\int_\Omega \int_0^t ((w - \overline{w})_t)^2 d\tau d\Omega\Big)^{1/2} \quad (\text{Hölder's in space}) \\ &\le \frac{3C}{2}\int_\Omega \int_0^t ((w - \overline{w})_t)^2\, d\tau\, d\Omega + \frac{C}{2}\int_\Omega |h - \overline{h}|^2\, d\Omega\,. \end{aligned} \tag{5.3}$$

Similarly, for we have for the p-term

$$\int_\Omega \int_t^T |(hp_t - \overline{h}\overline{p}_t)(p - \overline{p})_t|\, d\tau\, d\Omega \le \frac{3C}{2}\int_\Omega \int_t^T ((p - \overline{p})_t)^2 d\tau d\Omega + \frac{C}{2}\int_\Omega |h - \overline{h}|^2 d\Omega. \tag{5.4}$$

We now estimate the h-term using equations (5.1.a)-(5.1.b).

$$\begin{aligned}
&\int_\Omega |h-\overline{h}|^2 d\Omega \\
&\le \frac{1}{\beta^2}\int_\Omega \Big|\int_0^T (w_t p - \overline{w}_t\overline{p})\,d\tau\Big|^2 d\Omega \\
&= \frac{1}{\beta^2}\int_\Omega \Big|\int_0^T [(w-\overline{w})_t p + (p-\overline{p})\overline{w}_t d\tau]\Big|^2 d\Omega \\
&\le \frac{2}{\beta^2}\Big\{\int_\Omega [\big(\int_0^T (w-\overline{w})_t p d\tau\big)^2 + \big(\int_0^T (p-\overline{p})\overline{w}_t d\tau\big)^2] d\Omega\Big\} \qquad (5.5)\\
&\le \frac{2}{\beta^2}\Big\{\int_\Omega \Big[\int_0^T ((w-\overline{w})_t)^2 d\tau \int_0^T p^2 d\tau + \int_0^T (p-\overline{p})^2 d\tau \int_0^T (\overline{w}_t)^2 d\tau\Big] d\Omega\Big\} \\
&\le C\int_\Omega\int_0^T \big(((w-\overline{w})_t)^2 + (p-\overline{p})^2\big)\,d\tau\,d\Omega\,.
\end{aligned}$$

In the previous to the last inequality we used Hölder's in time, and in the last inequality the boundedness of p and $\overline{w}_t$. Bounding the last term in equation (5.2), we have

$$\begin{aligned}
&\int_\Omega\int_t^T (\overline{w}-w)(p-\overline{p})_t d\tau d\Omega \qquad (5.6)\\
&\quad \le \frac{1}{2}\int_\Omega\int_t^T (w-\overline{w})^2 d\tau d\Omega + \int_\Omega\int_t^T ((p-\overline{p})_t)^2 d\tau d\Omega.
\end{aligned}$$

Putting together equations (5.2)-(5.6),

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \big(((w-\overline{w})_t)^2(t) + ((p-\overline{p})_t)^2(t)\big)\,d\Omega \\
&+\frac{1}{2}a(w-\overline{w},w-\overline{w})(t) + \frac{1}{2}a(p-\overline{p},p-\overline{p})(t) \qquad (5.7)\\
&\quad \le \overline{C}\int_\Omega \Big\{\int_0^T ((w-\overline{w})_t)^2 d\tau + \int_0^T ((p-\overline{p})_t)^2 d\tau \int_0^T ((w-\overline{w})_t)^2 d\tau \\
&\quad + \int_0^T (p-\overline{p})^2 d\tau + \int_0^T (w-\overline{w})^2 d\tau\Big\} d\Omega\,.
\end{aligned}$$

Now taking a supremum in time on both sides of the inequality, we have that

$$\begin{aligned}
&\sup_{0\le t\le T}\Big\{\frac{1}{2}\int_\Omega \big(((w-\overline{w})_t)^2(t) + ((p-\overline{p})_t)^2(t)\big)\,d\Omega \\
&+\frac{1}{2}a(w-\overline{w},w-\overline{w})(t) + \frac{1}{2}a(p-\overline{p},p-\overline{p})(t)\Big\} \\
&\qquad \le \sup_{0\le t\le T}\overline{C}T\int_\Omega \big[((w-\overline{w})_t)^2(t) + ((p-\overline{p})_t)^2(t) \\
&\qquad + (p-\overline{p})^2(t) + (w-\overline{w})^2(t)\big] d\Omega.
\end{aligned}$$

Using Poincare's inequality,

$$\int_\Omega \left[(w-\overline{w})^2(t)+(p-\overline{p})^2(t)\right] d\Omega \le a(w-\overline{w},w-\overline{w})(t)+a(p-\overline{p},p-\overline{p})(t).$$

Finally, by taking T such that $\overline{C}T<\frac{1}{2}$, we have that

$$\sup_{0\le t\le T}\Big\{\int_\Omega \big(((w-\overline{w})_t)^2(t)+((p-\overline{p})_t)^2(t)\big)\,d\Omega + a(w-\overline{w},w-\overline{w})(t)+a(p-\overline{p},p-\overline{p})(t)\Big\}\le 0,$$

which implies

$$(w-\overline{w})=(p-\overline{p})=(w-\overline{w})_t=(p-\overline{p})_t=0 \Longrightarrow \tilde{w}=\hat{w} \text{ and } \tilde{p}=\hat{p}.$$

This completes present proof. □

Remark: In [7], we were able to solve the corresponding bilinear control problem, but with the controlled velocity coefficient, $h(t)$, being a function of time only. A main difference between our result here and that result is in the proofs of the existence and the uniqueness of the optimal control. The case of the controlled velocity coefficient, $h(x,y,t)$, is an open problem at this time.

Acknowledgment The second author was partially supported by the National Science Foundation.

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