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Periodic solutions for a class of non-coercive Hamiltonian systems *

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Abstract

We prove the existence of non-constant T-periodic orbits of the Hamiltonian system

$$\begin{aligned}\dot{q} &= H_p(t, p(t), q(t))\\ \dot{p} &= -H_q(t, p(t), q(t)),\end{aligned}$$

where H is a T-periodic function in t, non-convex and non-coercive in (p, q), and has the form $H(t, p, q) \sim |q|^\alpha(|p|^\beta - 1)$ with $\alpha > \beta > 1$.

1 Introduction

We study the existence of T-periodic solutions of the Hamiltonian system

$$\begin{aligned}\dot{q} &= H_p(t, p(t), q(t))\\ \dot{p} &= -H_q(t, p(t), q(t)).\end{aligned} \tag{1.1}$$

Here, $H(t, p, q) : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ ($N \geq 3$) is T-periodic in t and differentiable in (p, q). We also assume that H, H_p, H_q are continuous.

Most of the existence results use coercivity (i.e., $H(t, p, q) \to \infty$ as $|(p, q)| \to \infty$) or convexity assumptions in $H(t, .)$; see [1, 2, 3, 4, 5] and references therein. The purpose of this paper is to study non-coercive and non-convex Hamiltonians. Typically,

$$H(t, p, q) \sim |q|^\alpha(|p|^\beta - 1); \quad \alpha > \beta > 1.$$

To state our existence result, we introduce the following hypotheses. For constants $\alpha > \beta > 1$, $r > 0$, $a_1, \dots, a_8 > 0$ and functions A, $K_i \in C(\mathbb{R}^N, \mathbb{R})$ with $K_i(0) = 0$ ($i = 1, 2, 3$), we assume:

(H1) $H(t + \frac{T}{2}, p, q) = H(t, -p, -q)$ for all t, p, q;

(H2) (i) $H(t, p, q) \leq a_1|q|^\alpha|p|^\beta$ for all t, p, q;
(ii) $H(t, p, q) \geq a_2|q|^\alpha|p|^\beta - K_1(q)$ for all t, p, q;

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(H3) $-H(t,p,q)+H_p(t,p,q)p \geq a_3|q|^\alpha(|p|^\beta+1)-a_4$ for all t,p,q;

(H4) $|H_p(t,p,q)| \leq a_5|q|^\alpha(|p|^{\beta-1}+1)+a_6|q|$ for all t,p,q;

(H5) $|H_q(t,p,q)| \leq A(q)(|p|^\beta+1)$ for all t,p,q;

(H6) (i) $H_q(t,p,q)q - H_p(t,p,q)p \geq a_7 H(t,p,q)+K_2(q)$ for all $t,p,|q|\leq r$;
(ii) $|H_p(t,p,q)|^{\frac{\beta}{\beta-1}} \leq a_8|q|^{\frac{\alpha}{\beta-1}}\big(|q|^\alpha|p|^\beta+K_3(q)\big)$ for all $t,p,|q|\leq r$.

Our main result is as follows.

Theorem 1.1 *Under assumptions (H1)-(H6), System (1.1) has at least one non-constant T-periodic solution $(p(t),q(t))$ with $q(t)\neq 0$ for all t.*

Remark. If $H(t,p,q)=a(t)|q|^\alpha(|p|^\beta-1)$ with $\alpha>\beta>1$ and $a(t)\in C(\mathbb{R},\mathbb{R})$ is a $\frac{T}{2}$-periodic and positive function , then (H1)-(H6) hold.

Remark. The condition $\alpha>\beta$ is necessarily for the existence of non-constant T-periodic solution. More precisely, in case

$$H(t,p,q)=|q|^\alpha(|p|^\beta-1),$$

if $(p(t),q(t))$ is a non-constant T-periodic solution of (1.1), then
(i) $\alpha>\beta$;
(ii) there exists a constant $C>0$ such that

$$|q(t)|^\alpha(|p(t)|^\beta-1)=C>0 \text{ for all } t\in\mathbb{R}.$$

In particular, $q(t)\neq 0$ for all $t\in\mathbb{R}$.

Indeed, by (1.1) we have

$$\int_0^T p\dot{q}dt=\beta\int_0^T |q|^\alpha|p|^\beta dt=\alpha\int_0^T|q|^\alpha(|p|^\beta-1)dt.$$

Then

$$(\alpha-\beta)\int_0^T|q|^\alpha|p|^\beta dt=\alpha\int_0^T|q|^\alpha dt.$$

Since (p,q) is non-constant, one can see that $q\neq 0$ and $\alpha>\beta$. Also note that (ii) follows from the conservation of the energy.

To show the existence of a T-periodic solution of (1.1), we use a variational method; we introduce the functional

$$I(p,q)=\int_0^T[p\dot{q}-H(t,p,q)]dt$$

defined on the function space

$$E=\{(p,q)\in L^\gamma(0,T;\mathbb{R}^N)\times W^{1,\frac{\gamma}{(\gamma-1)}}(0,T;\mathbb{R}^N);\ q(0)=q(T)\}$$

where $\gamma = \alpha + \beta$. Critical points of $I(p,q)$ on E correspond to T-periodic solutions of (1.1). We remark that the correspondence is one-to-one.

Since it is difficult to verify the Palais-Smale compactness condition for $I(p,q)$, we introduce in the following section, modified functionals and a finite dimensional approximation. We will use a minimax argument.

2 Modified functionals and other preliminaries

As stated in the introduction, we will find a critical point of the functional $I(p,q)$ on $E = P \times Q$ where

$$P = L^{\gamma}(0,T;\mathbb{R}^N), \quad Q = \{q \in W^{1,\frac{\gamma}{(\gamma-1)}}(0,T;\mathbb{R}^N); q(0) = q(T)\}.$$

We set

$$\Lambda = \{q \in Q;\ q(t) \neq 0 \text{ for all } t\}$$

and introduce the modified functionals

$$I_{\delta}(p,q) = \int_0^T [p\dot{q} - H(t,p,q) + \frac{\delta}{|q|^{\gamma}}]dt,$$

$$I_{\delta,\varepsilon}(p,q) = \int_0^T [p\dot{q} - H(t,p,q) + \frac{\delta}{|q|^{\gamma}} + \varepsilon(|q|^{\gamma} - |p|^{\gamma})]dt$$

for $\delta,\ \varepsilon \in [0,1]$. Since $\gamma \geq \beta > 1$, by (H2), (H4), and (H5), we can see that $I_{\delta,\varepsilon} \in C^1(P \times \Lambda;\mathbb{R})$.

