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Minimizing pseudo-harmonic maps in manifolds *

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Abstract

In this work, we show some regularity and uniqueness results for generalized harmonic maps on target manifolds which are graphs of real-valued functions defined on ellipsoids. As an application, we prove a diffeomorphism property for such harmonic maps in two dimensions.

1 Introduction

Harmonic maps with values in a "convex" ball of a Riemannian manifold enjoy nice properties such as the regularity of weak solutions and their uniqueness. Such results were obtained in [7] concerning the regularity of a weakly harmonic map and in [8] for the uniqueness, under the hypothesis that the map takes its values in a geodesically convex ball whose radius is strictly less than $\frac{\pi}{2\sqrt{K}}$, where K is the upper bound for the sectional curvature. It turns out that these "convexity" conditions on the target are not necessary as shown by the result in [5] where similar conditions are obtained for maps into the upper hemisphere of a "flattened" ellipsoid of revolution.

Here we want to show that these regularity and uniqueness results are true also for more general targets than spheres or ellipsoids of revolution, namely for target manifolds which are graphs of real-valued functions defined on ellipsoids satisfying some property.

We also prove a diffeomorphism property for harmonic maps between a two-dimensional ball and such a two-dimensional manifold under suitable boundary conditions.

Our approach relies on a method of E. Sandier and J. Shafrir [13], which allows simplest proofs for the uniqueness and regularity theorems.

The class of target manifolds that we consider is defined as follows. Let $(,)$ be some scalar product in $\mathbb{R}^n$, different from the standard Euclidean scalar product $\langle,\rangle$ in general. For simplicity, we assume that $|y|^2 = (y,y) = \sum_{i=1}^n a_i y_i^2$ with $a_i > 0$. Denote $A = \{y \in \mathbb{R}^n/(y,y) < 1\}$. Clearly, A is a convex set. Let $\mathcal{N}$ be a hypersurface in $\mathbb{R}^{n+1}$, defined by $\mathcal{N} = \{(y, y_{n+1}) \in \mathbb{R}^{n+1}/y \in A, y_{n+1} = f(|y|^2)\}$, where f is a function in $C^\infty([0,1)) \cap C^0([0,1])$ such that $f(1) = 0$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $g : \partial\Omega \to \mathcal{N}$ be a

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prescribed $C^{2,\gamma}$ mapping with $\gamma > 0$. We consider the space $H^1_g(\Omega, \mathcal{N}) = \{v \in H^1(\Omega, \mathbb{R}^{n+1}) / v(x) \in \mathcal{N}$ a.e. $x \in \Omega$ and $v|_{\partial\Omega} = g\}$. Let $\langle , \rangle_{\mathbb{R}^{n+1}}$ be the standard Euclidean inner product on $\mathbb{R}^{n+1}$. We define on $H^1_g(\Omega, \mathcal{N})$ the energy functional

$$E(u) = \frac{1}{2} \int_\Omega e(u) dx = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x) \Big\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \Big\rangle_{\mathbb{R}^{n+1}} dx, \tag{1}$$

where $a_{ij}(x)$ satisfy the following conditions:

$$\exists\, \alpha > 0, \text{ such that } a_{ij}(x)\xi^i\xi^j \geq \alpha |\ \xi\ |^2, \quad \forall\, x \in \Omega, \forall\, \xi \in \mathbb{R}^N; \tag{2}$$

$$a_{ij}(x) \in C^{1,\gamma}(\bar{\Omega}, \mathbb{R}), \quad \text{for some } \gamma > 0 \text{ and } 1 \leq i, j \leq N; \tag{3}$$

$$a_{ij} = a_{ji}, \quad 1 \leq i, j \leq N. \tag{4}$$

The critical points of E satisfy in the sense of distributions the Euler equation

$$Lu + \sum_{i,j=1}^{N} a_{ij}(x) C\left(\frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x)\right) = 0, \quad \text{in } \Omega \tag{5}$$
$$u = g, \quad \text{on } \partial\Omega,$$

where $L = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ and $C(,)$ is the second fundamental form of $\mathcal{N}$. We remark that if $N \neq 2$, this problem is equivalent to studying harmonic maps from (Ω, g_{ij}) on $\mathcal{N}$, where $(g_{ij}) = (\det(a_{ij}))^{\frac{1}{N-2}} (a_{ij})^{-1}$. However, if $N = 2$, this problem concerns a more general problem which agrees to harmonic maps if and only if $\det(a_{ij}) = 1$.

Our first result is a uniqueness principle. Let $\mathcal{M}$ be a hypersurface such that $\overline{\mathcal{N}} \subset \mathcal{M}$. We will show the following result.

Theorem 1 *Assume that $g(\partial\Omega) \subset \overline{\mathcal{N}}$ and the following conditions are verified,*

$$f^{(1)} < 0 \ \textit{on}\ [0, 1); \tag{6}$$

$$f^{(2)} \leq 2(f^{(1)})^3 \ \textit{and}\ 2(f^{(1)}f^{(3)} - 3(f^{(2)})^2) + f^{(2)}f^{(1)} - 2(f^{(1)})^4 \geq 0 \ \textit{on}\ [0, 1); \tag{7}$$

$$a_i < 1 \quad \forall i = 1, \dots, n. \tag{8}$$

Then, the minimizer of E in $H^1_g(\Omega, \overline{\mathcal{N}})$ is unique. Moreover, if u is a critical point of E, whose image lies in a compact subset of $\mathcal{N}$, then u is a minimizer.

