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A PRIORI ESTIMATES FOR GLOBAL SOLUTIONS AND MULTIPLE EQUILIBRIA OF A SUPERLINEAR PARABOLIC PROBLEM INVOLVING MEASURES

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Abstract. We consider a noncoercive elliptic problem in a bounded domain with a power nonlinearity and measure data. It is known that the problem possesses a stable solution and we prove existence of three further solutions. The proof is based on uniform bounds of global solutions of the corresponding parabolic problem and on a topological degree argument.

1. Introduction

In this paper we consider the problem

$$
\begin{aligned}
u_t = \Delta u + |u|^{p-1}u + \mu, &\quad x \in \Omega,\ t > 0, \\
u = 0, &\quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), &\quad x \in \bar{\Omega},
\end{aligned}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain with $n \geq 2$, μ is a positive bounded Radon measure on Ω and

$$
p > 1, \quad p < \frac{n}{n-2} \ \text{ if } n > 2. \tag{1.2}
$$

The restriction $p(n-2) < n$ is not of technical nature, it is necessary for the local existence of the solution (see [7] or [26] and the references therein).

If $\mu = 0$ and $p > 1$, $p(n-2) < n+2$, then the Ljusternik-Schnirelman theory guarantees the existence of infinitely many stationary solutions of problem (1.1). A generalization of this result for $\mu \neq 0$, μ regular, was obtained under various additional assumptions on μ and p using perturbation methods in [25], [5], [23], [6] (see [9] and the references therein for the case of non-homogeneous boundary conditions and related problems). Variational methods were used also in [27] for the proof of existence of at least four solutions of the homogeneous Dirichlet problem for the equation $0 = \Delta u + f(u)$, where f was a superlinear (non-symmetric) C^1-function with subcritical growth (see [8] for additional properties of these solutions and further references).

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In the present paper, we assume (1.2) and we consider a general measure μ of the form

$$\mu = a\mu_0, \qquad \text{where } a \in \mathbb{R}^+ \text{ and } \mu_0 \text{ is a bounded positive Radon measure on } \Omega. \tag{1.3}$$

Denote

$$a^* = \sup\{a > 0 \,:\, (1.1) \text{ has a positive equilibrium}\}.$$

It follows from [7] (see also [4] for a more general setting) that $a^* > 0$. Assuming

$$0 < a < a^*, \tag{1.4}$$

we show existence of at least four stationary solutions of (1.1). We use a dynamical method which consists in looking for stationary solutions in the ω-limit sets of some global trajectories of (1.1). This approach does not require any symmetry of the problem so that one can use it also for more general problems (for example, $u_t = \Delta u + f(u) + \mu$, where f is as in [27]). In general, our method can yield different solutions from those obtained by variational methods: see [19], where it is used in the study of the Dirichlet problem for the equation $0 = \Delta u + u_+^p - u_-^q$ with $0 < q < 1 < p$, $p(n-2) < n+2$.

The crucial prerequisites for our approach are a priori estimates for global solutions of (1.1). In the case $\mu = 0$ and $p > 1$, $p(n-2) < n+2$, it is known that all global solutions of (1.1) are bounded and the corresponding bound depends only on a suitable norm of the initial function u_0, see [20]. In this paper we generalize the a priori estimates from [20] to the case $\mu \neq 0$ (under assumptions (1.2),(1.3),(1.4)) and then we use these estimates for the dynamical proof of existence of multiple equilibria. The main difficulty in this generalization and the subsequent application consists in the fact that the solutions of (1.1) are not regular enough for the direct use of the technical tools exploited in [20],[19] (for example, the standard Lyapunov functional is not well defined in our situation). These difficulties also rule out a straightforward use of variational methods for the proof of the multiplicity result.

Positive stationary solutions of problem (1.1) were studied by several authors, see references in [4]. If problem (1.1) has a positive equilibrium then there exists a minimal positive equilibrium v_1 of this problem (see [4]). It follows from [4, Theorems 1.2, 1.3] that assuming (1.2),(1.3),(1.4), problem (1.1) admits at least two positive equilibria. The proof of this multiplicity result was based on a priori estimates of positive stationary solutions and the computation of the Leary-Schauder index of the solution v_1. In the present paper, we shall use the local information on the solution v_1 and our a priori estimates of global solutions of (1.1) in order to prove the existence of equilibria v_2, v_3, v_4 such that $v_2 > v_1 > v_3$ and $v_4 - v_1$ changes sign.

Assumption (1.4) is crucial also for the proof of a priori estimates of global solutions of (1.1): instead of estimating the singular solution $u(t)$ (which need not be even continuous, in general), we estimate the difference $w(t) = u(t) - v_1$ which turns out to be a Hölder continuous function.

Let us mention that a priori estimates of global or periodic solutions of similar superlinear parabolic problems with regular data were already used for the proof of existence of positive stationary solutions (see [17], [10], [24]), sign-changing stationary solutions (see [19], [13]), infinitely many stationary solutions (see [22]), periodic solutions (see [11], [12], [16]), for establishing the blow-up rate of nonglobal solutions (see [15], [14]), and for the study of the boundary of domains of attraction of stable equilibria (see [18]).

This paper is organized as follows. Section 2 deals with existence and regularity of solutions of (1.1). Main results of the paper are stated in Theorem 3.1 (a priori estimates) and Theorem 4.5 (existence of multiple stationary solutions).

2. PRELIMINARIES

Let $q \in (1,\infty)$, $q' = q/(q-1)$, let $W_q^z(\Omega)$, $z \geq 0$, denote the usual Sobolev-Slobodeckii space and γ the trace operator, $\gamma : W_q^z(\Omega) \to W_q^{z-1/q}(\partial\Omega)$ for $z > 1/q$. For $\theta \in I_q := [-2,2] \setminus \{1/q + m \,:\, m \in \mathbb{Z}\}$ put

$$W^\theta := W^\theta_{q,\gamma} := \left\{ \begin{array}{ll} \{u \in W_q^\theta(\Omega) \,:\, \gamma u = 0\} & \text{if } 1/q < \theta, \\ W_q^\theta(\Omega) & \text{if } 0 \leq \theta < 1/q, \\ \big(W^{-\theta}_{q',\gamma}\big)' & \text{if } \theta < 0, \end{array} \right\} \tag{2.1}$$

and let $|\cdot|_{\theta,q}$ denote the norm in $W^\theta_{q,\gamma}$. The norm in $W^0_{q,\gamma} = L_q(\Omega)$ will be denoted simply by $|\cdot|_q$. The norm in the Hölder space $C^{0,\alpha}(\bar\Omega)$ will be denoted by $\|\cdot\|_{0,\alpha}$.

Let $M = M(\Omega)$ be the space of bounded Radon measures on Ω. The spaces $W^\theta_{q,\gamma}$ are ordered Banach spaces and $M(\Omega)$ is a Banach lattice (cf. [4, Section 5]). Moreover, $M(\Omega) \hookrightarrow W^\theta_{q,\gamma}$ provided $\theta < -n/q'$. For $u, v \in W^\theta_{q,\gamma}$, we write $u < v$ if $v - u$ belongs to the positive cone of $W^\theta_{q,\gamma}$ and $u \neq v$. We denote also $[u,v] = \{w : u \leq w \leq v\}$ and we put $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$. By c and C we denote positive constants which may vary from step to step; by $c_1, C_1, c_2, C_2, \dots$ we denote fixed positive constants.