To get the existence of a T-periodic solution for a symmetric Hamiltonians, we have to restrict our functionals to a subsets of E. We set

$$E_0 = \{(p,q) \in E;\ (p,q)(t + \frac{T}{2}) = -(p,q)(t)\}$$

with norm

$$\|(p,q)\|_{E_0} = \|p\|_{\gamma} + \|\dot{q}\|_{\frac{\gamma}{\gamma-1}}$$

where

$$\|u\|_s = (\int_0^T |u(t)|^s dt)^{1/s} \text{ for all } s \geq 1.$$

For $m \in \mathbb{N}$, we define

$$\begin{gathered} P_m = Q_m \\ = \big\{p(t) = \sum_{|j|\leq m} \theta_j e^{\frac{2i\pi jt}{T}};\ p(t + \frac{T}{2}) = -p(t), \theta_j \in \mathbb{C}^N, \theta_{-j} = \bar{\theta_j}, |j| \leq m\big\}, \\ E_m = P_m \times Q_m, \\ \Lambda_m = \{q \in Q_m;\ q(t) \neq 0 \text{ for all } t\}, \\ \partial\Lambda_m = \{q \in Q_m;\ q(t_0) = 0 \text{ for some } t_0\} \end{gathered}$$

and we consider the restriction of $I_{\delta,\varepsilon}(p,q)$:

$$I_{\delta,\varepsilon,m} = I_{\delta,\varepsilon}/_{P_m\times\Lambda_m} : P_m \times \Lambda_m \to \mathbb{R}.$$

The main reason for introducing such subspaces are the following Lemmas.

Lemma 2.1 *For any $u \in Q$ such that $u(t+\frac{T}{2}) = -u(t)$, we have*

$$\|u\|_\infty \le \int_0^T |\dot{u}|dt.$$

Proof. Let $u \in Q$ such that $u(t+\frac{T}{2}) = -u(t)$. Then for all $t \in [0,T]$, we have

$$|u(t)| = \frac{1}{2}|u(t+\frac{T}{2}) - u(t)| = \frac{1}{2}|\int_t^{t+\frac{T}{2}} \dot{u}\, ds| \le \int_0^T |\dot{u}|ds.$$

Thus we obtain the desired result. ♢

Lemma 2.2 *Suppose $(p,q) \in P_m \times \Lambda_m$ is such that*

$$I'_{\delta,\varepsilon,m}(p,q)(h,k) = 0 \quad \textit{for all } (h,k) \in E_m. \tag{2.1}$$

Then (p,q) is a critical point for $I_{\delta,\varepsilon,m}$.

Proof. It is sufficient to remark that, by (H1), $I'_{\delta,\varepsilon,m}(p,q) \in E_m$. Since $I'_{\delta,\varepsilon,m}(p,q)$ belongs also to $E_m^\perp$ from 2.1, we have the conclusion. ♢

The proof of Theorem 1.1 will be done as follows: In section 3, we introduce a minimax method to $I_{\delta,\varepsilon,m}$. For $\delta,\varepsilon \in]0,1]$ and $m \in \mathbb{N}$, we establish the existence of a sequence $(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m$ such that

$$I'_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) = 0, \tag{2.2}$$

$$I_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \le \bar{c} \tag{2.3}$$

where $\bar{c} > 0$ is a constant independent of δ, ε and m. From 2.2-2.3, we can find uniform estimates for $(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m})$ and we can extract, in section 4, a subsequence converging to $(p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) \in (P\times\Lambda)\cap E_0$. Next in Section 5, we pass to the limit as $\varepsilon \to 0$ and obtain a critical points $(p_\delta, q_\delta) \in (P\times\Lambda)\cap E_0$ of I_δ such that

$$I_\delta(p_\delta, q_\delta) \le \bar{c}. \tag{2.4}$$

Finally in Section 6, we pass to the limit as $\delta \to 0$. Lemma 2.1 plays a essential role to obtain a non-constant T-periodic solution $(p,q) = \lim(p_\delta, q_\delta)$ of (1.1).

In the sequel, we use the projection operator

$$\begin{gathered}\mathrm{proj}_m : L^s(0,T;\mathbb{R}^N) \to \mathrm{span}\{e^{\frac{2i\pi jt}{T}}; |j| \le m\}\,,\\ (\mathrm{proj}_m u)(t) = \sum_{|j|\le m} \theta_j e^{\frac{2i\pi jt}{T}} \quad \text{for} \quad u(t) = \sum_{j\in\mathbb{Z}} \theta_j e^{\frac{2i\pi jt}{T}}.\end{gathered}$$

Lemma 2.3 *For any* $s \in]1, +\infty[$, *there exists a constant* $K_s > 0$ *independent of* $m \in \mathbb{N}$ *such that*

$$\|\mathrm{proj}_m u\|_s \le K_s \|u\|_s \quad \textit{for all} \quad u \in L^s(0, T; \mathbb{R}^N).$$

This lemma is a special case of Steckin's theorem [6, Theorem 6.3.5]. In sections 3, 4, 5, and 6, we will assume (H1)-(H6).

3 A minimax method for $I_{\delta,\varepsilon,m}$

In this part, we study the existence of critical points in $P_m \times \Lambda_m$ of $I_{\delta,\varepsilon,m}$ for $\delta, \varepsilon \in]0, 1]$ and $m \in \mathbb{N}$. First, we give some a priori estimates and verify the Palais-Smale condition (PS) for $I_{\delta,\varepsilon,m}$.

Lemma 3.1 *(i) For any* $M_1 > 0$, *there exists a constant* $C_0 = C_0(M_1) > 0$ *independent of* $\delta, \varepsilon \in]0, 1]$ *and* $m \in \mathbb{N}$ *such that: If* $(p, q) \in P_m \times \Lambda_m$ *satisfies*

$$I_{\delta,\varepsilon,m}(p, q) \le M_1, \tag{3.1}$$

$$I'_{\delta,\varepsilon,m}(p, q) = 0, \tag{3.2}$$

then

$$\int_0^T |q|^\alpha |p|^\beta dt + \int_0^T |q|^\alpha dt \le C_0,$$

$$\varepsilon \int_0^T \big(|q|^\gamma + |p|^\gamma\big) dt + \delta \int_0^T \frac{1}{|q|^\gamma} dt \le C_0.$$

(ii) For any $\delta, \varepsilon \in]0, 1]$ *and* $m \in \mathbb{N}$, *if* $(p_j, q_j)_{j=1}^{\infty} \subset P_m \times \Lambda_m$ *satisfies*

$$(p_j, q_j) \to (p_0, q_0) \in P_m \times \partial\Lambda_m,$$

then $I_{\delta,\varepsilon,m}(p_j, q_j) \to +\infty$.
(iii) For any $\delta, \varepsilon \in]0, 1]$ *and* $m \in \mathbb{N}$, $I_{\delta,\varepsilon,m}$ *satisfies the condition (PS) on* $P_m \times \Lambda_m$; *i.e., if* $(p_j, q_j)_{j\in\mathbb{N}} \subset P_m \times \Lambda_m$ *satisfies* $I_{\delta,\varepsilon,m}(p_j, q_j) \to c > 0$ *and* $(I_{\delta,\varepsilon,m})'(p_j, q_j) \to 0$, *then* (p_j, q_j) *possesses a subsequence converging in* E_m *to some* $(p, q) \in P_m \times \Lambda_m$.