This result is a generalized variant of [5]. In fact, if we set $f = \sqrt{1-t}$, we find that $\mathcal{N}$ is the upper hemisphere of the ellipsoid. In this paper, we will adopt the same strategy as in [13]. Our approach relies on a convex inequality for the energy functional E.

The second result is a regularity theorem. We will show the following.

Theorem 2 *Under the same hypotheses as in Theorem 1, assume that $u \in H^1_g(\Omega, \mathcal{N})$, whose image belongs to a compact subset of $\mathcal{N}$, and that u is a critical point of E, i. e., u is a weak solution of (5). Then u is C^2 on Ω.*

This theorem is a variant of a result of S. Hildebrandt, H. Kaul and K-O. Widman, who have proved in [7] the same result for a harmonic map in a geodesic ball of radius $r < \frac{\pi}{2\sqrt{K}}$ where K is an upper bound of the sectional curvature of the manifold. By the uniqueness principle, we need only prove it for the minimizing maps. So it is relatively easy to obtain the regularity property with the help of ϵ-regularity due to R. Schoen and Uhlenbeck [15]. Note that if $\mathcal{N}$ is a hemisphere of a flattened ellipsoid, we find again the result of [5].

In the third part, we will use these two first results to study the problem of diffeomorphism. Let $n = N = 2$ and $\Omega = B$ be the unit disc in $\mathbb{R}^2$. Assume that $g : \partial\Omega \to \mathcal{N}$ is a convex Jordan curve $\mathcal{N} \cap \{x_3 = \alpha > 0\}$. We show the following theorem.

Theorem 3 *Under the above assumptions, assume that $u \in H^1_g(\Omega, \mathcal{N})$ is a critical point of E. Then, u is a diffeomorphism.*

Here, we argue by a continuity method, due to J. Jost [9]. We connect the critical point with a harmonic map. Using a result of Hartman and Wintner [3], we conclude our claim. In the last section, we will treat the limit case. We set $f(t) = \sqrt{1-t}$ and $a_1 < \ldots < a_r < a_{r+1} = \ldots = a_n = 1$. In general, the uniqueness principle fails, as E. Sandier and I. Shafrir have proved for the case where $a_1 = \ldots = a_n = 1$ and $a_{ij} = \delta_{ij}$. With the same procedure as in [13], we will establish a criterion on the boundary condition for which the uniqueness principle holds.

2 Proof of Theorem 1

First, we give a basic inequality, which is a variant of the inequality in [13].

Lemma 1 *Assume that (6) and (7) in the theorem 1 hold and that*

$$a_i \leq 1. \tag{8'}$$

Then function

$$G(v, w) = (w, w) + (2f'(|v|^2)(v, w))^2$$

is convex over the set $A = \{(v, w) \in \mathbb{R}^n \times \mathbb{R}^n / (v, v) < 1\}$. Moreover, we have that for any (v^0, w^0), $(v^1, w^1) \in A$, with $(v^0, w^0) \neq (v^1, w^1)$,

$$G\left(\frac{v^0 + v^1}{2}, \frac{w^0 + w^1}{2}\right) = \frac{1}{2}(G(v^0, w^0) + G(v^1, w^1))$$

holds only if for any $0 \leq t \leq 1$ we have

i) $$\delta w + 4(f'(|v^t|^2))^2(v^t, w^t)\delta v = 0, \tag{9}$$

ii) $$(\delta v, w^t)f'(|v^t|^2) + 4[f^{(2)}(|v^t|^2) - f'(|v^t|^2)^3](v^t, w^t)(\delta v, v^t) = 0$$

and then $G(v^t, w^t) = G(v^0, w^0)$ for $0 \leq t \leq 1$. Here we wrote $(v^t, w^t) = (1-t)(v^0, w^0) + t(v^1, w^1)$ and $(\delta v, \delta w) = (v^0 - v^1, w^0 - w^1)$. Furthermore, if $f^{(2)} = 2(f^{(1)})^3$ and $f^{(1)}f^{(3)} = 3(f^{(2)})^2$, i) and ii) are also the sufficient conditions.

Proof. Define $F(s) = G(v^0 + s\delta v, w^0 + s\delta w)$. A calculation leads to

$$\begin{aligned} F'(t) &= 2(w^t, \delta w) + 8[((\delta v, w^t) + (v^t, \delta w))f'(|v^t|^2) \\ &\quad +2f^{(2)}(|v^t|^2)(v^t, \delta v)(v^t, w^t)]f'(|v^t|^2)(v^t, w^t) \end{aligned}$$

and

$$\begin{aligned} 2F^{(2)}(t) &= |\delta w + 4f'(|v^t|^2)(v^t, w^t)\delta v|^2 \\ &\quad +4[4f^{(2)}(|v^t|^2)(v^t, \delta v)(v^t, w^t) + ((\delta v, w^t) + (v^t, \delta w))f'(|v^t|^2)]^2 \\ &\quad +8[f'(|v^t|^2)f^{(2)}(|v^t|^2) - 2(f'(|v^t|^2))^4](v^t, w^t)^2(\delta v, \delta v) \\ &\quad +16[f'(|v^t|^2)f^{(3)}(|v^t|^2) - 3(f^{(2)}(|v^t|^2))^2](v^t, w^t)^2(v^t, \delta v)^2. \end{aligned}$$