Let $A_2 : W^2 \to W^0 : u \mapsto -\Delta u$. It is well known that A_2 is an isomorphism of W^2 onto W^0 and it generates an analytic semigroup in W^0. Moreover, the operator A_2 can be extended to an isomorphism $A_0 : W^0 \to W^{-2}$ such that the $W^{\theta-2}$-realization A_θ of A_0 is an isomorphism of W^θ onto $W^{\theta-2}$ and it generates an analytic semigroup e^{-tA_θ} in $W^{\theta-2}$ for any $\theta \in I_q$, $\theta \geq 0$,

$$|e^{-tA_\theta}u|_{\eta,q} \leq Ce^{-ct}\big(|u|_{\eta,q} \wedge t^{(\theta-\eta)/2-1}|u|_{\theta-2,q}\big) \tag{2.2}$$

for any $\eta \in I_q \cap (\theta-2,\theta)$ (see [2]).

The results of [2, Section 12] imply that problem (1.1) admits a unique maximal solution $u \in C\big([0,T), W^z_{q,\gamma}\big)$ satisfying the variation-of-constants formula

$$u(t) = e^{-tA_z}u_0 + \int_0^t e^{-(t-\tau)A_z}(|u(\tau)|^{p-1}u(\tau) + \mu)\,d\tau \tag{2.3}$$

provided $u_0 \in W^z_{q,\gamma}$ and

$$-\frac{n}{p} \leq z - \frac{n}{q} < 2 - n, \quad q > 1, \quad z \geq 0, \quad z \in I_q. \tag{2.4}$$

The existence of a unique u satisfying (2.3) can be proved directly in the following way. Condition (2.4) guarantees $W^z_{q,\gamma} \hookrightarrow L_p(\Omega)$ and $L_1(\Omega) \hookrightarrow M(\Omega) \hookrightarrow W^{z-2+\varepsilon}_{q,\gamma}$

for some $\varepsilon > 0$, hence the mapping $F : W^z_{q,\gamma} \to W^{z-2+\varepsilon}_{q,\gamma} : u \mapsto |u|^{p-1}u + \mu$ is well defined and Lipschitz continuous. Now using (2.2) we obtain

$$\begin{aligned}
|e^{-tA_z}u_0|_{z,q} &\le C|u_0|_{z,q} \le C, \\
|e^{-tA_z}F(u)|_{z,q} &\le Ct^{-1+\varepsilon/2}|F(u)|_{z-2+\varepsilon,q} \\
&\le Ct^{-1+\varepsilon/2}\big(1+|u|^p_{z,q}\big), \\
|e^{-tA_z}\big(F(u)-F(v)\big)|_{z,q} &\le Ct^{-1+\varepsilon/2}\big(1+|u|^{p-1}_{z,q}+|v|^{p-1}_{z,q}\big)|u-v|_{z,q}.
\end{aligned} \tag{2.5}$$

These inequalities easily imply that the operator

$$R(u)(t) = e^{-tA_z}u_0 + \int_0^t e^{-(t-\tau)A_z}F\big(u(\tau)\big)\,d\tau$$

is a contraction in an appropriate ball of the Banach space $C\big([0,T], W^z_{q,\gamma}\big)$ if T is small enough. The fixed point of R is the solution of (2.3), hence of (1.1).

Solutions of (1.1) are not continuous, in general. Anyhow, if $u, v : [0,T] \to W^z_{q,\gamma}$ are two solutions of (1.1) with initial conditions u_0, v_0, respectively, then the difference $w(t) = u(t) - v(t)$ is Hölder continuous for $t > 0$ and its $C^{0,\alpha}(\bar\Omega)$-norm (where $\alpha > 0$ is sufficiently small) can be estimated by the $W^z_{q,\gamma}$-norm of $w(0)$. More precisely, the following lemma is true.

Lemma 2.1. *Let $u, v : [0,T] \to W^z_{q,\gamma}$ be two solutions of* (1.1) *with initial conditions $u(\cdot,0) = u_0$, $v(\cdot,0) = v_0$. Put $w = u - v$ and denote*

$$K_u = \sup_{\tau\in[0,T]} |u(\tau)|_{z,q}. \tag{2.6}$$

There exist $r > n$ and $\alpha > 0$ such that $w(t) \in C^{0,\alpha}(\bar\Omega) \cap W^1_{r,\gamma}$ for any $t > 0$ and

$$|w(t)|_{z,q} + |w(t)|_{1,r} + \|w(t)\|_{0,\alpha} \le c(t_0, T, K_u \vee K_v)|w(0)|_{z,q} \tag{2.7}$$

for any $t \in [t_0, T]$ and $t_0 > 0$. Moreover, $w \in C^{0,\tilde\alpha}([t_0,T], C^{0,\alpha}(\bar\Omega) \cap W^1_{r,\gamma})$ for some $\tilde\alpha > 0$ and the norm of w in this space can be bounded by a constant depending on $t_0, T, K_u \vee K_v$.

Proof. Let $\tilde z, q$ satisfy (2.4) (with z replaced by $\tilde z$), $\tilde z > z$. Estimating the $W^{\tilde z}_{q,\gamma}$-norm in (2.3) we obtain

$$\begin{aligned}
|u(t)|_{\tilde z,q} &\le Ct^{z-\tilde z}|u_0|_{z,q} + C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-1+\tilde\varepsilon/2}\big|F\big(u(\tau)\big)\big|_{\tilde z-2+\tilde\varepsilon}\,d\tau \\
&\le C(K_u)(1+t^{z-\tilde z}),
\end{aligned}$$

where $\tilde\varepsilon > 0$ is small enough. Using the imbedding $W^{z_1}_{q_1,\gamma} \hookrightarrow W^{z_2}_{q_2,\gamma}$ if $z_1 - \frac{n}{q_1} > z_2 - \frac{n}{q_2}$ and $z_1 \ge z_2$ and repeating the estimate above with different $z, \tilde z, q$, if necessary, we get

$$|u(t)|_{\tilde z,\tilde q} \le C(\delta, K_u) \quad \text{for any } t \in [\delta, T], \tag{2.8}$$

whenever $\tilde z, \tilde q$ satisfy (2.4). Analogous estimates and the generalized Gronwall inequality [1, Theorem II.3.3.1] imply

$$|w(t)|_{\tilde z,\tilde q} \le C(\delta, T, K_u, K_v)|w(0)|_{z,q} \quad \text{for any } t \in [\delta, T]. \tag{2.9}$$

Estimates (2.8) and (2.9) imply that we may assume both $z = 1$ and the boundedness of $u(\tau), v(\tau)$, $\tau \in [0,T]$, in $W^{\tilde z}_{\tilde q,\gamma}$ for any $\tilde z, \tilde q$ satisfying (2.4). In particular, $u(\tau), v(\tau)$ are bounded in $W^1_{q,\gamma}$ for any $q < n/(n-1)$, hence in $L_r(\Omega)$ for any $r < n/(n-2)$.