Proof. (i) Let $\delta, \varepsilon \in]0, 1]$ and $m \in \mathbb{N}$. We assume $(p, q) \in P_m \times \Lambda_m$ satisfies 3.1 and 3.2 for $M_1 > 0$. We have

$$I'_{\delta,\varepsilon,m}(p, q)(p, 0) = \int_0^T [p\dot{q} - H_p(t, p, q)p - \varepsilon\gamma |p|^\gamma] dt.$$

Hence,

$$\begin{aligned} &I_{\delta,\varepsilon,m}(p, q) - I'_{\delta,\varepsilon,m}(p, q)(p, 0) \qquad (3.3)\\ &= \int_0^T [-H(t, p, q) + H_p(t, p, q)p + \frac{\delta}{|q|^\gamma} + \varepsilon |q|^\gamma + \varepsilon(\gamma - 1)|p|^\gamma] dt. \end{aligned}$$

By the assumptions 3.1 and 3.2, we get

$$\int_0^T [-H(t,p,q) + H_p(t,p,q)p + \frac{\delta}{|q|^\gamma} + \varepsilon|q|^\gamma + \varepsilon(\gamma-1)|p|^\gamma]dt \leq M_1.$$

From (H3), it follows that

$$\int_0^T [a_3|q|^\alpha(|p|^\beta + 1) - a_4 + \frac{\delta}{|q|^\gamma} + \varepsilon|q|^\gamma + \varepsilon(\gamma-1)|p|^\gamma]dt \leq M_1.$$

Thus we obtained (i).
(ii) By (H2)(i), we have for all $(p,q) \in P_m \times \Lambda_m$

$$I_{\delta,\varepsilon,m}(p,q) \geq \int_0^T [p\dot{q} - a_1|q|^\alpha|p|^\beta + \varepsilon(|q|^\alpha - |p|^\gamma)]dt + \delta\int_0^T \frac{1}{|q|^\gamma}dt. \qquad (3.4)$$

Since $\delta\int_0^T \frac{1}{|q_j|^\gamma}dt \to \infty$, we get the conclusion easily.
(iii) Let $(p_j,q_j)_{(j\in\mathbb{N})} \subset P_m \times \Lambda_m$ be a sequence satisfying the assumptions of the condition (PS).We may assume that

$$I_{\delta,\varepsilon,m}(p_j,q_j) \to c, \qquad (3.5)$$

$$\|I'_{\delta,\varepsilon,m}(p_j,q_j)\|_{E_m^\star} \to 0. \qquad (3.6)$$

We prove that (p_j,q_j) possesses a convergent subsequence to some $(p,q) \in P_m \times \Lambda_m$. By (H3) and 3.3-3.6, for large j,

$$\begin{aligned}\int_0^T [a_3|q_j|^\alpha(|p_j|^\beta + 1) - a_4]dt + \delta\int_0^T \frac{1}{|q_j|^\gamma}dt \\ +\varepsilon\int_0^T |q_j|^\gamma dt + \varepsilon(\gamma-1)\int_0^T |p_j|^\gamma dt \quad &\leq \quad 2c + \|p_j\|_\gamma .\end{aligned}$$

Thus, for some constant $C_1 > 0$ independent of j,

$$\int_0^T |q_j|^\alpha dt,\ \int_0^T |p_j|^\gamma dt \leq C_1 \quad \text{for all } j \in \mathbb{N}.$$

Since $\dim E_m < \infty$, we can extract a subsequence - still indexed by (p_j,q_j) -, such that $(p_j,q_j) \to (p,q) \in E_m$. By (ii), we necessarily have $q \in \Lambda_m$.

Next, we apply to $I_{\delta,\varepsilon,m}$ a minimax argument related to the one in [7]. This argument will play an important role in obtaining a critical points $(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m$ with uniform upper bound of critical values. We define

$$\Gamma_m = \{A(p,\xi) \in C\big(P_m \times S^{N-2}, P_m \times \Lambda_m\big);\ A(p,\xi) = \big(p,\sigma_0(\xi)\big)\text{for large } \|p\|_\beta\}$$

where

$$\sigma_0 : S^{N-2} = \{\xi = (\xi_1,\dots,\xi_{N-1}) \in \mathbb{R}^{N-1} : \sum_{j=1}^{N-1} |\xi_j|^2 = 1\} \to Q_m$$

is given by

$$\sigma_0(\xi)(t) = \cos\frac{2\pi t}{T}(\xi_1, \dots, \xi_{N-1}, 0) + \sin\frac{2\pi t}{T}(0, \dots 0, 1).$$

We remark that $A_0(p,\xi) = (p, \sigma_0(\xi)) \in \Gamma_m$ and $\Gamma_m \neq \emptyset$. Then we define the minimax values of $I_{\delta,\varepsilon,m}$ as follows

$$c_{\delta,\varepsilon,m} = \inf_{A\in\Gamma_m} \sup_{(p,\xi)\in P_m\times S^{N-2}} I_{\delta,\varepsilon,m}(A(p,\xi)).$$

Proposition 3.1 *For any* $\delta, \varepsilon \in]0,1]$ *and* $m \in \mathbb{N}$*, there exists a constant* $\underline{c}(\delta,\varepsilon) > 0$ *such that*

$$c_{\delta,\varepsilon,m} \geq \underline{c}(\delta,\varepsilon) > 0.$$

To prove this proposition, we need the following result.

Lemma 3.2 *For any* $A \in \Gamma_m$ *and* $\lambda > 0$*, we have*

$$A(P_m \times S^{N-2}) \cap \mathcal{D}_{m,\lambda} \neq \emptyset$$

where

$$\mathcal{D}_{m,\lambda} = \{(p,q) \in P_m \times \Lambda_m;\ p = \lambda \mathrm{proj}_m(|\dot q|^{\frac{1}{\gamma-1}-1}\dot q)\}.$$

The proof of this lemma will be given in the appendix.

