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L^p perturbations in delay differential equations *

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Abstract

We present extensions and new proofs for the asymptotic formulae and theorems in Cooke [3] for differential equations with variable delay. Explicit asymptotic formulae, examples and comparisons with the classical results are given here.

1 Introduction

This paper is motivated by a result given by Cooke [3] for the delay differential equation

$$x'(t) = ax(t - r(t)), \tag{1}$$

where $a \in \mathbb{C}$, $r : [0, +\infty[\to [0, R]$, $r(t)$ is a continuous and nonnegative for $t \geq \sigma$ such that the following conditions are satisfied

i) $r(t) \to 0$ and r' is bounded as $t \to +\infty$;

ii) $\inf_{t\geq\sigma}[t - r(t)] > -\infty$. Also, for σ sufficiently large assume that: There is a constant c_1 independent of σ such that

$$|a - a^2 r(t) - a \exp\Big(\int_{t-r(t)}^{t} [a + a^2 r(\tau)]d\tau\Big)| \leq c_1 r(t)^2,\ t \geq \sigma.$$

iii) $r \in L^p$ for some $p : 1 \leq p \leq 2$;

iv) Let $\tau_1 = \tau_1(\sigma)$ be defined by $\tau_1 - r(\tau_1) = \sigma$, $t - r(t) > \sigma \forall t > \tau_1$.

a. $8|a| \int_{\tau_1}^{t} r(s)ds \leq r(t)$, $\tau_1 \leq t$, $t - r(t) \leq \tau_1$,

b. $4|c_1| \int_{\tau_1}^{t} r(s)ds \leq r(t)^{-(2-p)}$, $t - r(t) \leq \tau_1 \leq t$,

c. $8|a| \int_{t-r(t)}^{t} r(s)ds \leq r(t)^p$, $t \geq \sigma$.

Now we state Cooke's theorem.

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Theorem 1 *Under the above assumptions, every solution of (1) satisfies*

$$x(t) = \exp\Big(at - a^2 \int_\sigma^t r(\tau)d\tau\Big)(c + o(1)), \tag{2}$$

as $t \to +\infty$. Moreover, for each $c \in \mathbb{C}$, there exists a solution x of (1) satisfying (2).

The equation (1) studied by Cooke has several additional hypotheses. For example, not only $r \in L^p$ for $1 \le p \le 2$ and bounded, but also $\int_{t-r(t)}^t r(s)ds \le cr(t)^p$ (see Assumption iv.c above) which implies $\int_{t-r(t)}^t r(s)ds \in L^1$. Now, consider a continuous function $r(t)$ such that

$$r(t) - \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} k_n(t) \in L^1,$$

where

$$k_n(t) = \begin{cases} 1 & \text{for } n - (1/\sqrt[3]{n}) \le t \le n \\ 0 & \text{otherwise.} \end{cases}$$

Here, $r \in L^2$, but $\int_{t-r(t)}^t r(s)ds$ is not in L^1. We will also study (1) under this hypothesis. In general, we will study the asymptotic behavior of solutions to the delay-differential equation

$$x'(t) = \lambda(t)x(t - r(t)) \quad t \ge 0, \tag{3}$$

where $r, \lambda : [0, +\infty[\to \mathbb{C}$ are continuous functions such that $r(t) \in [0, R]$, $R > 0$. These hypotheses will be assumed for the rest of this paper. We will be also interested in the condition

(H) For some p in $[1, 2]$,

$$|\lambda(t)|\Big[\int_{t-r(t)}^t |\lambda(s)| \exp\Big(- \operatorname{Re} \int_{s-r(s)}^t \lambda(\xi)d\xi\Big)ds\Big] \in L^p(R, \mathbb{C}).$$

Remark 1. For equation (1), condition (H) takes the form

$$\int_{t-r(t)}^t \exp\big(- \operatorname{Re}(a)(t + r(s) - s)\big)ds,$$

which is satisfied when $r \in L^p$ and r is bounded. The requirement of condition (H) is weaker than the conditions of Theorem 1 (see Corollary 1 below).

Remark 2. For the equation

$$x'(t) = \lambda(t)x(t - r), \tag{4}$$

with r a positive constant, condition (H) takes the form

$$|\lambda(t)| \int_{t-r}^t |\lambda(s)| \exp\Big(- \operatorname{Re} \int_{s-r}^t \lambda(\xi)d\xi\Big)ds \in L^p.$$

This is satisfied when $\lambda \in L^q$, for some q in $[1, 4]$. Note that $p \ge \max\{1, q/2\}$ is required.

Remakr 3. Condition (H) is easily satisfied when $\lambda(t) \geq 0$ is a not decreasing function and $\lambda(t)^2 r(t) \in L^p$, for some $p : 1 \leq p \leq 2$. In fact, for $t \geq R$, let

$$H(t) = \lambda(t)\Big[\int_{t-r(t)}^{t} \lambda(s) e^{-\int_{s-r(s)}^{t} \lambda(\xi)d\xi} ds\Big].$$

Then

$$H(t) \leq \lambda(t) e^{-\int_R^t \lambda(\xi)d\xi}\Big[\int_{t-r(t)}^{t} \lambda(t) e^{\int_R^{s-r(s)} \lambda(\xi)d\xi} ds\Big] \leq \lambda(t)^2 r(t).$$

This paper is organized as follows: in Section 2 we give some preliminaries and asymptotic results for integral-differential equations. Our main results are given in Section 3. Some consequences and examples are given in Section 4. The asymptotic formulae of this paper are explicit and easy to apply. Other asymptotic studies can be found in [1, 2, 7, 10, 11, 12].