The function w solves the equation $w_t = \Delta w + \Phi(u,v)$ in Ω with

$$|\Phi(u,v)| = \big||u|^{p-1}u - |v|^{p-1}v\big| \le C|w|\varphi, \quad \varphi := \big(|u|+|v|\big)^{p-1},$$

where the function $\varphi(t)$ is bounded in $L_s(\Omega)$ for some $s > \frac{n}{2}$. Put $Q = q$, $R = 1$ and choose $\beta > 0$ and $\varepsilon > 0$ small. Then

$$L_R(\Omega) \hookrightarrow W^{\beta-1+\varepsilon}_{Q,\gamma} \quad \text{and} \quad W^{1+\beta}_{Q,\gamma} \hookrightarrow L_{sR/(s-R)}$$

due to

$$\frac{1}{R} < \frac{1}{Q} + \frac{1-\beta}{n} \quad \text{and} \quad \frac{1}{R} \ge \frac{1}{Q} + \frac{1}{s} - \frac{1+\beta}{n}, \quad s > R, \tag{2.10}$$

hence

$$\begin{aligned}|\Phi\big(u(\tau),v(\tau)\big)|_{\beta-1+\varepsilon,Q} &\le C|\Phi\big(u(\tau),v(\tau)\big)|_R = C|w(\tau)\varphi(\tau)|_R \\ &\le C|w(\tau)|_{sR/(s-R)}|\varphi(\tau)|_s \le C|w(\tau)|_{1+\beta,Q}.\end{aligned}$$

This and the variation-of-constants formula imply

$$\begin{aligned}|w(t)|_{1+\beta,Q} &\le Ct^{-\beta}|w_0|_{1,Q} + C\int_0^t (t-\tau)^{-1+\varepsilon/2}|\Phi\big(u(\tau),v(\tau)\big)|_{\beta-1+\varepsilon,Q}\,d\tau \\ &\le Ct^{-\beta}|w_0|_{1,Q} + C\int_0^t (t-\tau)^{-1+\varepsilon/2}|w(\tau)|_{1+\beta,Q},\end{aligned}$$

so that the Gronwall inequality implies

$$|w(t)|_{1+\beta,Q} \le C_1(t)|w_0|_{1,Q},$$

where $C_1(t)$ is bounded for t lying in compact subsets of $(0,T]$. Now $W^{1+\beta}_{Q,\gamma} \hookrightarrow W^1_{\tilde Q,\gamma}$ for some $\tilde Q > Q$. If $\tilde Q \le n$ then repeating the estimates above with Q replaced by $\tilde Q$, R by $\tilde R$ and β by $\tilde\beta$ such that (2.10) remains true, we obtain

$$|w(2t)|_{1+\tilde\beta,\tilde Q} \le C_2(t)|w(t)|_{1,\tilde Q} \le CC_1(t)C_2(t)|w_0|_{1,Q}.$$

A standard bootstrap argument yields the estimate of w in $W^1_{Q,\gamma}$ for some $Q > n$, hence in $C^{0,\alpha}(\bar\Omega)$ for some $\alpha > 0$. This shows (2.7). Notice that an upper bound for the bootstrap procedure is given by $1/Q > 1/s - 1/n$.

The Hölder continuity of $w : [t_0,T] \to W^1_{Q,\gamma}$ for some $Q > n$ follows from the variation-of-constants formula, the estimates above and the estimates

$$\begin{gathered}\Big|\int_{t_1}^{t_2} e^{-(t_2-\tau)A_z}\Phi(u(\tau),v(\tau))\,d\tau\Big|_{1,Q} \le C\int_{t_1}^{t_2}(t_2-\tau)^{-1+\varepsilon/2}\,d\tau \le C(t_2-t_1)^{\varepsilon/2}, \\ |e^{-(t_2-t_1)A_z}f - f|_{1,Q} \le C(t_2-t_1)^{\beta/2}|f|_{1+\beta,Q},\end{gathered}$$

where $T \ge t_2 > t_1 \ge t_0$ and $f := \int_0^{t_1} e^{-(t_1-\tau)A_z}\Phi(u(\tau),v(\tau))\,d\tau$ (cf. also [1, Theorem II.5.3.1]). $\square$

Remark 2.2. Let u and v be solutions of (1.1) on $[0,T]$ with initial conditions $u(\cdot,0)=u_0$ and $v(\cdot,0)=v_0$, where $u_0,v_0\in W^z_{q,\gamma}$ and z,q satisfy (2.4). Let $\mu_k\to\mu$ in $M(\Omega)$ and $u_{0,k}\to u_0$, $v_{0,k}\to v_0$ in $W^z_{q,\gamma}$. Let u_k,v_k be solutions of (1.1) with μ replaced by μ_k and initial conditions $u_k(\cdot,0)=u_{0,k}$, $v_k(\cdot,0)=v_{0,k}$. Put $w=u-v$, $w_k=u_k-v_k$. Then the variation-of-constants formula, estimates (2.5), Gronwall's inequality and obvious modifications of estimates in the proof of Lemma 2.1 imply that u_k,v_k are well defined on $[0,T]$ for k large enough, $\sup_{t\in[0,T]}|u(t)-u_k(t)|_{z,q}\to 0$ as $k\to\infty$ and $\sup_{t\in[t_0,T]}|w(t)-w_k(t)|_{1,r}\to 0$ as $k\to\infty$ for some $r>n$ and any $t_0>0$.

The following theorem follows from [21, Theorem 3.1].

Theorem 2.3. *Let z,q satisfy* (2.4) *and $u\in C\big([0,T),W^z_{q,\gamma}\big)$ be the maximal solution of* (1.1). *Let $u(t)$ be bounded in $L_r(\Omega)$ for $t\in[0,T)$, where $r>\frac{n}{2}(p-1)$, $r>1$. Then $T=+\infty$ and $u(t)$ is bounded in $W^z_{q,\gamma}$ for $t\in[0,\infty)$.*

3. A PRIORI ESTIMATES

The main result of this section is the following

Theorem 3.1. *Assume* (1.2),(1.3),(1.4). *Let u be a global solution of* (1.1) *and let z,q satisfy* (2.4). *Then $|u(t)|_{z,q}\le c$, where c depends only on the norm of u_0 in $W^z_{q,\gamma}$.*

Proof. Assumption (1.4) guarantees existence of the minimal positive stationary solution v_1. Let u be a global solution of (1.1) and put $w(t):=u(t)-v_1$.

The functions v_1 and w are solutions of the following problems

$$\begin{gathered} 0=\Delta v_1+v_1^p+\mu,\quad x\in\Omega,\\ v_1=0,\quad x\in\partial\Omega, \end{gathered}\tag{3.1}$$

and

$$\begin{gathered} w_t=\Delta w+h(w),\quad x\in\Omega,\ t>0,\\ w=0,\quad x\in\partial\Omega,\ t>0,\\ w(x,0)=u_0(x)-v_1(x),\quad x\in\bar\Omega \end{gathered}\tag{3.2}$$

where

$$h(w):=|w+v_1|^{p-1}(w+v_1)-v_1^p.$$

Due to Lemma 2.1, we have

$$w(t)\in W^1_{r,\gamma}\hookrightarrow C^{0,\alpha}(\bar\Omega)$$

for some $r>n$, $\alpha>0$ and any $t>0$. Moreover, putting

$$f(w):=\frac{1}{p+1}\big(|w+v_1|^{p+1}-v_1^{p+1}\big)-wv_1^p,$$

the regularity of w and the mean value theorem imply

$$|f(w)|\le p\big(|w|+|v_1|\big)^{p-1}|w|^2\le C(w)\big(1+|v_1|^{p-1}\big)\in L_s(\Omega)$$

for some $s > \frac{n}{2}$, since $v_1 \in L_r(\Omega)$ for any $r < \frac{n}{n-2}$ (see [4]). Consequently, the energy functional

$$E(w) := \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega f(w)\,dx \tag{3.3}$$

is well defined along the solution $w = w(t)$. Multiplying the equation in (3.2) by w and integrating by parts yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w(t)^2\,dx = -2E\big(w(t)\big) + \int_\Omega g\big(w(t)\big)\,dx, \tag{3.4}$$

where

$$g(w) = \frac{p-1}{p+1}|w+v_1|^{p+1} - |w+v_1|^{p-1}(w+v_1)v_1 + wv_1^p + \frac{2}{p+1}v_1^{p+1}.$$