Lemma 3.3 *For sufficiently small* $\lambda_\varepsilon > 0$*, there exists a constant* $c(\delta,\varepsilon) > 0$ *such that*

$$I_{\delta,\varepsilon,m}(p,q) \geq c(\delta,\varepsilon) > 0 \quad \text{for all } (p,q) \in \mathcal{D}_{m,\lambda_\varepsilon}$$

where $\mathcal{D}_{m,\lambda_\varepsilon}$ *is given in Lemma 3.2.*

Proof. Let $(p,q) \in \mathcal{D}_{m,\lambda}$. We recall that $\gamma = \alpha+\beta$. By the Young's inequality,

$$a_1 \int_0^T |q|^\alpha |p|^\beta dt \leq \frac{\alpha}{\gamma}\varepsilon \int_0^T |q|^\gamma dt + \frac{\beta}{\gamma}\Big(\frac{a_1}{\varepsilon^{\frac{\alpha}{\gamma}}}\Big)^{\frac{\gamma}{\beta}} \int_0^T |p|^\gamma dt.$$

Thus, from 3.4,

$$I_{\delta,\varepsilon,m}(p,q) \geq \int_0^T p\dot q dt - a(\varepsilon)\int_0^T |p|^\gamma dt + \delta \int_0^T \frac{1}{|q|^\gamma} dt$$

where $a(\varepsilon) = \varepsilon + \frac{\beta}{\gamma}(\frac{a_1}{\varepsilon^{\frac{\alpha}{\gamma}}})^{\frac{\gamma}{\beta}} > 0$. Since $(p,q) \in \mathcal{D}_{m,\lambda}$,

$$\int_0^T p\dot q dt = \lambda \int_0^T |\dot q|^{\frac{\gamma}{\gamma-1}} dt. \tag{3.7}$$

Moreover, by Lemma 2.1 and Lemma 2.3

$$T^{\frac{1}{\gamma}} \|\dot q\|_{\frac{\gamma}{\gamma-1}} \geq \int_0^T |\dot q| dt \geq \|q\|_\infty, \tag{3.8}$$

$$\int_0^T |p|^\gamma dt = \lambda^\gamma \|\mathrm{proj}_m(|\dot q|^{\frac{1}{\gamma-1}-1}\dot q)\|_\gamma^\gamma \le \lambda^\gamma K_\gamma^\gamma \|\dot q\|_{\frac{\gamma}{\gamma-1}}^{\frac{\gamma}{\gamma-1}}. \tag{3.9}$$

By 3.7 and 3.9, we get

$$I_{\delta,\varepsilon,m}(p,q) \ge (\lambda - a(\varepsilon)K_\gamma^\gamma\lambda^\gamma)\|\dot q\|_{\frac{\gamma}{\gamma-1}}^{\frac{\gamma}{\gamma-1}} + \delta\int_0^T \frac{1}{|q|^\gamma}dt.$$

Taking λ_ε small enough so that $A_\varepsilon = \lambda_\varepsilon - a(\varepsilon)K_\gamma^\gamma\lambda_\varepsilon^\gamma > 0$, from 3.8, for all $(p,q) \in \mathcal{D}_{m,\lambda_\varepsilon}$, we have

$$I_{\delta,\varepsilon,m}(p,q) \ge \inf_{q\in\Lambda}\big(\frac{A_\varepsilon}{T^{\frac{1}{\gamma-1}}}\|q\|_\infty^{\frac{\gamma}{\gamma-1}} + \frac{\delta T}{\|q\|_\infty^\gamma}\big) = c(\delta,\varepsilon) > 0\,.$$

Proof of Proposition 3.1 Let $\lambda_\varepsilon > 0$ be as in Lemma 3.3. By Lemma 3.2, we have

$$A(P_m \times S^{N-2}) \cap \mathcal{D}_{m,\lambda_\varepsilon} \neq \emptyset \quad \text{for all } A \in \Gamma_m.$$

Thus, we find that

$$\begin{aligned} c_{\delta,\varepsilon,m} &= \inf_{A\in\Gamma_m}\ \sup_{(p,\xi)\in P_m\times S^{N-2}} I_{\delta,\varepsilon,m}(A(p,\xi)) \\ &\ge \inf_{(p,q)\in\mathcal{D}_{m,\lambda_\varepsilon}} I_{\delta,\varepsilon,m}(p,q) \\ &\ge c(\delta,\varepsilon) > 0. \end{aligned}$$

We choose $\underline{c}(\delta,\varepsilon) = c(\delta,\varepsilon)$, we get the desired result. ♢

Now, we prove an existence result

Proposition 3.2 *For any $\delta,\varepsilon \in]0,1]$ and $m \in \mathbb{N}$, we have*
(i)

$$0 < \underline{c}(\delta,\varepsilon) \le c_{\delta,\varepsilon,m} \le \bar c$$

where $\bar c$ is independent of δ,ε and m.
(ii) If $\|p\|_\beta$ is sufficiently large, then for all $\xi \in S^{N-2}$,

$$I_{\delta,\varepsilon,m}(A_0(p,\xi)) \le 0\,.$$

(iii) There exists a critical point $(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m$ of $I_{\delta,\varepsilon,m}$ such that

$$I_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) = c_{\delta,\varepsilon,m}.$$

Proof. (i) By (H2)(ii), we have

$$\begin{aligned} I_{\delta,\varepsilon,m}(A_0(p,\xi)) &\le \int_0^T |p||\frac{d}{dt}\sigma_0(\xi)|dt - a_2\int_0^T |\sigma_0(\xi)|^\alpha|p|^\beta dt \\ &\quad + \int_0^T K_1(\sigma_0(\xi))dt + \int_0^T (\frac{1}{|\sigma_0(\xi)|^\gamma} + |\sigma_0(\xi)|^\gamma)dt \\ &\le k_1\|p\|_\beta - k_2\|p\|_\beta^\beta + k_3 \end{aligned} \tag{3.10}$$

for some positive constants k_1, k_2, k_3 independent of δ, ε and m. Since $\beta > 1$, there exists a constant $\bar{c} > 0$ independent of δ, ε and m such that

$$c_{\delta,\varepsilon,m} \leq \sup_{(p,\xi)\in P_m\times S^{N-2}} I_{\delta,\varepsilon,m}(A_0(p,\xi)) \leq \bar{c}.$$

(ii) follows clearly from 3.10.
(iii) Since $I_{\delta,\varepsilon,m}$ satisfies the (PS) condition and property (ii) of Lemma 3.1, then by a standard argument using the deformation theorem and (ii), we can see that $c_{\delta,\varepsilon,m} > 0$ is a critical value of $I_{\delta,\varepsilon,m}$. By Lemma 2.2, we get (iii). ♢

As a corollary to (i) of Lemma 3.1 and the uniform estimates of $c_{\delta,\varepsilon,m}$, we have the following statements.