Hence, $F^{(2)}(t) \geq 0$ on $[0,1]$ since $(v^t, \delta v)^2 \leq (\delta v, \delta v)(v^t, v^t) \leq (\delta v, \delta v)$. The convexity of G follows. Finally, note that if $F(s) = (1-s)F(0) + sF(1)$ for some $s \in (0,1)$, then $F^{(2)}(t) \equiv 0$ for all $t \in [0,1]$. Hence, we achieve the proof. ♢

We will adapt the notation in [13]. So that every $u \in \overline{\mathcal{N}}$ can be written in the form $u = u^h + u^v e_{n+1}$, where $u^h \in A$ and $u^v = f(|u^h|^2)$. Then for u_0, $u_1 \in H^1(\Omega, \overline{\mathcal{N}})$ and $0 \leq t \leq 1$, we define the map $u^t(x) = (1-t)u^0 \oplus tu^1$ by

$$[(1-t)u^0 \oplus tu^1]^h(x) = (1-t)(u^0)^h(x) + t(u^1)^h(x),$$

and $[(1-t)u^0 \oplus tu^1]^v(x) = f(|(u^t)^h|^2)$.

Lemma 2 *For any $u^0, u^1 \in H^1(B, \overline{\mathcal{N}})$, let $u^t = (1-t)u^0 \oplus tu^1$ for $0 \leq t \leq 1$. Then $u_t \in H^1(B, \overline{\mathcal{N}})$ and*

$$e(u^t) \leq (1-t)e(u^0) + te(u^1) \quad \text{a. e. on } \Omega. \tag{10}$$

Remark. Here, $H^1(\Omega, \overline{\mathcal{N}}) = \{u \in H^1(\Omega, \mathcal{M})$ and $u(x) \in \overline{\mathcal{N}}$ a. e. on $\Omega\}$, where $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{n+1}$ such that $\overline{\mathcal{N}} \subset \mathcal{M}$.

Proof. We will decompose e into two parts: $e(v) = F_1(v) + F_2(v)$, where

$$F_1(v)(x) = \sum_{i,j=1}^{N} a_{ij}(x) \left[\left\langle \frac{\partial v^h}{\partial x_i}, \frac{\partial v^h}{\partial x_j} \right\rangle_{\mathbb{R}^n} - \left(\frac{\partial v^h}{\partial x_i}, \frac{\partial v^h}{\partial x_j} \right) \right]$$

and

$$F_2(v)(x) = \sum_{i,j=1}^{N} a_{ij}(x) \left[\left(\frac{\partial v^h}{\partial x_i}, \frac{\partial v^h}{\partial x_j} \right) + \frac{\partial v^v}{\partial x_i} \frac{\partial v^v}{\partial x_j} \right].$$

Clearly, $F_1(u^t)(x) \leq (1-t)F_1(u^0)(x) + tF_1(u^1)(x)$, for all $x \in \Omega$, since $\langle , \rangle_{\mathbb{R}^n} - (,)$ is a positive bilinear form on $\mathbb{R}^n$. Now we fix $x \in \Omega$, then there exists $C(x) = (c_{ij}(x)) \in GL(N, \mathbb{R})$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi^i\xi^j = \sum_{i=1}^{N} \Big(\sum_{j=1}^{N} c_{ij}(x)\xi^j \Big)^2 \text{ for all } \xi \in \mathbb{R}^N.$$

Let us first suppose that for some $\epsilon > 0$,

$$|(u^0)^h|, |(u^1)^h| \leq 1 - \epsilon.$$

It is clear that $|(u^t)^h| \leq 1 - \epsilon$ for $0 \leq t \leq 1$, or that

$$\frac{\partial (u^t)^v}{\partial x_j} = 2f'(|(u^t)^h|^2)((u^t)^h, \partial (u^t)^h/\partial x_j), \text{ for } j = 1, \dots, N.$$

Hence,

$$F_2(u^t)(x) = \sum_{i=1}^{N} G\Big((u^t)^h, \sum_{j=1}^{N} c_{ij}(x)\partial (u^t)^h/\partial x_j\Big).$$

Therefore, (10) follows by lemma 1. In the general case, for getting the result, we use an approximation argument. Define a map $P_\lambda : \overline{\mathcal{N}} \to \overline{\mathcal{N}}$ depending on $\lambda > 0$ by

$$P_\lambda(u^h, u^v) = ((1-\lambda)u^h, f((1-\lambda)^2|u^h|^2)).$$

Obviously, $P_\lambda(\overline{\mathcal{N}}) \subset \overline{\mathcal{N}} \cap \{u_{n+1} \geq \epsilon(\lambda)\}$, where $\epsilon(\lambda)$ is strictly positive and P_λ converges in $C^1(\overline{\mathcal{N}}, \overline{\mathcal{N}})$ norm to the identity mapping as λ goes to 0. For any $\lambda > 0$, let $u^0_\lambda = P_\lambda \circ u^0$ and $u^1_\lambda = P_\lambda \circ u^1$. Setting $u^t_\lambda = (1-t)u^0_\lambda \oplus t u^1_\lambda$, we have that for any $0 \leq t \leq 1$ and a.e. x in Ω,

$$F_2(u^t_\lambda)(x) \leq (1-t)F_2(u^0_\lambda)(x) + tF_2(u^1_\lambda)(x). \tag{11}$$