We shall show in Lemma 3.2 that there exist positive constants $c_0, c_1, \dots, c_4$ such that

$$\begin{gathered} g(w) \geq c_0|w|^{p+1} - c_1 w^2 v_1^{p-1}, \\ c_2|w|^{p+1} + c_3 w^2 v_1^{p-1} \geq f(w) \geq c_4|w|^{p+1}. \end{gathered} \tag{3.5}$$

Assume that $\varepsilon > 0$. Integrating inequalities in (3.5) and using the estimate

$$\int_\Omega w^2 v_1^{p-1}\,dx \leq |w|_{r_1}^2 |v_1|_{r_2}^{p-1} = C|w|_{r_1}^2 \leq \varepsilon|\nabla w|_2^2 + C_\varepsilon|w|_2^2 \tag{3.6}$$

(where $r_1 < \frac{2n}{n-2}$ and $r_2 < \frac{n}{n-2}$ are suitable exponents required by the corresponding Hölder inequality) one obtains

$$\begin{gathered} \int_\Omega g(w)\,dx \geq \int_\Omega \big(C_0|w|^{p+1} - C_1 w^2 - \varepsilon|\nabla w|^2\big)\,dx, \\ \int_\Omega \big(C_2|w|^{p+1} + C_3 w^2 + \varepsilon|\nabla w|^2\big)\,dx \geq \int_\Omega f(w)\,dx \geq \int_\Omega C_4|w|^{p+1}\,dx, \end{gathered} \tag{3.7}$$

where $C_0, C_1, \dots, C_4$ are positive constants (and C_1, C_3 depend on ε). The choice of r_1, r_2 in (3.6) is possible due to

$$2\frac{n-2}{2n} + (p-1)\frac{n-2}{n} < 1.$$

Now (3.4), (3.7) and the choice $\varepsilon \leq 1/4 \wedge C_0/(8C_2)$ imply

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega w(t)^2\,dx &\geq -2(1+2\varepsilon)E\big(w(t)\big) + \tilde{C}_0\int_\Omega |w|^{p+1}\,dx - \tilde{C}_1\int_\Omega w^2\,dx \\ &\geq -2(1+2\varepsilon)E\big(w(t)\big) + \hat{C}_0\Big(\int_\Omega w^2\,dx\Big)^{(p+1)/2} - \hat{C}_1. \end{aligned} \tag{3.8}$$

Let $t_1 < t_2$ be fixed positive numbers and let μ_k be smooth positive functions, $\mu_k \to \mu$ in $M(\Omega)$. Denote by $(1.1)_k$ problem (1.1) with μ replaced by μ_k. Then problem $(1.1)_k$ admits a classical solution u_k defined on $[0, t_2]$ for k large enough (cf. Remark 2.2). Moreover, for k large enough, [4, Theorem 6.3] implies existence of positive stationary solutions $v_{1,k}$ of $(1.1)_k$ such that $v_{1,k} \to v_1$ in $W^z_{q,\gamma}$. Set

$w_k = u_k - v_{1,k}$ and denote by $(3.2)_k$ problem (3.2) with w replaced by w_k and $h(w)$ by $h_k(w_k) := |w_k + v_{1,k}|^{p-1}(w_k + v_{1,k}) - v_{1,k}^p$. Multiplying $(3.2)_k$ by $\partial_t w_k$ and integrating over $(x,t) \in Q := \Omega \times (t_1, t_2)$ yields

$$\int_{t_1}^{t_2} \int_\Omega (w_k)_t^2 \, dx \, dt = E_k\big(w_k(t_1)\big) - E_k\big(w_k(t_2)\big), \tag{3.9}$$

where

$$E_k(w_k) := \frac{1}{2} \int_\Omega |\nabla w_k|^2 \, dx - \int_\Omega f_k(w_k) \, dx,$$
$$f_k(w_k) := \frac{1}{p+1}\big(|w_k + v_{1,k}|^{p+1} - v_{1,k}^{p+1}\big) - w_k v_{1,k}^p.$$

Since the right-hand side of (3.9) is uniformly bounded due to Remark 2.2, we may assume that $(w_k)_t$ converges weakly in $L_2(Q)$ to some function $\tilde{w}$. Remark 2.2 implies the pointwise convergence of w_k to w in $\bar{Q}$, hence $h(w_k) \to h(w)$ in $L_r(Q)$ for any $r < n/[p(n-2)]$ (recall that $v_1 \in L_R(\Omega)$ for any $R < n/(n-2)$). Passing to the limit in the weak formulation of $(3.2)_k$ shows $\tilde{w} = w_t$. Thus, passing to the limit in (3.9) gives

$$\int_{t_1}^{t_2} \int_\Omega w_t^2 \, dx \, dt \le E\big(w(t_1)\big) - E\big(w(t_2)\big). \tag{3.10}$$

Consequently, the function $t \mapsto E\big(w(t)\big)$ is nonincreasing. Now (3.8) and the global existence of w imply both

$$\big|E\big(w(t)\big)\big| \le c \tag{3.11}$$

and

$$\int_\Omega w^2(t) \, dx \le c \tag{3.12}$$

(otherwise the function $y(t) := \int_\Omega w^2(t) \, dx$ has to blow up in finite time). Estimates (3.11) and (3.10) entail

$$\int_0^\infty \int_\Omega w_t^2 \, dx \, dt \le c. \tag{3.13}$$

Now (3.3),(3.7),(3.11) and (3.4),(3.7),(3.11) show that

$$\int_\Omega |\nabla w|^2 \, dx \le c\Big(1 + \int_\Omega |w|^{p+1} \, dx\Big) \le c\Big(1 + \frac{1}{2}\frac{d}{dt}\int_\Omega w^2 \, dx\Big),$$

hence

$$\int_\Omega |w|^{p+1} \, dx + \int_\Omega |\nabla w|^2 \, dx \le c\Big(1 + \Big|\int_\Omega w w_t \, dx\Big|\Big). \tag{3.14}$$

Squaring (3.14) and integrating over time yields

$$\int_t^{t+1} \Big(\int_\Omega |w|^{p+1} \, dx\Big)^2 dt + \int_t^{t+1} \Big(\int_\Omega |\nabla w|^2 \, dx\Big)^2 dt \le c, \tag{3.15}$$

where we have used

$$\int_t^{t+1} \Big|\int_\Omega w w_t \, dx\Big|^2 dt \le \int_t^{t+1} |w|_2^2 |w_t|_2^2 \, dt \le c \int_t^{t+1} |w_t|_2^2 \, dt \le c$$

(see (3.12) and (3.13)). Estimates (3.13), (3.15) and [10, the proof of Proposition 2] imply uniform bounds (depending only on $|u_0|_{z,q}$) for $|w(t)|_r$ if $r < 6n/(3n-4)$. Since $v_1 \in L_s(\Omega)$ for any $s < n/(n-2)$, the last estimate, (1.2) and Theorem 2.3 imply the a priori bound for $|u(t)|_{z,q}$ if $n > 2$.

If $n = 2$ then one can make a bootstrap argument based on maximal regularity as in [20] to get a priori bound in $L_r(\Omega)$ for any $r > 1$. Since it is not completely clear which estimate corresponds to [20, (16)] in our case (and also for the reader's convenience) we repeat the whole argument from [20].