Corollary 3.1 *Let $(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \in P_m \times \Lambda_m$ be a critical point of $I_{\delta,\varepsilon,m}$ obtained by Proposition 3.2.Then, there exists a constant $C_2 > 0$ independent of δ, ε and m, such that for all $\delta, \varepsilon \in]0,1]$ and $m \in \mathbb{N}$, we have*

(i) $$\int_0^T |q_{\delta,\varepsilon,m}|^\alpha |p_{\delta,\varepsilon,m}|^\beta dt + \int_0^T |q_{\delta,\varepsilon,m}|^\alpha dt \leq C_2,$$

(ii) $$\varepsilon \int_0^T \big(|q_{\delta,\varepsilon,m}|^\gamma + |p_{\delta,\varepsilon,m}|^\gamma\big) dt \leq C_2,$$

(iii) $$\delta \int_0^T \frac{1}{|q_{\delta,\varepsilon,m}|^\gamma} dt \leq C_2.$$

4 Limiting process as $m \to \infty$

Proposition 4.1 *For any $\delta, \varepsilon \in]0,1]$, $(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m})$ possesses a subsequence converging in E to $(p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) \in (P \times \Lambda) \cap E_0$. Moreover,*

$$I_{\delta,\varepsilon}(p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) \leq \bar{c}, \tag{4.1}$$
$$I'_{\delta,\varepsilon}(p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) = 0\,. \tag{4.2}$$

Proof. By (ii) of Corollary 3.1, we can extract a subsequence - still indexed by m- such that

$$(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \rightharpoonup (p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) \quad \text{weakly in } L^\gamma(0,T;\mathbb{R}^N).$$

We remark that $I'_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) = 0$ is equivalent to

$$\dot{q}_{\delta,\varepsilon,m} = \mathrm{proj}_m[H_p(t, p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) + \varepsilon\gamma|p_{\delta,\varepsilon,m}|^{\gamma-2} p_{\delta,\varepsilon,m}], \tag{4.3}$$

$$\dot{p}_{\delta,\varepsilon,m} = \text{-}\mathrm{proj}_m[H_q(t, p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) + \delta\gamma \frac{q_{\delta,\varepsilon,m}}{|q_{\delta,\varepsilon,m}|^{\gamma+2}} - \varepsilon\gamma|q_{\delta,\varepsilon,m}|^{\gamma-2} q_{\delta,\varepsilon,m}]. \tag{4.4}$$

By (H4) and Lemma 2.3, we have from 4.3

$$\begin{aligned} \|\dot{q}_{\delta,\varepsilon,m}\|_{\frac{\gamma}{\gamma-1}} \leq\ & K_{\frac{\gamma}{\gamma-1}}[a_5\|(|q_{\delta,\varepsilon,m}|^\alpha |p_{\delta,\varepsilon,m}|^{(\beta-1)})\|_{\frac{\gamma}{\gamma-1}} + a_5\|q_{\delta,\varepsilon,m}\|^\alpha_{\alpha\frac{\gamma}{\gamma-1}} \\ & + a_6\|q_{\delta,\varepsilon,m}\|_{\frac{\gamma}{\gamma-1}} + \varepsilon\gamma\|p_{\delta,\varepsilon,m}\|_\gamma^{\gamma-1}]. \end{aligned}$$

Using a Hölder's inequality and (i)-(ii) of Corollary 3.1, we can find a constant $C_3 > 0$ independent of $m \in \mathbb{N}$, such that

$$\|q_{\delta,\varepsilon,m}\|_{W^{1,\frac{\gamma}{\gamma-1}}(0,T;\mathbb{R}^N)} \leq C_3.$$

Thus we can see from (iii) of Corollary 3.1 that

$$q_{\delta,\varepsilon,m} \to q_{\delta,\varepsilon} \in \Lambda \text{ uniformly in } [0,T]. \tag{4.5}$$

On the other hand, by (H5) and Lemma 2.3, we have from 4.4

$$\begin{aligned}\|\dot{p}_{\delta,\varepsilon,m}\|_{\frac{\gamma}{\gamma-1}} \quad \leq \quad & K_{\frac{\gamma}{\gamma-1}}[\|A(q_{\delta,\varepsilon,m})|p_{\delta,\varepsilon,m}|^{\beta}\|_{\frac{\gamma}{\gamma-1}} + \|A(q_{\delta,\varepsilon,m})\|_{\frac{\gamma}{\gamma-1}} \\ & +\gamma\|\delta\frac{q_{\delta,\varepsilon,m}}{|q_{\delta,\varepsilon,m}|^{\gamma+2}} - \varepsilon|q_{\delta,\varepsilon,m}|^{\gamma-2}q_{\delta,\varepsilon,m}\|_{\frac{\gamma}{\gamma-1}}].\end{aligned}$$

Using 4.5, we find

$$\|p_{\delta,\varepsilon,m}\|_{W^{1,\frac{\gamma}{\gamma-1}}(0,T;\mathbb{R}^N)} \leq C_4$$

where $C_4 > 0$ is a constant independent of m. The injection $W^{1,\frac{\gamma}{\gamma-1}}(0,T;\mathbb{R}^N) \subset L^{\gamma}(0,T;\mathbb{R}^N)$ is compact, thus we have

$$p_{\delta,\varepsilon,m} \to p_{\delta,\varepsilon} \text{ strongly in } L^{\gamma}(0,T;\mathbb{R}^N) \text{ and uniformly in } [0,T]. \tag{4.6}$$

By (i) and (iii) of Proposition 3.2, we deduce that

$$\begin{gathered} I_{\delta,\varepsilon}(p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) = \lim_{m\to\infty} I_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m}) \leq \bar{c}, \\ I'_{\delta,\varepsilon}(p_{\delta,\varepsilon,}, q_{\delta,\varepsilon})(h,k) = \lim_{m\to\infty} I'_{\delta,\varepsilon,m}(p_{\delta,\varepsilon,m}, q_{\delta,\varepsilon,m})(h,k) = 0 \end{gathered}$$

for all sums

$$h = \sum_{|j|\leq n} \theta_j e^{\frac{2i\pi jt}{T}}\ ,\ k = \sum_{|j|\leq n} \psi_j e^{\frac{2i\pi jt}{T}} \quad (\theta_j, \psi_j \in \mathbf{C}^N).$$

Therefore, $I'_{\delta,\varepsilon}(p_{\delta,\varepsilon}, q_{\delta,\varepsilon})(h,k) = 0$ for all $(h,k) \in E$.

5 Limiting process as $\varepsilon \to 0$

We take the limit as $\varepsilon \to 0$ to obtain a critical point $(p_\delta, q_\delta) \in (P \times \Lambda) \cap E_0$ of I_δ with uniform upper bound for critical values. As a consequence to Corollary 3.1, and 4.5, 4.6 we have the following lemma.