Now we pass to the limit as λ goes to 0. The right hand side converges in L^1 to $(1-t)F_2(u^0) + tF_2(u^1)$. By coerciveness and (11), u^t_λ remains bounded in H^1. After choosing a subsequence, we may assume that $u^t_\lambda \to v^t$ weakly in H^1 for some $v^t \in H^1$. But $u^t_\lambda \to u^t$ a.e. in B and hence $u^t = v^t$. Let $K \subset \Omega$ a subset of Ω. By weak lower semi-continuity, we deduce from (11) that

$$\begin{aligned}
\int_K F_2(u^t) &\leq \liminf_{\lambda \to 0} \int_K F_2(u^t_\lambda) \\
&\leq \lim_{\lambda \to 0} \int_K (1-t)F_2(u^0_\lambda) + tF_2(u^1_\lambda) \\
&= \int_K (1-t)F_2(u^0) + tF_2(u^1).
\end{aligned}$$

But it is well known that

$$⨍_{B(x,r)} v \to v(x) \text{ a.e. in } B \text{ as } r \to 0 \text{ for any } v \in L^1,$$

where $B(x,r) = \{y \in \mathbb{R}^N \,|\, |y - x| < r\}$. Consequently, (10) holds. $\diamondsuit$

Proof of Theorem 1. We assume that u^0 and u^1 are two distinct minimizers with the same boundary data g. Thanks to lemma 2, we obtain

$$E(u^t) \leq (1-t)E(u^0) + tE(u^1) = E(u^0),$$

which implies for $0 \leq t \leq 1$,

$$F_1(u^t)(x) = (1-t)F_1(u^0)(x) + tF_1(u^1)(x) \quad \text{a. e. } x \in \Omega,$$

that is,

$$d(u^0)^h(x) = d(u^1)^h(x) \quad \text{a. e. } x \in \Omega,$$

since $a_i < 1$ for $1 \leq i \leq n$. Hence we conclude $u^0 = u^1$. This contradiction terminates the first part of our claim. Now let u^0 be the minimizer and u^1 be a critical point of E which agrees with u^0 on $\partial\Omega$. Assume that u^0 and u^1 lie in a compact subset of $\mathcal{N}$. Denote $I(t) = E(u^t)$ for any $0 \leq t \leq 1$. Obviously, $I(t) \in C^1([0,1])$ and is convex by lemma 2. But $I'(0) \geq 0 = I'(1)$. Thus, it follows that $I(t) \equiv I(0)$, that is, u^1 is the minimizer. ♢

Remarks 1.) $u \in H^1_g(\Omega, \overline{\mathcal{N}})$ is a minimizer in $H^1_g(\Omega, \overline{\mathcal{N}})$. Thus u verifies the Euler-Lagrange equation (5). Indeed, let C be a contraction from $\mathcal{M}$ onto $\overline{\mathcal{N}}$ such that $C|_{\overline{\mathcal{N}}} = Id$. So we have $E(C(v)) \leq E(v)$, that is,

$$\inf_{v \in H^1_g(\Omega, \overline{\mathcal{N}})} E(v) = \inf_{v \in H^1_g(\Omega, \mathcal{M})} E(v).$$

2.) The existence of a minimizer is obtained by the minimizing method.

3 Proof of Theorem 2

Now we consider a weak pseudo-harmonic map u, that is a solution of (5), whose image lies into a compact subset of $\mathcal{N}$. So there exists $\alpha_0 > 0$ with

$$u_{n+1} \geq \alpha_0 \quad \text{a.e.} \quad \text{on } \Omega.$$

In view of theorem 1, it is sufficient to prove it for the minimizing maps. Thanks to a result due to R. Schoen and K. Ulenbeck [15], there exists $\epsilon_0 > 0$ such that $\int_{B_r(x)} e(u) \leq \epsilon_0$ for any $x \in \Omega$ and for all $r > 0$ such that $B_r(x) \subset \Omega$, then u is regular on $B_{r/2}(x)$. Here, we will use the arguments in [7] (see also in [6]). For any bounded domain Ω in which the relations of coefficients (a_{ij}) are defined as above, then we have an associate Green function G, which satisfies

$$\forall \xi \in C_c^\infty(\Omega, \mathbb{R}), \forall y \in \Omega, \quad \xi(y) = -\int_\Omega L\xi(x) G(x,y) dx. \tag{12}$$

Fixed $y \in \Omega$, there exists $\tau_0 > 0$ such that $B_{\tau_0}(y) \subset \Omega$. Then we can consider

$$G^\tau(x,y) = \fint_{B_\tau(y)} G(x,z) dz, \quad \text{for all } \tau \in (0, \tau_0]. \tag{13}$$

It follows from (12) that

$$\fint_{B_\tau(y)} \xi(z)dz = -\int_\Omega \sum_{ij=1}^N \partial_i(a_{ij}(x)\partial_j\xi(x))G^\tau(x,y)dx,$$

where G^τ is an approximation to the Green function in the sense that

$$\lim_{\tau\to 0} G^\tau = G(x,y) \quad \text{for } x \neq y, x, y \in \Omega. \tag{14}$$