We already know by (3.11) that

$$-C \le \frac{1}{2}\int_\Omega |\nabla w(t)|^2\,dx - \int_\Omega f\big(w(t)\big)\,dx \le C. \tag{3.16}$$

Moreover, (3.15) implies

$$\sup_{t\ge t_0}\int_t^{t+1} |w(s)|_{p+1}^{(p+1)r}\,ds < C \tag{3.17}$$

for any $t_0 > 0$ and $r = 2$. The interpolation theorem in [10, Appendice], (3.13) and (3.17) imply

$$\sup_{t\ge t_0} |w(t)|_\lambda < C \tag{3.18}$$

for any

$$\lambda < \lambda(r) := p + 1 - \frac{p-1}{r+1}.$$

Due to Theorem 2.3 and the definition of w, estimate (3.18) guarantees the required bound in $W^z_{q,\gamma}$ if

$$\lambda(r) > \frac{n}{2}(p-1) = p - 1,$$

or, equivalently,

$$p < p(r) := 2r + 3.$$

Fix $t_0 \in (0,1)$. Our bootstrap argument is as follows: assuming (3.17) for some $r \ge 2$, we shall show the same estimate for some $\tilde r > r$ (with the difference $\tilde r - r$ bounded away from zero). Thus, after finitely many steps we prove (3.17) with some r satisfying $2r + 3 > p$ which will conclude the proof.

Hence, let (3.17) be true for some $r \ge 2$. Then (3.18) is true for $\lambda < \lambda(r)$. Choose $\lambda \in \big(2, \lambda(r)\big)$ and denote

$$\theta = \frac{p+1}{p-1}\,\frac{\lambda-2}{\lambda}, \quad \lambda' = \frac{\lambda}{\lambda-1} \quad \text{and} \quad p_1 = \frac{p+1}{p}.$$

Then $\theta \in (0,1)$ and $\lambda' \in (p_1, 2)$ due to $\lambda < p + 1$. Moreover,

$$\frac{\theta}{p_1} + \frac{1-\theta}{2} = \frac{1}{\lambda'}.$$

Using (3.14), Hölder's inequality, (3.18) and interpolation, we obtain

$$\begin{aligned} |w(t)|_{1,2}^2 &\le C|\nabla w(t)|_2^2 \le C\big(1 + |w(t)w_t(t)|_1\big) \\ &\le C\big(1 + |w_t(t)|_{\lambda'}\big) \le C\big(1 + |w_t(t)|_{p_1}^{\theta}|w_t(t)|_2^{1-\theta}\big). \end{aligned} \tag{3.19}$$

Let $t \geq t_0$ and $\delta \in (0, t_0/2)$ be given. Then (3.15) implies

$$\int_{t-\delta}^{t-\delta/2} |w(s)|_{1,2}^2 \, ds < C,$$

hence there exists $\tau_1 \in (t-\delta, t-\delta/2)$ such that

$$|w(\tau_1)|_{1,2} < C_5,$$

where C_5 depends on δ but it does not depend on t. Given $\tilde{q} < n/(n-1) = 2$, the last estimate and $v_1 \in W^1_{\tilde{q}}(\Omega)$ (cf. [4]) imply $|u(\tau_1)|_{1,\tilde{q}} < \tilde{C}_5$, where $\tilde{C}_5 = \tilde{C}_5(C_5, v_1, \tilde{q})$. The existence proof for (1.1) based on (2.5) shows that there exists $\delta_1 = \delta_1(C_5, \tilde{C}_5) > 0$ small ($\delta_1 \leq \delta/4$) such that u and w stay bounded on $(\tau_1, \tau_1 + 2\delta_1)$ in $W^1_{\tilde{q},\gamma}$ by a constant $C_6 = C_6(C_5, \tilde{C}_5, \delta_1)$. By Lemma 2.1, $w(t)$ stays bounded in $C(\bar{\Omega})$ on $(\tau_1 + \delta_1, \tau_1 + 2\delta_1)$ by a constant $C_7 = C_7(C_6, \delta_1)$. Since $v_1 \in L_s(\Omega)$ for any s and $|h(w)| \leq \tilde{C}|v_1|^{p-1}$ if $|w| \leq C$, the function h stays bounded in $L_\rho(\Omega)$ (for some $\rho > p_1$) on $(\tau_1 + \delta_1, \tau_1 + 2\delta_1)$ by a constant $C_8 = C_8(C_7, v_1, \rho)$. Now standard estimates in the variation-of-constants formula for w on $(\tau_1 + \delta_1, \tau_1 + 2\delta_1)$ imply

$$|w(\tau_1 + 2\delta_1)|_{2-\varepsilon,\rho} \leq C_9,$$

where $C_9 = C_9(C_8, \delta_1, \varepsilon)$ and $\varepsilon > 0$ is small. Choose $\varepsilon < 2/p_1 - 2/\rho$. Then $W^{2-\varepsilon}_\rho(\Omega) \hookrightarrow X_P$, where $X_P := (E_0, E_1)_{1-1/P,P}$ is the real interpolation space between $E_0 = L_{p_1}(\Omega)$ and $E_1 = W^2_{p_1}(\Omega)$, and $P > 1$ is arbitrary. Consequently,

$$\|w(\tau)\|_{X_P} \leq C_{10}, \tag{3.20}$$

where $\tau := \tau_1 + 2\delta_1 \in (t-\delta, t)$. Notice that given $t \geq t_0$ and $\delta \in (0, t_0/2)$ we have found $\tau \in (t-\delta, t)$ and $C_{10} = C_{10}(\delta, v_1, |u_0|_{z,q}, P)$ such that (3.20) is true and C_{10} is independent of w and t.

We have $1 - \theta = \frac{2}{p-1}\big(\frac{p+1}{\lambda} - 1\big) < \frac{2}{r}$ for λ sufficiently close to $\lambda(r)$ since the last inequality is satisfied for $\lambda = \lambda(r)$. Now choose $\tilde{r} > r$ such that

$$\beta := \frac{2}{(1-\theta)\tilde{r}} > 1$$

and notice that $\theta\tilde{r}\beta' > 1$ where $\beta' = \beta/(\beta-1)$. Next we use (3.7) and (3.16), then (3.19), Hölder's inequality, (3.13), maximal Sobolev regularity (see [1, Theorem III.4.10.7]), (3.20) and inequality $|h(w)| \leq C(|w|^p + |v_1|^p)$ to get

$$\begin{aligned}
\int_\tau^{t+1} |w(s)|_{p+1}^{\tilde{r}(p+1)} \, ds &\leq C\Big(1 + \int_\tau^{t+1} |w(s)|_{1,2}^{2\tilde{r}} \, ds\Big) \\
&\leq C\Big(1 + \int_\tau^{t+1} |w_t(s)|_{p_1}^{\theta\tilde{r}} |w_t(s)|_2^{(1-\theta)\tilde{r}} \, ds\Big) \\
&\leq C\Big(1 + \Big(\int_\tau^{t+1} |w_t(s)|_{p_1}^{\theta\tilde{r}\beta'} \, ds\Big)^{1/\beta'} \underbrace{\Big(\int_\tau^{t+1} |w_t(s)|_2^2 \, ds\Big)^{1/\beta}}_{\leq C}\Big) \\
&\leq C\Big(1 + \Big(\int_\tau^{t+1} |h(w(s))|_{p_1}^{\theta\tilde{r}\beta'} \, ds\Big)^{1/\beta'} + \|w(\tau)\|_{X_P}^{\theta\tilde{r}}\Big) \\
&\leq C\Big(1 + \Big(\int_\tau^{t+1} |w(s)|_{p+1}^{p\theta\tilde{r}\beta'} \, ds\Big)^{1/\beta'}\Big),
\end{aligned}$$

where $P = \theta\tilde{r}\beta'$. Now we see that the last estimate implies (3.17) with $\tilde{r}$ instead of r provided $p\theta\tilde{r}\beta' \le \tilde{r}(p+1)$, that is if $\theta\beta' \le p_1$. This condition is equivalent to

$$p \le \frac{\lambda(\tilde{r}-1)-\tilde{r}}{\tilde{r}-2}. \tag{3.21}$$

Considering $\tilde{r} \to r+$ and $\lambda \to \lambda(r)-$ we see that it is sufficient to verify

$$p(r-2) < \lambda(r)(r-1) - r,$$

which is equivalent to $(p-1)2r > 0$. Consequently, the sufficient condition for bootstrap is satisfied and we are done. Note that the possibility of choosing $\tilde{r} - r$ bounded away from zero follows by an easy contradiction argument. $\square$