Lemma 5.1 *For any $\delta, \varepsilon \in]0,1]$, $(p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) \in (P \times \Lambda) \cap E_0$ satisfies*

(i) $$\int_0^T |q_{\delta,\varepsilon}|^{\alpha}|p_{\delta,\varepsilon}|^{\beta}dt + \int_0^T |q_{\delta,\varepsilon}|^{\alpha}dt \leq C_2,$$

(ii) $$\varepsilon\int_0^T \big(|q_{\delta,\varepsilon}|^{\gamma} + |p_{\delta,\varepsilon}|^{\gamma}\big)dt \leq C_2,$$

(iii) $$\delta\int_0^T \frac{1}{|q_{\delta,\varepsilon}|^{\gamma}}dt \leq C_2.$$

Proposition 5.1 *For any* $\delta \in]0,1]$, $(p_{\delta,\varepsilon}, q_{\delta,\varepsilon})$ *possesses a subsequence converging in* E *to* $(p_\delta, q_\delta) \in (P \times \Lambda) \cap E_0$. *Moreover,*

$$I'_\delta(p_\delta, q_\delta) = 0,$$

$$I_\delta(p_\delta, q_\delta) \le \bar{c}.$$

Proof. Since $I'_{\delta,\varepsilon}(p_{\delta\varepsilon}, q_{\delta,\varepsilon}) = 0$, we have

$$\dot{q}_{\delta,\varepsilon} = H_p(t, p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) + \varepsilon\gamma |p_{\delta,\varepsilon}|^{\gamma-2} p_{\delta,\varepsilon}, \tag{5.1}$$

$$\dot{p}_{\delta,\varepsilon} = -[H_q(t, p_{\delta,\varepsilon}, q_{\delta,\varepsilon}) + \delta\gamma \frac{q_{\delta,\varepsilon}}{|q_{\delta,\varepsilon}|^{\gamma+2}} - \varepsilon\gamma |q_{\delta,\varepsilon}|^{\gamma-2} q_{\delta,\varepsilon}]. \tag{5.2}$$

By (H4) and 5.1, we can see from (i)-(ii) of Lemma 5.1 that

$$\begin{aligned}
\int_0^T |\dot{q}_{\delta,\varepsilon}| dt \;\; &\le \;\; a_5[\int_0^T |q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^{\beta-1} dt + \int_0^T |q_{\delta,\varepsilon}|^\alpha dt] + a_6 \int_0^T |q_{\delta,\varepsilon}| dt \\
&\quad + \varepsilon\gamma \int_0^T |p_{\delta,\varepsilon}|^{\gamma-1} dt \\
&\le \;\; C_5
\end{aligned}$$

where $C_5 > 0$ is a constant independent of ε. Thus, we deduce that $(q_{\delta,\varepsilon})_\varepsilon$ is bounded in $L^\infty(0, T; \mathbb{R}^N)$.
By (H4) and (5.1) again, we have

$$\begin{aligned}
||\dot{q}_{\delta,\varepsilon}||_{\frac{\gamma}{\gamma-1}} \;\; &\le \;\; a_5 ||(|q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^{\beta-1})||_{\frac{\gamma}{\gamma-1}} + a_5 ||q_{\delta,\varepsilon}||^\alpha_{\alpha\frac{\gamma}{\gamma-1}} \\
&\quad + a_6 ||q_{\delta,\varepsilon}||_{\frac{\gamma}{\gamma-1}} + \varepsilon\gamma ||p_{\delta,\varepsilon}||^{\gamma-1}_\gamma.
\end{aligned}$$

Here we will apply the Hölder's inequality

$$||fg||_s \le ||f||_{s\mu} ||g||_{s\nu}$$

with $f(t) = |q_{\delta,\varepsilon}|^{\frac{\alpha}{\beta}}$, $g(t) = (|q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^\beta)^{\frac{\beta-1}{\beta}}$, $s = \frac{\gamma}{\gamma-1}$, $\mu = \frac{(\gamma-1)\beta}{\alpha}$ and $\nu = \frac{(\gamma-1)\beta}{(\beta-1)\gamma}$.
We verify that $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Then we have

$$\begin{aligned}
||(|q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^{\beta-1})||_{\frac{\gamma}{\gamma-1}} \;\; &= \;\; ||(|q_{\delta,\varepsilon}|^{\frac{\alpha}{\beta}})(|q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^\beta)^{\frac{\beta-1}{\beta}}||_{\frac{\gamma}{\gamma-1}} \\
&\le \;\; ||(|q_{\delta,\varepsilon}|^{\frac{\alpha}{\beta}})||_{\frac{\gamma\beta}{\alpha}} ||(|q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^\beta)^{\frac{\beta-1}{\beta}}||_{\frac{\beta}{\beta-1}} \\
&= \;\; ||q_{\delta,\varepsilon}||^{\frac{\alpha}{\beta}}_\gamma ||(|q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^\beta)||_1^{\frac{\beta-1}{\beta}} \\
&\le \;\; C_6
\end{aligned}$$

where $C_6 > 0$ is a constant independent of ε.

Finally $(q_{\delta,\varepsilon})_\varepsilon$ is bounded in $W^{1,\frac{\gamma}{\gamma-1}}(0,T;\mathbf{R}^N)$. That is we can extract a subsequence -still indexed by ε- such that

$$q_{\delta,\varepsilon} \to q_\delta \in \Lambda \quad \text{uniformly in } [0,T]. \tag{5.3}$$

Since $\int_0^T |q_{\delta,\varepsilon}|^\alpha |p_{\delta,\varepsilon}|^\beta dt \leq C_2$, we get

$$\int_0^T |p_{\delta,\varepsilon}|^\beta dt \leq C_7 \tag{5.4}$$

for some constant $C_7 > 0$ independent of ε. By (H5) and 5.2-5.4, there exists a constant $C_8 > 0$ independent of ε such that

$$\begin{aligned}\int_0^T |\dot{p}_{\delta,\varepsilon}| dt &\leq \int_0^T A(q_{\delta,\varepsilon})\big(|p_{\delta,\varepsilon}|^\beta + 1\big) dt + \gamma \int_0^T \big(\frac{1}{|q_{\delta,\varepsilon}|^{\gamma+1}} + |q_{\delta,\varepsilon}|^{\gamma-1}\big) dt \\ &\leq C_8\end{aligned}$$

and

$$\int_0^T |\dot{p}_{\delta,\varepsilon}|^\gamma dt \leq C_8.$$

So we can extract a subsequence -still indexed by ε- such that

$$p_{\delta,\varepsilon} \to p_\delta \text{ strongly in } L^\gamma(0,T;\mathbb{R}) \text{ and uniformly in } [0,T]. \tag{5.5}$$

By 5.3 and 5.5, a passage to the limit on 4.1-4.2 similar as in Section 4 completes the proof.