(see [11], [7] and [6]). Moreover, there exists strictly positive constants K_1, K_2 and K_3 such that

$$0 \le G(x,y) \le K_1|x-y|^{2-N}; \tag{15}$$

$$G(x,y) \ge K_2|x-y|^{2-N} \quad \text{if } |x-y| \le \frac{3}{4}d(y,\partial\Omega); \tag{16}$$

$$|\nabla_x G(x,y)| \le K_3|x-y|^{1-N}; \tag{17}$$

$$G^\tau(x,y) \le 2^{N-2}K_1|x-y|^{2-N} \quad \text{if } \tau < \frac{1}{2}|x-y|; \tag{18}$$

$$\text{if } d(y,\partial\Omega) > \tau, x \to G^\tau(x,y) \in H_0^1(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R}), \tag{19}$$

where x and y are in Ω. Without loss of generality, assume that $B(0,r) \subset \Omega$ for some $r > 0$. Taking the map $G^\tau(x,0)$ as a test function in (5), we obtain

$$\int_\Omega G^\tau(x,0)Lu^v(x) + \lambda(x)G^\tau(x,0)dx = 0, \tag{20}$$

where

$$\begin{aligned} \lambda \;=\; & \Big(-2f'(|u^h|^2)\sum_{ij=1}^N a_{ij}\langle\partial_i u^h, \partial_j u^h\rangle_{\mathbb{R}^n} \\ & -4f^{(2)}(|u^h|^2)\sum_{ij=1}^N a_{ij}\langle u^h, \partial_i u^h\rangle_{\mathbb{R}^n}\langle u^h, \partial_j u^h\rangle_{\mathbb{R}^n}\Big) \\ & \div\Big(1 + 4(f')^2(|u^h|^2)\sum_{i=1}^n a_i^2 u_i^2\Big). \end{aligned}$$

Let $\omega \in H^1(\Omega,\mathbb{R})$ be the solution of the equation

$$\begin{gathered} L\omega = 0, \quad \text{in } \Omega \\ \omega = u^v, \quad \text{on } \partial\Omega. \end{gathered}$$

Clearly, $u^v - \omega \in H_0^1(\Omega)$ and from (12) we deduce that

$$-\int_\Omega G^\tau(x,0)Lu^v(x)dx = -\int_\Omega G^\tau(x,0)L(u^v-\omega)dx = \fint_{B_\tau(0)} (u^v-\omega)dx.$$

Consequently,

$$\int_\Omega G^\tau(x,0)\lambda(x) = \fint_{B_\tau(0)} (u^v - \omega)dx \le f(0).$$

Hence, we obtain

$$\int_\Omega G^\tau(x,0)|\nabla u^h|^2 \le Cf(0), \tag{21}$$

since $\lambda(x) \ge \alpha|\nabla u^h|^2(x)$. Using Fatou's lemma and passing to the limit in (21) as $\tau \to 0$, we deduce

$$\int_\Omega G(x,0)|\nabla u^h|^2 \le Cf(0).$$

It follows from (16) that

$$\lim_{\epsilon\to 0}\int_{B_\epsilon(0)} |\nabla u^h(x)|^2|x|^{2-N}dx = 0.$$

On the other hand, remark that u belongs to a compact subset of $\mathcal{N}$, which implies

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{N-2}}\int_{B_\epsilon(0)} |\nabla u(x)|^2 dx = 0.$$

This completes the present proof.

Remarks 1.) If $f \in C^2([0,1])$ and $f'(1) < 0$, we have the same conclusion in $H^1_g(\Omega,\overline{\mathcal{N}})$. Moreover, replacing (8) by (8') and (7) by $f^{(2)} \le 0$, our result is also right for minimizing maps.
2.) For $N = 2$, we will give the proof in the following section (see also [4]).

4 Proof of Theorem 3

In this part, we will use a similar strategy as in [9] (see also [4]). In order to prove our result, we will consider the following energy functional

$$\begin{aligned} E_t(v) &= \frac{1}{2}\int_B \sum_{i,j=1}^2 \Big[(1-t)\delta_{ij} + ta_{ij}(x)\Big]\Big\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j}\Big\rangle dx \\ &= \frac{1}{2}\int_B \sum_{i,j=1}^2 a_{ij}(t,x)\Big\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_j}\Big\rangle dx. \end{aligned} \tag{22}$$

We consider $G : [0,1] \times \partial B \to \mathcal{N}$ to be a $C^{2,\gamma}$ map such that

$$G(t,.) \text{ is a diffeomorphism ;} \tag{23}$$

$$G(1,.) = g; \tag{24}$$

$$G(t,.) = \{u_3 = \alpha(t)\} \cap \mathcal{N}, \text{ where } \alpha(0) \text{ is a number near to } f(0). \tag{25}$$

Let $I_t = \inf_{v \in H^1(B,\mathcal{N})} E_t(v)$. Denote $u^t \in H^1(B,\mathcal{N})$ the unique minimum of E_t in $H^1_{G(t,.)}(B,\mathcal{N})$ given by Theorem 1, then u^t satisfies:

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u^t}{\partial x_j}\right) + \lambda_t u^t = 0, \quad \text{in } B \tag{26}$$
$$u^t = G_t, \quad \text{on } \partial B,$$

where $\lambda_t = \sum_{i,j=1}^{2} a_{ij}(t,x) C \left\langle \frac{\partial u^t}{\partial x_i}(x), \frac{\partial u^t}{\partial x_j}(x)\right\rangle$. First we show the following lemma.