Lemma 3.2. *The functions f, g from the proof of Theorem 3.1 satisfy* (3.5) *for any $w \in \mathbb{R}$ and $v_1 > 0$.*

Proof. Since f and g can be viewed as positively homogeneous functions of two variables w, v_1 and $v_1 > 0$ one can put $v_1 = 1$. Consequently, we have to show

$$\begin{gathered} g_1(w) \ge c_0|w|^{p+1} - c_1 w^2, \\ c_2|w|^{p+1} + c_3 w^2 \ge f_1(w) \ge c_4|w|^{p+1}, \end{gathered}$$

where

$$\begin{aligned} f_1(w) &= \frac{1}{p+1}\big(|w+1|^{p+1} - 1\big) - w, \\ g_1(w) &= \frac{p-1}{p+1}|w+1|^{p+1} - |w+1|^{p-1}(w+1) + w + \frac{2}{p+1}. \end{aligned}$$

First let us show $f_1(w) \ge c_4|w|^{p+1}$. If $w > -1$ then $f_1'(w) = (w+1)^p - 1$ has the same sign as w and $f_1(w) = 0$, hence $f_1(w) > 0$ if $w > -1$, $w \ne 0$. Obviously, $f_1(w) \ge -\frac{1}{p+1} - w > 0$ if $w \le -1$. Since $f_1(w) \approx \frac{p}{2}w^2$ as $w \to 0$, there exists $\delta_1 > 0$ such that $f_1(w) \ge |w|^{p+1}$ for $|w| \le \delta_1$. Since $f_1(w)/|w|^{p+1} \to \frac{1}{p+1}$ as $|w| \to \infty$, there exists $K_1 > \delta_1$ such that $f_1(w) \ge \frac{1}{2(p+1)}|w|^{p+1}$ for $|w| \ge K_1$. The function f_1 is positive and continuous on the compact set $M_1 := [-K_1, -\delta_1] \cup [\delta_1, K_1]$, hence there exists $\varepsilon > 0$ such that $f_1(w) \ge \varepsilon K_1^{p+1} \ge \varepsilon|w|^{p+1}$ for $w \in M_1$. Consequently, it is sufficient to choose $c_4 = 1 \wedge \varepsilon \wedge \frac{1}{2(p+1)}$.

The same arguments as above show $f_1(w) \le c_2|w|^{p+1} + c_3 w^2$ if c_2, c_3 are sufficiently large.

The inequality for g_1 is equivalent to $G_1(w) + c_1 w^2 \ge G_2(w)$, where

$$\begin{aligned} G_1(w) &= \frac{p-1}{p+1}|w+1|^{p+1} + w + \frac{2}{p+1}, \\ G_2(w) &= |w+1|^{p-1}(w+1) + c_0|w|^{p+1}. \end{aligned}$$

Fix $c_0 < \frac{p-1}{p+1}$ and assume $c_1 \ge 1$. Since $G_1(w) - G_2(w) = o(w^2)$ as $w \to 0$, there exists $\delta_2 > 0$ such that $G_1(w) + c_1 w^2 \ge G_2(w)$ for $|w| \le \delta_2$ (and δ_2 does not depend on $c_1 \ge 1$). Since $G_1(w)/G_2(w) \to \frac{p-1}{(p+1)c_0} > 1$ as $|w| \to \infty$, there exists $K_2 > \delta_2$ such that $G_1(w) \ge G_2(w)$ for $|w| \ge K_2$. Since the function G_2 is bounded on the compact set $M_2 := [-K_2, -\delta_2] \cup [\delta_2, K_2]$ by some constant D_2, the choice $c_1 > D_2/\delta_2^2$ guarantees $c_1 w^2 \ge D_2 \ge G_2(w)$ on M_2. $\square$

4. STATIONARY SOLUTIONS

In this section we consider the problem

$$\begin{aligned} 0 = \Delta u + |u|^{p-1}u + \mu, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain, $n \geq 2$, p satisfies (1.2), and μ satisfies (1.3) and (1.4). Recall that assumption (1.4) guarantees the existence of the minimal positive solution v_1 of (4.1).

We fix $z = 1$ and q satisfying (2.4) and denote

$$X = W^z_{q,\gamma}, \quad Y = W^{z-2}_{q,\gamma} \quad \text{and} \quad Z = L_p(\Omega).$$

Notice that $X \hookrightarrow Z \hookrightarrow L_1(\Omega) \hookrightarrow M(\Omega) \hookrightarrow Y$. Recall also from Section 2 that $A := A_z : X \to Y$ is a linear isomorphism and denote

$$F(u) = |u|^{p-1}u + \mu \quad \text{and} \quad S = A^{-1}F.$$

The results of [4] imply that $A^{-1} \geq 0$, $F : Z \to Y$ and $S : Z \to X$ are nondecreasing, S is compact. The solutions of (4.1) correspond to the fixed points of the operator $S|_X : X \to X$. We denote by $\mathcal{E}$ the set of all solutions of (4.1).

In our study we shall use also the semiflow generated by problem (1.1). The considerations in Section 2 imply that this semiflow can be considered both in X and in Z. Due to [4, Theorem 5.1] and [1, Theorem II.6.4.1], this semiflow is order preserving.

We call $u \in Z$ a **supersolution** of (4.1) if $u \geq S(u)$ and $(1-e^{-tA})(u-S(u)) \geq 0$ for all $t > 0$. If $u \in X$ then these conditions may be replaced by a single condition $Au \geq Fu$: this follows from the following facts: $A^{-1} \geq 0$, $e^{-tA} \geq 0$, $\frac{1-e^{-tA}}{t}w \to Aw$ if $t \to 0$, $w \in X$, and $(1-e^{-tA})w = \int_0^t e^{-sA}Aw\,ds \geq 0$ if $w \in X$, $Aw \geq 0$. The subsolution is defined in an analogous way. One of the basic properties of sub- and supersolutions is formulated in the following

Proposition 4.1. *If $u^+ \in Z$ is a supersolution of (4.1) and $u_0 \in X$, $u_0 \leq u^+$, then the solution $u : [0, T_{\max}) \to X$ of (1.1) satisfies $u(t) \leq u^+$ for any $t \in [0, T_{\max})$, where $T_{\max}$ is the maximal existence time of this solution. Analogous assertion is true for subsolutions.*

Proof. The solution u can be (locally) obtained as the limit of the sequence $\{u_k\}$, where $u_1(t) \equiv u_0$ and

$$u_{k+1}(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F\big(u_k(s)\big)\,ds.$$

(cf. the existence proof in Section 2). We shall show by induction that $u_k(t) \leq u^+$. Obviously, $u_1(t) \leq u^+$. Hence assume $u_k(t) \leq u^+$. Then $F(u_k(s)) \leq F(u^+) = AS(u^+)$, so that

$$\begin{aligned} u_{k+1}(t) &\leq e^{-tA}u^+ + \int_0^t e^{-(t-s)A}AS(u^+)\,ds \\ &= e^{-tA}u^+ + S(u^+) - e^{-tA}S(u^+) \leq u^+. \quad \square \end{aligned}$$

In what follows, we shall construct a subsolution v_ε and a supersolution v^ε such that

$$v^\varepsilon \ge v_1 + \varepsilon, \quad v_\varepsilon \le -\varepsilon \quad \text{and} \quad [v_\varepsilon, v^\varepsilon] \cap \mathcal{E} = \{v_1\}.$$

Due to [4, Section 12], the operator $F : Z \to Y$ is of the class C^1 and the operator $u \mapsto u - S'(v_1)u$ is an isomorphism considered both as an operator $X \to X$ and $Z \to Z$. Consequently, v_1 is an isolated stationary solution of (1.1) both in X and in Z.