6 Proof of Theorem 1.1

We take a limit as $\delta \to 0$ to obtain a T-periodic solution of (1.1). Let $(p_\delta, q_\delta) \in (P \times \Lambda) \cap E_0$ be a critical point of $I_\delta(p,q)$ obtained by Proposition 5.1. By Lemma 5.1, 5.3 and 5.5, we have

Lemma 6.1 *For any* $\delta \in]0,1]$,

(i) $$\int_0^T |q_\delta|^\alpha |p_\delta|^\beta dt + \int_0^T |q_\delta|^\alpha dt \leq C_2,$$

(ii) $$\delta \int_0^T \frac{1}{|q_\delta|^\gamma} dt \leq C_2.$$

By (i) of Lemma 6.1, we can extract a subsequence -still indexed by δ- such that

$$q_\delta \rightharpoonup q \quad \text{weakly in} \quad L^\alpha(0,T;\mathbb{R}^N).$$

We also remark that $I'_\delta(p_\delta, q_\delta) = 0$ is equivalent to

$$\dot{q}_\delta = H_p(t, p_\delta, q_\delta), \tag{6.1}$$

$$\dot{p}_\delta = -[H_q(t, p_\delta, q_\delta) + \delta\gamma \frac{q_\delta}{|q_\delta|^{\gamma+2}}]. \tag{6.2}$$

Lemma 6.2 $q_\delta \to q \in \Lambda$ *uniformly in* $[0,T]$.

Proof. By (H4) and 6.1, we have

$$\int_0^T |\dot q_\delta| dt \le a_5 \int_0^T |q_\delta|^\alpha |p_\delta|^{\beta-1} dt + a_5 \int_0^T |q_\delta|^\alpha dt + a_6 \int_0^T |q_\delta| dt.$$

Using (i) of Lemma 6.1, we can see that $\|q_\delta\|_{W^{1,1}(0,T;\mathbb{R}^N)}$ is bounded. Thus we can find a constant $C_9 > 0$ independent of δ, such that

$$\int_0^T |\dot q_\delta|^{\frac{\beta}{\beta-1}} dt \le C_9.$$

Consequently, we obtain $q_\delta \to q$ uniformly in $[0,T]$.
We now argue indirectly and suppose that

$$q(t_0) = 0 \quad \text{for some } \ t_0 \in [0,T].$$

We may assume $t_0 = 0$. By 6.1, for any $t \in]0,T]$ we have

$$|\log|q_\delta(t)| - \log|q_\delta(0)\| \le \int_0^t \frac{|\dot q_\delta(s)|}{|q_\delta(s)|} ds = \int_0^t \frac{|H_p(s,p_\delta,q_\delta)|}{|q_\delta|} ds. \tag{6.3}$$

By (H4),

$$\int_0^t \frac{|H_p(s,p_\delta,q_\delta)|}{|q_\delta|} ds \le a_5 \int_0^t |q_\delta|^{\alpha-1} |p_\delta|^{\beta-1} ds + a_5 \int_0^t |q_\delta|^{\alpha-1} ds + a_6 T.$$

Since $\alpha > \beta > 1$ and $\int_0^T |q_\delta|^\alpha |p_\delta|^\beta dt \le C_2$, there exists a constant $C_{10} > 0$ independent of δ, such that

$$\int_0^t \frac{|H_p(s,p_\delta,q_\delta)|}{|q_\delta|} ds \le C_{10}. \tag{6.4}$$

Passing to the limit in 6.3, we see that $q_\delta \to 0$ uniformly in $[0,T]$. By 6.1-6.2, we have

$$\begin{aligned} I_\delta(p_\delta,q_\delta) &= \int_0^T H_p(t,p_\delta,q_\delta)p_\delta dt - \int_0^T H(t,p_\delta,q_\delta)dt + \delta \int_0^T \frac{1}{|q_\delta|^\gamma} dt \\ &= \int_0^T H_q(t,p_\delta,q_\delta)q_\delta dt - \int_0^T H(t,p_\delta,q_\delta)dt + \delta(\gamma+1) \int_0^T \frac{1}{|q_\delta|^\gamma} dt. \end{aligned}$$

Hence

$$\int_0^T [H_q(t,p_\delta,q_\delta)q_\delta - H_p(t,p_\delta,q_\delta)p_\delta] dt + \delta\gamma \int_0^T \frac{1}{|q_\delta|} dt = 0.$$

From (H6)(i) and (H2)(ii), it follows that

$$a_7 a_2 \int_0^T |q_\delta|^\alpha |p_\delta|^\beta dt - a_7 \int_0^T K_1(q_\delta) dt + \int_0^T K_2(q_\delta) dt + \delta\gamma \int_0^T \frac{1}{|q_\delta|^\gamma} dt \le 0$$

for small δ. Since $q_\delta \to 0$ uniformly in $[0,T]$, we find

$$\int_0^T |q_\delta|^\alpha |p_\delta|^\beta dt \to 0 \text{ as } \delta \to 0. \tag{6.5}$$

Thus we can see from 6.1, 6.5 and (H6)(ii),

$$\begin{aligned} \int_0^T \frac{|\dot q_\delta|^{\frac{\beta}{\beta-1}}}{|q_\delta|^{\frac{\alpha}{\beta-1}}} dt &= \int_0^T \frac{|H_p(t,p_\delta,q_\delta)|^{\frac{\beta}{\beta-1}}}{|q_\delta|^{\frac{\alpha}{\beta-1}}} dt \\ &\le a_8 \int_0^T [|q_\delta|^\alpha |p_\delta|^\beta + K_3(q_\delta)] dt \\ &\to 0 \text{ as } \delta \to 0. \end{aligned} \tag{6.6}$$

In other hand, we have from Lemma 2.1

$$\begin{aligned} \int_0^T \frac{|\dot q_\delta|^{\frac{\beta}{\beta-1}}}{|q_\delta|^{\frac{\alpha}{\beta-1}}} dt &\ge \frac{\left(\int_0^T |\dot q_\delta| dt\right)^{\frac{\beta}{\beta-1}}}{T^{\frac{1}{\beta-1}} \|q_\delta\|_\infty^{\frac{\alpha}{\beta-1}}} \\ &\ge \frac{1}{T^{\frac{1}{\beta-1}} \|q_\delta\|_\infty^{\frac{\alpha-\beta}{\beta-1}}} \\ &\to +\infty \text{ as } \delta \to 0. \end{aligned}$$

This is a contradiction to 6.6 which proves the Lemma 6.2.