Lemma 3 *We have* $u^t \in C^{2,\gamma}(\bar{B},\mathbb{R}^3)$ *and*

$$\|u^t\|_{C^{2,\gamma}} \leq C_1.$$

Proof. We write E_t as follows

$$E_t(v) = \frac{1}{2}\int_B \sum_{i,j=1}^{2} a_{ij}(t,x)\Big[\left\langle \frac{\partial v^h}{\partial x_i}, \frac{\partial v^h}{\partial x_j}\right\rangle + (f'(|v^h|^2))^2\big(v^h, \frac{\partial v^h}{\partial x_i}\big)\big(v^h, \frac{\partial v^h}{\partial x_j}\big)\Big]dx.$$

Obviously, there exists some $\beta > 0$ such that $\big((u^t)^h, (u^t)^h\big) \leq 1 - \beta$ for any $t \in [0,1]$ and $x \in \bar{B}$. Thanks to a result of Jost and Meier (see [10]), we deduce that

$$\|u^t\|_{W^{1,q}} \leq C, \text{ for some } q > 2.$$

Recalling that u^t satisfies the equations (26), it follows that

$$\|u^t\|_{W^{2,q}} \leq C.$$

Consequently,

$$\|u^t\|_{W^{1,\frac{2q}{4-q}}} \leq C\|u^t\|_{W^{2,q}} \leq C, \quad \text{if } q < 4.$$

Now iterating the above procedure and using Sobolev's embedding theorem, we obtain

$$\|u^t\|_{C^{1,\gamma}} \leq C.$$

Hence, using Schauder's estimates, we complete the proof. $\diamondsuit$

Lemma 4 *With the above notation, we have* $\operatorname{rank}(\nabla u^t(x)) = 2$, *for any* $t \in [0,1]$ *and* $x \in \partial B$.

Proof. Denote $L_t = \sum_{i,j=1}^{2} \partial_i(a_{ij}(t,x)\partial_j)$. Using (26), we state $L_t((u^t)_3) \leq 0$, and the strong maximum principle implies

$$(u^t)_3(x) > \alpha(t) \quad \forall x \in B, \text{ or } (u^t)_3 \equiv \alpha(t).$$

The latter is incompatible with $L_t((u^t)_3) < 0$ on ∂B. Hence, the claim follows from Hopf's maximum principle. $\diamondsuit$

Lemma 5 ([17]) *Assume that U and V are smooth, bounded domains in $\mathbb{R}^2$, diffeomorphic to B. Let $\varphi : \bar{U} \to \bar{V}$ be a C^1 map such that*

$$\det(\nabla\varphi(x)) > 0, \quad \forall x \in U.$$

Moreover, suppose that $\varphi|_{\partial U}$ is a diffeomorphism from ∂U to ∂V. Then φ is a diffeomorphism.

Lemma 6 ([9, theorem 5.1.1]) *We have* $\operatorname{rank}(\nabla u^0(x)) = 2$*, for all $x \in \bar{B}$, and u_0 is a diffeomorphism.*

Lemma 7 ([4]) *the mapping $F_* : [0,1] \to C^{1,\beta}(\bar{B},\mathcal{N}) \cap H^1(B,\mathcal{N})$ such that $t \mapsto u^t$ is continuous.*

The proof of this lemma is a consequence of Theorem 1 and lemma 3.

Now, we define the set

$$T_1 = \{t \in [0,1],\ u^t \text{ is a diffeomorphism }\}. \tag{27}$$

It suffices to prove that T_1 is open, closed and not empty.
Step 0. $0 \in T_1$. This is just the statement of lemma 6.
Step 1. T_1 is open. It follows from lemma 5 and 7.
Step 2. T_1 is also closed. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence converging to t. Assume that u^{t_n} are diffeomorphic, for all $n \in \mathbb{N}$. We assume that

$$\exists\, x_0 \in \bar{B}, \quad \text{such that} \quad \operatorname{Rank}(\nabla(u^t)(x_0)) \le 1.$$

We shall use geodesic parallel coordinates based on a geodesic arc c through $q = u^t(x_0)$ as in [9]. In these coordinates (v_1, v_2), for $v^2 \equiv 0$, v_1 is the arclength parameter of c, where as the curves $v_1 \equiv \mathit{const}$ are geodesics normal to c parametrized by arclength v_2, consequently the curves $v_2 \equiv \mathit{const}$ are parallel curves of c. Moreover, we can choose the coordinates such that $\partial_z(v_2 \circ u^t)(q) = 0$. In these coordinates, we have for the metric tensor

$$g_{11}(v_1, 0) = 1, \quad g_{12}(v_1, v_2) = 0, \quad g_{22}(v_1, v_2) = 1,$$

therefore, the only non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11}\partial_{x_1} g_{11},\\
\Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2} g^{11}\partial_{x_2} g_{11},\\
\Gamma^2_{11} &= -\frac{1}{2} g^{22}\partial_{x_2} g_{11},
\end{aligned}$$

Hence, equations (26) for u^t take the form

$$\begin{gathered}
L_t v_1 = -\Gamma^1_{11}\left(\sum_{ij=1}^{2} a_{ij}\partial_{x_i} v_1 \partial_{x_j} v_1\right) - \Gamma^1_{12}\left(\sum_{ij=1}^{2} a_{ij}\partial_{x_i} v_1 \partial_{x_j} v_2\right),\\
L_t v_2 = -\Gamma^2_{11}\sum_{ij=1}^{2} a_{ij}\partial_{x_i} v_1 \partial_{x_j} v_1.
\end{gathered}$$

Applying a result due to Hartman and Wintner [3], we obtain that

$$\partial_z(v_1 \circ u^t)(z) = a_1(z - z_1)^m + o(|z - z_1|^m), \text{ for some } m \geq 1 \text{ and } a \in \mathbb{C}^*,$$

where z_1 are coordinates of q. Consequently,

$$\deg(\partial_z(v_1 \circ u^t), B(z_1, r_1), 0) = m \geq 1, \text{ for some } r_1 > 0,$$

which implies $\deg(\partial_z(v_1 \circ u^{t_n}), B(z_1, r_1), 0) = m \geq 1$, for some sufficiently large n. Hence, there exists $z_2 \in B(z_1, r_1)$ such that $\partial_z(v_1 \circ u^{t_n})(z_2) = 0$ by the property of degree. This contradicts that u^{t_n} is diffeomorphic. Therefore, the assertion follows from Lemma 5.