Similarly, the operator $\tilde{F} : Z \to Y : u \mapsto |u|^p + \mu$ is C^1 and $\tilde{F}'(v_1) = F'(v_1)$, hence the implicit function theorem guarantees the unique solvability of the equation $u = A^{-1}\tilde{F}(u+\varepsilon)$ in the neighbourhood of v_1 if $\varepsilon > 0$ is small enough. Denoting this solution by u^ε, the function $v^\varepsilon := u^\varepsilon + \varepsilon$ satisfies $0 = \Delta v^\varepsilon + |v^\varepsilon|^p + \mu$ in Ω and $v^\varepsilon = \varepsilon$ on $\partial\Omega$. Since $\Delta v^\varepsilon \le 0$, we have $u^\varepsilon = v^\varepsilon - \varepsilon \ge 0$, hence $u^\varepsilon = S(u^\varepsilon + \varepsilon)$ and $v^\varepsilon - S(v^\varepsilon) = \varepsilon$. Consequently, v^ε is a supersolution of problem (4.1).

Next we show that

$$\text{any positive supersolution } v^+ \text{ of (4.1) fulfils } v^+ \ge v_1. \tag{4.2}$$

Indeed, assuming the contrary and denoting $y = v_1 \wedge v^+$ we have $S(y) \le S(v_1) = v_1$ and $S(y) \le S(v^+) \le v^+$, hence $S(y) \le y$. Since 0 is a subsolution of (4.1), and the operator $S : [0, y] \to [0, y]$ is nondecreasing and compact, there exists a solution of the problem $u = S(u)$ in the order interval $[0, y]$ (see [3, Corollary 6.2]). Since $y < v_1$, we obtain a contradiction with the minimality of v_1.

We have $v^\varepsilon \ge v_1$, $v^\varepsilon - v_1 = \varepsilon$ on $\partial\Omega$ and $\Delta(v^\varepsilon - v_1) = v_1^p - (v^\varepsilon)^p \le 0$ in Ω, therefore $v^\varepsilon \ge v_1 + \varepsilon$. Finally, choosing ε small enough we may assume $[v_1, v^\varepsilon] \cap \mathcal{E} = \{v_1\}$.

Similarly as above, the implicit function theorem used for the problem

$$\begin{aligned} 0 = \Delta u + |u|^{p-1}u, &\quad x \in \Omega, \\ u = 0, &\quad x \in \partial\Omega \end{aligned} \tag{4.3}$$

yields the existence of a unique solution v_ε of the problem $0 = \Delta v + |v|^{p-1}v$ in Ω, $v = -\varepsilon$ on $\partial\Omega$, in the neighbourhood of the zero solution of (4.3). Obviously, v_ε is a subsolution of both (4.3) and (4.1). Standard regularity estimates imply that the $C(\bar{\Omega})$-norm of $v_\varepsilon + \varepsilon$ can be bounded by $C\varepsilon^p$ (where C is a given constant), hence $v_\varepsilon < 0$ if $\varepsilon > 0$ is small enough. Since $\Delta v_\varepsilon = -v_\varepsilon |v_\varepsilon|^{p-1} > 0$ in Ω and $v_\varepsilon = -\varepsilon$ on $\partial\Omega$, we have $v_\varepsilon \le -\varepsilon$ in $\bar{\Omega}$. As before, the order interval $[v_\varepsilon, 0]$ does not contain any solution of (4.3) different from 0 if ε is small enough.

Next we show that

$$\text{any supersolution } v^+ \text{ of (4.1) satisfying } v^+ \ge v_\varepsilon \text{ fulfils } v^+ \ge v_1. \tag{4.4}$$

Assume the contrary and let $v^+ \ge v_\varepsilon$ be a supersolution of (4.1), $v^+ \not\ge v_1$. Then v^+ cannot be positive due to (4.2). Since v^+ is also a supersolution of (4.3), the function $y := v^+ \wedge 0 < 0$ is a supersolution of (4.3) as well. Consequently, we can find a solution of (4.3) between the subsolution v_ε and the supersolution y, a contradiction.

Notice that (4.4) and $[v_1, v^\varepsilon] \cap \mathcal{E} = \{v_1\}$ imply $[v_\varepsilon, v^\varepsilon] \cap \mathcal{E} = \{v_1\}$.

Now denote D_A the domain of attraction of the equilibrium v_1 (that is D_A is the set of all initial conditions $u_0 \in X$ for which the solution $u(t)$ of (1.1) exists globally and tends to v_1 in X as $t \to +\infty$). Summarizing the considerations above we obtain the following

Lemma 4.2. *The sets* $\{v \in X : v \geq v_\varepsilon\}$ *and* $\{v \in X : v \leq v^\varepsilon\}$ *are invariant under the semiflow defined by* (1.1)*. The set* $[v_\varepsilon, v^\varepsilon] \cap X$ *is a subset of* D_A*. The set* D_A *is open in* X*.*

Proof. The invariance follows from Proposition 4.1.

If $u_0 \in [v_\varepsilon, v^\varepsilon]$ then the corresponding solution $u(t)$ of (1.1) stays in the same order interval, hence it is global due to Theorem 2.3. The existence of the Lyapunov functional (see (3.10)), the boundedness of $u(t)$ and the compactness of the semiflow imply that the ω-limit set $\omega(u_0)$ of this solution is a nonempty compact set consisting of equilibria. Since $\omega(u_0) \subset [v_\varepsilon, v^\varepsilon]$ and $[v_\varepsilon, v^\varepsilon] \cap \mathcal{E} = \{v_1\}$, we have $\omega(u_0) = \{v_1\}$.

Now let $u_0 \in D_A$, $\delta > 0$, $K := \|v_1\|_X + 1$ and $\eta = \frac{\varepsilon}{2c}$, where $c = c(\delta, \delta, K)$ is the constant from (2.7). Since $u(t) \to v_1$ in X as $t \to \infty$, there exists $t_1 > 0$ such that $\|u(t) - v_1\|_X \leq \eta$ if $t \geq t_1$. Put $T = t_1 + \delta$. Then (2.7) implies

$$\|u(T) - v_1\|_{0,\alpha} \leq c(\delta, \delta, K)\|u(t_1) - v_1\|_X \leq \frac{\varepsilon}{2}.$$

If $y_0 \in X$, $\|y_0 - u_0\|_X \leq \frac{1}{2}$, and y is the solution of (1.1) with the initial condition y_0, then there exists $\beta > 0$ (independent of y_0) such that $y(t)$ exists and $\|y(t) - u(t)\|_X \leq 1$ for $t \leq \beta$. If we require $\|y_0 - u_0\|_X \leq \frac{\varepsilon}{2c}$, where $c = c(\beta, T, K_u + 2)$ is the constant from (2.7), then (2.7) implies existence of $y(t)$ for $t \leq T$ and the estimate

$$\|y(T) - u(T)\|_{0,\alpha} \leq c(\beta, T, K_u + 2)\|u_0 - y_0\|_X \leq \frac{\varepsilon}{2},$$

hence

$$\|y(T) - v_1\|_{0,\alpha} \leq \|y(T) - u(T)\|_{0,\alpha} + \|u(T) - v_1\|_{0,\alpha} \leq \varepsilon,$$

so that $y(T) \in [v_\varepsilon, v^\varepsilon] \cap X \subset D_A$. This implies that the set D_A is open in X. $\square$