Lemma 6.3 *There exists a constant C_{11} independent of $\delta \in]0,1]$ such that*

$$\|p_\delta\|_{W^{1,\gamma}(0,T;\mathbb{R}^N)} \le C_{11}.$$

Proof. Since $q_\delta \to q \in \Lambda$ uniformly in $[0,T]$ and $\int_0^T |q_\delta|^\alpha |p_\delta|^\beta dt \le C_2$, there exists a constant $C_{12} > 0$ independent of $\delta \in]0,1]$ such that

$$\int_0^T |p_\delta|^\beta dt \le C_{12}.$$

By (H5) and 6.2, one deduce that $\int_0^T |\dot p_\delta| dt$ is bounded. Thus we can see for some constant $C_{11} > 0$ independent of $\delta \in]0,1]$

$$\|p_\delta\|_{W^{1,\gamma}(0,T;\mathbb{R}^N)} \le C_{11}.$$

We complete the proof of Theorem 1.1 as follows: By Lemmas 6.2 and 6.3, we can extract a subsequence -still indexed by δ- such that $p_\delta \to p$ strongly in $L^\gamma(0,T;\mathbb{R}^N)$ and $(p_\delta, q_\delta) \to (p,q) \in (P \times \Lambda) \cap E_0$ uniformly in $[0,T]$. Since $I'_\delta(p_\delta, q_\delta) = 0$, we get

$$I'(p,q)(h,k) = 0 \quad \text{for all } \ (h,k) \in E.$$

That is $(p,q) \in (P \times \Lambda) \cap E_0$ is a non-constant T-periodic solution of (1.1).

7 Remarks on the prescribed energy problem

If $H(t,p,q)$ does not depend on t, then the energy surface

$$S_h = H^{-1}(h) = \{(p,q) \in \mathbb{R}^N \times \mathbb{R}^N;\ H(p,q) = h\}\ (h > 0)$$

is not compact for such Hamiltonian functions. Moreover, S_h is equal to

$$\tilde{H}^{-1}(1) = \{(p,q) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\};\ \tilde{H}(p,q) = 1\}$$

where

$$\tilde{H}(p,q) = \frac{H(p,q) - h}{|q|^\alpha} + 1. \tag{7.1}$$

It is clear that, if $H(p,q) \sim |q|^\alpha(|p|^\beta - 1)$, then

$$\tilde{H}(p,q) \sim |p|^\beta - \frac{h}{|q|^\alpha}. \tag{7.2}$$

In the last few years, the existence of periodic solutions of singular Hamiltonian systems has been studied via variational methods under the situation related to two-body problem in celestial mechanics. That is, situation $\tilde{H}(p,q)$ is of the form

$$\tilde{H}(p,q) = \frac{1}{2}|p|^2 + V(q)$$

where $V(q) \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ and $V(q) \to -\infty$ as $q \to 0$. See [8, 9, 10] and references therein. Results dealing with more general singular Hamiltonians of the form (7.2) can be found in [7, 11] for fixed period problems, and in [12, 13] for fixed energy problems.

According to the fundamental lemma of Rabinowitz (see [1] and [14, lemma 3.1]), it follows that the Hamiltonian system (1.1) has, for H and $\tilde{H}$ which are related by 7.1, the same orbits on S_h. Therefore, under suitable conditions on H including $|q|^\alpha(|p|^\beta - 1)$ with $\alpha > \beta > 1$, the theorem of [12] carries a non-collision orbit of the singular Hamiltonian system

$$\begin{gathered} \dot{q} = \tilde{H}_p(p(t), q(t)) \\ \dot{p} = -\tilde{H}_q(p(t), q(t)) \\ \tilde{H}(p,q) = 1, \end{gathered}$$

which corresponds to a non-constant periodic solution of (1.1) with energy h.

Appendix: Proof of Lemma 3.2

The proof of Lemma 3.2 is a special case of [7, lemma 3.1]. We fix $A \in \Gamma_m$ and take $R > 0$ such that

$$R > \lambda \max_{\xi \in S^{N-2}} \|\mathrm{proj}_m |\frac{d}{dt}(\sigma_0(\xi))(t)|^{\frac{1}{\gamma-1}-1} \frac{d}{dt}(\sigma_0(\xi))(t)\|_\beta,$$

$$A(p,\xi) = (p, \sigma_0(\xi)) \quad \text{if } \|p\|_\beta \geq R.$$

We note that

$$A(p,\xi) = \big(x(p,\xi), y(p,\xi)\big), \tag{A.1}$$

$$B(\rho) = \{p \in P_m;\ \|p\|_\beta \leq \rho\}, \quad \rho > 0.$$

Then we define the function $\phi(\rho) \in C(\mathbb{R}, [0,1])$ such that

$$\phi(\rho) = \begin{cases} 1, & \rho \leq R, \\ 0, & \rho \geq 2R. \end{cases}$$

Using the notation (A.1), we define a mapping

$$F : P_m \times S^{N-2} \times [0,T]/\{0,T\} \sim P_m \times S^{N-2} \times S^1 \to P_m \times S^{N-1}$$

by

$$F(p,\xi,t) = \big(x(p,\xi) - \lambda\phi(\|p\|_\beta)\mathrm{proj}_m(|\dot y(p,\xi)|^{\frac{1}{\gamma-1}-1}\dot y(p,\xi)), \tilde\sigma(\xi)(t)\big)$$

where $\tilde\sigma(\xi)(t) = \frac{\sigma(\xi)(t)}{|\sigma(\xi)(t)|}$ and

$$\sigma(\xi)(t) = (3 + \cos\frac{2\pi t}{T})(\xi_1, \dots, \xi_{N-1}, 0) - (3, 0, \dots, 0) + (0, \dots, 0, \sin\frac{2\pi t}{T}).$$

We remark that $F(p,\xi,t) = (p, \tilde\sigma(\xi)(t))$ for $\|p\|_\beta \geq 2R$ and the degree of the map $\tilde\sigma : S^{N-2} \times S^1 \to S^{N-1}$ is not equal to zero.
Thus, there exists $R' \geq 2R$ such that the degree of the mapping

$$F : \big(B(R') \times S^{N-2} \times S^1; \partial B(R') \times S^{N-2} \times S^1\big) \to \big(B(R') \times S^{N-1}; \partial B(R') \times S^{N-1}\big)$$

is not equal to zero. Then it follows the existence of (p,ξ) such that

$$x(p,\xi) - \lambda\phi(\|p\|_\beta)\mathrm{proj}_m\big(|\dot y(p,\xi)|^{\frac{1}{\gamma-1}-1}\dot y(p,\xi)\big) = 0.$$

By the definition of R, we have necessarily $\|p\|_\beta \leq R$. That is

$$x(p,\xi) = \lambda\mathrm{proj}_m\big(|\dot y(p,\xi)|^{\frac{1}{\gamma-1}-1}\dot y(p,\xi)\big)$$

and then

$$A(P_m \times S^{N-2}) \bigcap \mathcal{D}_{m,\lambda} \neq \emptyset.$$

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