Remark. We have a more general result, that is, our result also holds for a convex curve in $\overline{\mathcal{N}}$.

5 The limit case

In this section, we consider the limit case; that is, $f(t) = \sqrt{1-t}$. Hence,

$$\overline{\mathcal{N}} = \Big\{ x \in \mathbb{R}^{n+1}; x_{n+1} \geq 0 \text{ and } x_{n+1}^2 + \sum_{i=1}^n a_i^2 x_i^2 = 1 \Big\}$$

is an upper hemisphere of a n-dimensional ellipsoid. Assume that for some r with $1 \leq r \leq n$,

$$a_1 \leq a_2 \leq \ldots \leq a_r < 1 \quad a_{r+1} = \ldots = a_n = 1\,. \tag{28}$$

In general, Theorem 1 fails. We will show that uniqueness depends on the boundary data as in [13]. Let $\{e_1, \ldots, e_{n+1}\}$ denote a basis of $\mathbb{R}^{n+1}$ and P (resp. P_1) the projection from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^{n-r+1}$ (resp. $\mathbb{R}^r$) defined as follows

$$P(x_1, \ldots, x_{n+1}) = (x_{r+1}, \ldots, x_n, x_{n+1}),$$
$$P_1(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_r).$$

For any map g with values in $\overline{\mathcal{N}}$, we define $Rank(P \circ g)$ to be the smallest integer k with $0 \leq k \leq n - r + 1$ such that the image of g lies in a k-dimensional vector subspace of $\mathbb{R}^{n-r+1}$. With the same procedure as in [13], we will prove the following result.

Theorem 4 *Let $g : \partial\Omega \to \overline{\mathcal{N}}$ be a $C^{2,\gamma}$ map for some $\gamma > 0$. Then uniqueness of the minimizer for the boundary data g fails if and only if*
(I) $k = Rank(P \circ g) \leq n - r - 1$ and $Image(g) \subset \overline{\mathcal{N}} \cap \{x_{n+1} = 0\}$,
(II) the $\overline{\mathcal{N}}^k$ valued minimization problem

$$(P_g^k) \quad \inf\{E(v), v \in H_g^1(\Omega, \overline{\mathcal{N}}^k)\}$$

has a solution u such that $\operatorname{Rank}(P \circ u) = k+1$*, where*

$$\overline{\mathcal{N}}^k = \left\{ x \in \mathbb{R}^{r+k+1}; x_{r+k+1} \geq 0 \text{ and } \sum_{i=1}^{r} a_i^2 x_i^2 + \sum_{i=r+1}^{r+k+1} x_i^2 = 1 \right\}.$$

Moreover, when uniqueness fails, let u^0 be any one of the minimizers, the set of the minimizers is obtained by composition of u^0 with any rotation of $\mathbb{R}^{n+1}$ that leaves the k-dimensional vector subspace containing $P \circ g(\partial\Omega)$ and $P_1(\mathbb{R}^{n+1})$ invariant.

We divide our proof in several steps.
Step 1 consists of the following lemma.

Lemma 8 *We assume that u^0 and u^1 are two distinct $\overline{\mathcal{N}}$-valued minimizers with same boundary data $g \in C^{2,\gamma}$. Then for every $0 < t < 1$, $u^t(x) = [(1-t)u^0 \oplus tu^1](x)$ is also a minimizer which is $C^{2,\gamma}$ in Ω and $(u^t)^v > 0$ in Ω.*

Proof. Clearly, it follows from lemma 2 that u^t is a minimizer for any $0 < t < 1$. Fix some $0 < t < 1$ and so by results of R. Schoen and K. Uhlenbeck [15] and [16], it is $C^{2,\gamma}$ near the boundary and in Ω outside of a closed set M of Hausdorff dimension at most $N-3$. We know that $-\triangle(u^t)_{n+1} \geq 0$ and $(u^t)_{n+1} \geq 0$ in $\Omega \backslash M$. Applying the strong maximum principle in $\Omega \backslash M$, we have either $(u^t)_{n+1} > 0$ in $\Omega \setminus M$ or $(u^t)_{n+1} \equiv 0$ in $\Omega \setminus M$. The latter would imply $u^0 \equiv u^1$ in $\Omega \setminus M$ since $\Omega \setminus M$ is connect, and then in Ω, which contradicts our assumptions. Thus, $(u^t)^v > 0$ in $\Omega \backslash M$. However, $\Omega \backslash M$ contains a neighborhood of $\partial\Omega$. Therefore, using theorem 1, we conclude the claim. $\diamondsuit$

Step 2. consists of the following lemma.