Lemma 4.3. *Let* V *be a finite dimensional subspace of* X*. Then the set* $D_A \cap V$ *is bounded.*

Proof. We shall proceed by contradiction. Assume $u_k \in D_A \cap V$, $\|u_k\|_X \to \infty$ as $k \to \infty$. Then $\|u_k - v_1\|_X \to \infty$ as well. Since the solution $U_k(t)$ of (1.1) with the initial condition u_k fulfils $U_k(t) - v_1 \in W^1_{2,\gamma}$ for $t > 0$ and $W^1_{2,\gamma} \hookrightarrow X$, we may assume $u_k - v_1 \in W^1_{2,\gamma}$ and $\alpha_k := |u_k - v_1|_{1,2} \to \infty$. Denote $w_k = (u_k - v_1)/\alpha_k$, $A_k = \int_\Omega |\nabla w_k|^2\,dx$, $B_k = \int_\Omega |w|_k^{p+1}\,dx$. The sequence w_k is bounded in $W^1_{2,\gamma}$ and belongs to a finite dimensional subspace, hence it is relatively compact and we may assume $w_k \to w$ in $W^1_{2,\gamma}$, $|w|_{1,2} = 1$. Moreover, we have $A_k \leq 1$, $B_k \to \int_\Omega |w|^{p+1}\,dx > 0$. Using (3.3),(3.7) we obtain

$$E(u_k - v_1) = E(\alpha_k w_k) \leq \alpha_k^2 A_k - \alpha_k^{p+1} C_4 B_k \leq -\hat{C}_1$$

for k sufficiently large, where $\hat{C}_1$ is the constant from (3.8). Since $t \mapsto E\big((U_k - v_1)(t)\big)$ is nonincreasing, (3.8) implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (U_k - v_1)^2\,dx \geq \hat{C}_0\Big(\int_\Omega (U_k - v_1)^2\,dx\Big)^{(p+1)/2} + \hat{C}_1,$$

so that U_k cannot exist globally, a contradiction. $\square$

In what follows put

$$V = \{v_1 + \alpha_1 y_1 + \alpha_2 y_2 \,:\, \alpha_1, \alpha_2 \in \mathbb{R}\},$$

where $y_1, y_2 \in X$ are continuous functions such that $y_1 > 0$ and y_2 changes sign. Denote $\mathcal{E}^+ = \{v \in \mathcal{E} \,:\, v > v_1\}$ and $\mathcal{E}^- = \{v \in \mathcal{E} \,:\, v < v_1\}$. Let ∂D_A be the boundary of D_A in X and $\partial_V D_A$ be the boundary of $D_A \cap V$ in V. Obviously, $\partial_V D_A \subset \partial D_A$. Our a priori assumptions imply that any solution of (1.1) with the initial condition belonging to ∂D_A exists globally and is bounded in X. Consequently, its ω-limit set consists of equilibria. We put

$$\partial_V D_A^{\pm} = \{u \in \partial_V D_A \,:\, \omega(u) \cap \mathcal{E}^{\pm} \neq \emptyset\}.$$

Lemma 4.4. *If $u_0 \in \partial_V D_A^{\pm}$ then $\omega(u_0) \subset \mathcal{E}^{\pm}$. The sets $\partial_V D_A^{\pm}$ are open in $\partial_V D_A$.*

Proof. Let $u_0 \in \partial_V D_A^+$. Then there exists $v^+ \in \mathcal{E}^+$ and $t_k \to \infty$ such that the solution u of (1.1) fulfils $u(t_k) \to v^+$ in X. Choosing $\delta > 0$, inequality (2.7) implies $\|u(t_k + \delta) - v^+\|_{0,\alpha} \to 0$ as $k \to \infty$, hence

$$u(t_k + \delta) \geq v^+ - \frac{\varepsilon}{2} \geq v_\varepsilon + \frac{\varepsilon}{2} \geq v_\varepsilon \tag{4.5}$$

for k large enough. Fix k and put $T = t_k + \delta$. We have $u(t) \geq v_\varepsilon$ for $t \geq T$, hence $\omega(u_0) \subset \mathcal{E}^+$. Moreover, if y denotes the solution of (1.1) with the initial condition $y_0 \in \partial_V D_A$ then (2.7) implies

$$\|(u - y)(T)\|_{0,\alpha} \leq \frac{\varepsilon}{2} \quad \text{provided} \quad \|u_0 - y_0\|_X < \eta, \tag{4.6}$$

where $\eta > 0$ is small enough (cf. the proof of Lemma 4.2). Estimates (4.5) and (4.6) imply $y \geq v_\varepsilon$, hence $\omega(y_0) \subset \mathcal{E}^+$. Consequently, the set $\partial_V D_A^+$ is open in $\partial_V D_A$.

The proofs in the case $\partial_V D_A^-$ are analogous. $\square$

Theorem 4.5. *Assume* (1.2), (1.3), (1.4) *and let v_1 be the minimal positive solution of* (4.1). *Then there exist solutions v_2, v_3, v_4 of* (4.1) *such that $v_2 > v_1 > v_3$ and the function $v_4 - v_1$ changes sign.*

Proof. The proof is similar to the corresponding proof in [19].

Due to Lemmata 4.2-4.3, there exists $u_0 \in \partial D_A$, $u_0 > v_1$. The a priori estimates from Section 3 guarantee that the ω-limit set $\omega(u_0)$ is a nonempty compact subset consisting of equlibria (cf. the proof of Lemma 4.2). Proposition 4.1 and (4.4) imply $\omega(u_0) \subset \mathcal{E}^+$, hence there exists $v_2 \in \mathcal{E}^+$. Similarly one obtains the existence of $v_3 \in \mathcal{E}^-$.

The existence of v_4 will be shown by contradiction. Hence, assume the contrary; then the compact set $\partial_V D_A$ can be written as a union of two open disjoint subsets $\partial_V D_A^{\pm}$. Consequently, both $\partial_V D_A^+ = \partial_V D_A \setminus \partial_V D_A^-$ and $\partial_V D_A^-$ are compact and their distance is positive. Moreover, $\partial_V D_A^{\pm} \cap \{v_1 \mp \lambda y_1 \,:\, \lambda > 0\} = \emptyset$ so that the following homotopy

$$H(t,u) = \begin{cases} v_1 + (1-2t)(u - v_1) + 2t y_1, & u \in \partial_V D_A^+,\ t \in [0,1/2] \\ v_1 + (1-2t)(u - v_1) - 2t y_1, & u \in \partial_V D_A^-,\ t \in [0,1/2] \\ v_1 + (2-2t) y_1 + (2t-1) y_2, & u \in \partial_V D_A^+,\ t \in [1/2,1] \\ v_1 + (2-2t)(-y_1) + (2t-1) y_2, & u \in \partial_V D_A^-,\ t \in [1/2,1], \end{cases}$$

fulfils $H(t,u) \neq v_1$ for $u \in \partial_V D_A$. The homotopy invariance property of the topological degree in V yields

$$1 = \deg\left(H(0,\cdot), v_1, D_A \cap V\right) = \deg\left(H(1,\cdot), v_1, D_A \cap V\right) = 0,$$

which is a contradiction. $\square$

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