Lemma 9 *Under the above assumptions,*

$$P_1 \circ u^0 = P_1 \circ u^1, \tag{29}$$

and for $0 < t < 1$ and $i = 1, \dots, N$,

$$\frac{\partial^2 u}{\partial t \partial x_i} = -\frac{(u^h, \partial u^h / \partial x_i)}{1 - |u^h|^2} \frac{\partial u}{\partial t}. \tag{30}$$

Proof. We fix $0 < t_0 < t < t_1 < 1$ and denote $\tilde{u}^0 = (1-t_0)u^0 \oplus t_0 u^1$, $\tilde{u}^1 = (1-t_1)u^0 \oplus t_1 u^1$. Then we can write $u^t = \frac{t_1 - t}{t_1 - t_0} \tilde{u}^0 \oplus \frac{t - t_0}{t_1 - t_0} \tilde{u}^1$ and $(\tilde{u}^0)^v, (\tilde{u}^1)^v > 0$ in Ω. Obviously,

$$F_1(\tilde{u}^t) = (1-t)F_1(\tilde{u}^0) + tF_1(\tilde{u}^1), \tag{31}$$

and

$$F_2(\tilde{u}^t) = (1-t)F_2(\tilde{u}^0) + tF_2(\tilde{u}^1), \tag{32}$$

Thus, from (31), we obtain (29). Using lemma 1, we deduce that

$$\sum_{j=1}^{N} c_{ij}(x) \frac{\partial^2 u^h}{\partial t \partial x_j} = -\frac{\left(u^h, \sum_{j=1}^{N} c_{ij}(x) \frac{\partial u^h}{\partial x_j}\right)}{1 - |u^h|^2} \frac{\partial u^h}{\partial t}, \text{ for } 1 \leq i \leq N,$$

that is,

$$\frac{\partial^2 u^h}{\partial t \partial x_j} = -\frac{\big(u^h, \partial u^h/\partial x_j\big)}{1-|u^h|^2}\frac{\partial u^h}{\partial t}.$$

Therefore, as the same arguments as in [13] (see also [4]), we get

$$\frac{\partial^2 u^v}{\partial t \partial x_j} = -\frac{(u^h, \partial u^h/\partial x_j)}{1-|u^h|^2}\frac{\partial u^v}{\partial t}$$

Step 3. In fact, (30) can be written as

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u/\partial t}{u^v}\right) = 0, \text{ for } j = 1, .., N,$$

which implies

$$\frac{\partial u}{\partial t}(t,x) = u^v(t,x)\alpha(t), \tag{33}$$

for some map $\alpha : (0,1) \to \mathbb{R}^{n+1}$. On the other hand,

$$\left(\frac{\partial u^h}{\partial t}, u^h\right) + u^v\frac{\partial u^v}{\partial t} = 0,$$

and so we get

$$\big(\alpha^h, u^h(t,x)\big) + \alpha^v u^v(t,x) = 0. \tag{34}$$

Thus,

$$\langle P\circ u(t,x), P\circ\alpha(t)\rangle = 0 \text{ and } P_1\circ\alpha(t) = 0, \tag{35}$$

since $\frac{\partial P_1\circ u}{\partial t} = 0$. Or, from (34), we claim that e_{n+1} and $\alpha(t)$ are not proportional. Then, by considering $x \in \partial\Omega$, it follows from (33) and (35) that

$$u^v(x,t) = 0 \text{ and } \langle P\circ u(t,x), P\circ\alpha(\tfrac{1}{2})\rangle = 0 \quad \text{on } \partial\Omega.$$

Lemma 10 *Suppose that u^0 and u^1 are two distinct minimizers for the same boundary data g. Then*

$$Rank(P\circ u^0) = Rank(P\circ u^1) \le n - r.$$

Proof. Suppose that $\mathrm{Rank}(P\circ u^0) = m_0+1$ and $\mathrm{Rank}(P\circ u^1) = m_1+1 \le m_0$, and that $P\circ u^i \subset S_i$ where S_i is a m_i-dimensional sub-sphere. Without loss of generality, we can assume that $S_1 \subset S_0$ after a rotation. Then u^0 and u^1 are also two $\overline{\mathcal{N}}^{m_0}$-valued minimizers (here we will replace u^0 by $(u^0)^h + |(u^0)^v|e_{m_0+1}$ if it is necessary). From (35), we deduce that $\langle P\circ u^0, P\circ\alpha(0)\rangle = 0$, that is, $\mathrm{Rank}(P\circ u^0) \le m_0$, a contradiction. ♢

Step 4. In the following, $\overline{\mathcal{N}}^k$ is a submanifold of $\overline{\mathcal{N}}$ in a natural way. In fact, let u be a minimizer for the problem P^k_g. Denote $g_\lambda = P_\lambda\circ g$. Let u_λ be the unique pseudo-harmonic map with the boundary data g_λ and in particular

$u_\lambda(\Omega) \subset \overline{\mathcal{N}}^k$. Assume that $u_\lambda \rightharpoonup \tilde{u}$ in H^1 and let w be a $\overline{\mathcal{N}}$-minimizer for the boundary g. Therefore, we obtain

$$E(\tilde{u}) \leq \liminf_{\lambda \to 0} E(u_\lambda) \leq \liminf_{\lambda \to 0} E(P_\lambda \circ w) = E(w).$$

This means that $\overline{\mathcal{N}}^k$-valued map $\tilde{u}$ is minimizing among $\overline{\mathcal{N}}$-valued maps. Suppose that the minimizer u of the problem P_g^k has rank k. Then by lemma 10 and the above result, all $\overline{\mathcal{N}}$-valued minimizers have rank k. If the $\overline{\mathcal{N}}$-valued minimization problem has two solutions, then so does the $\overline{\mathcal{N}}^k$-valued problem. But this means $Rank(P \circ g) \leq k-1$ which is false. Therefore $P \circ u$ has rank $k+1$. So we complete the part of necessity.

Step 5. Let u be the unique minimizer of the problem P_g^k. Let R be a rotation which leaves the vector space containing $Image(P \circ g)$ and $P_1(\mathbb{R}^{n+1})$invariant. So $R \circ u$ is another minimizer. Hence, this terminate the part of sufficiency. The rest of the theorem is evident.

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