2 Preliminaries

In this section, we consider

$$z'(t) = c(t) \int_{t-r(t)}^{t} b(s)[z(t) - z(s - r(s))]ds + F(z)(t), \tag{5}$$

for $t \geq 3R$, where $b, c : [2R, +\infty[\to \mathbb{C}$ are continuous complex-valued functions, and $F : L^\infty([2R, +\infty[, \mathbb{C}) \to L^\infty([3R, +\infty[, \mathbb{C})$ is a linear function. We will assume in this paper that the following hypotheses are satisfied:

(H1) for some p in $[1, 2]$, $|c(t)| \int_{t-r(t)}^{t} |b(s)|ds \in L^p([3R, +\infty[, \mathbb{C})$

(H2) For $t \geq 3R$ and $h(t) \in L^1([3R, +\infty[, \mathbb{C})$,

$$|F(z)(t)| \leq h(t) \sup_{t-r(t) \leq s \leq t} |z(s - r(s))|.$$

Consider the homogeneous integral-differential equation

$$u'(t) = c(t) \int_{t-r(t)}^{t} b(s)[u(t) - u(s - r(s))]ds, \quad t \geq 3R. \tag{6}$$

Note that (5) is a perturbation of (6). Also note that given $\xi \in \mathbb{C}$, $u(t) = \xi$ is a solution of (6). Let $f(t) = |c(t)| \int_{t-r(t)}^{t} |b(s)|ds$ for $t \geq 3R$, and

$$\alpha(t) = 2f(t) \int_{t-R}^{t} f(\tau)d\tau, \tag{7}$$

for $t \geq 4R$. Since b and c satisfy (H1), $f \in L^p$ and $\int_{t-r(t)}^{t} f(\tau)d\tau \in L^{\frac{p}{p-1}}$ [6, pp. 17-21]. Thus, $\alpha \in L^1([4R, +\infty[, \mathbb{C})$.

Lemma 1 *Suppose that the continuous complex-valued functions b, c satisfy (H1). Then, all the solutions of (6) are convergent. Furthermore, they have the form*

$$u(t) = \xi + O\Big(\int_t^{+\infty} \alpha(\xi) d\xi\Big),$$

as $t \to +\infty$ for some $\xi \in \mathbb{C}$ where α is given by (7).

Proof. We observe that all the solutions of (6) satisfy

$$u'(t) = c(t) \int_{t-r(t)}^{t} b(s) \Big[\int_{s-r(s)}^{t} c(\tau) \Big[\int_{\tau-r(\tau)}^{\tau} b(\xi)[u(\tau) - u(\xi - r(\xi))] d\xi\Big] d\tau\Big] ds, \tag{8}$$

for $t \geq R$. Assume that $u : \mathbb{R} \to \mathbb{R}$ with $\|u\|_t := \sup_{s \leq t} |u(s)|$ is bounded. Then

$$|u'(t)| \leq 2f(t) \Big[\int_{t-r(t)}^{t} f(\tau) d\tau\Big] \|u\|_t, \tag{9}$$

where $t \geq 4R$. Integrating this equation $[4R, t]$ we have,

$$|u(t)| \leq |u(4R)| + \int_{4R}^{t} \alpha(\xi) \|u\|_\xi d\xi.$$

Hence,

$$|u(\tilde{t})| \leq |u(4R)| + \int_{4R}^{t} \alpha(\xi) \|u\|_\xi d\xi,$$

where $4R \leq \tilde{t} \leq t$. Thus,

$$\|u\|_t \leq |u(4R)| + \int_{4R}^{t} \alpha(\xi) \|u\|_\xi d\xi$$

and by Gronwall's inequality,

$$\|u\|_t \leq |u(4R)| \exp\Big(\int_{4R}^{t} \alpha(\xi) d\xi\Big). \tag{10}$$

Since $\alpha \in L^1$ and u is bounded, the right side of equality (9) belongs to L^1. Therefore u is convergent. $\square$

Example 1. Consider the homogeneous integral-differential equation

$$tu'(t) = \int_{t-1}^{t} \sin(s^2)[u(t) - u(s-1)] ds, \tag{11}$$

with $t \geq 1$. Clearly, given $\xi \in \mathbb{C}$, $u(t) = \xi$ is a solution of (11). Moreover, by the above lemma all the solutions of (11) are convergent and have the form

$$u(t) = \xi + O\Big(\int_t^{+\infty} \Big[\frac{1}{\tau} \int_{\tau-r(\tau)}^{\tau} |\sin(s^2)| ds\Big] \Big[\int_{\tau-r(\tau)}^{\tau} \frac{1}{\zeta} \int_{\zeta-r(\zeta)}^{\zeta} |\sin(s^2)| ds \, d\zeta\Big] d\tau\Big).$$

Lemma 2 *Assume that the complex-valued functions b, c, h, k satisfy (H1) and (H2). Then the solutions of (5) are convergent and satisfy*

$$z(t) = \xi + O\Big(\int_t^{+\infty} |h(\tau)| \int_{\tau - r(\tau)}^{\tau} |k(s)| ds d\tau \Big) \tag{12}$$

as $t \to +\infty$. Conversely, given $\xi \in \mathbb{C}$, there is a solution $z(t) = z_\xi(t)$ of (5), defined for $t \geq 3R$ such that it satisfies (12).

Proof: Let $u(t) = u(t; 3R, \varphi)$ be the unique solution of the homogeneous equation

$$u'(t) = c(t) \int_{t-r(t)}^{t} b(s)[u(t) - u(s - r(s))] ds,$$

for $t \geq 3R$ such that $u(t; 3R, \varphi) = \varphi(t)$ for $t \leq 3R$. Then, by a variation of constants formula (5), with initial condition $z(t) = \varphi(t)$ for $t \leq 3R$, is equivalent to

$$z(t) = u(t; 3R, \varphi) + \int_{3R}^{t} u(t; \tau, \chi_\tau) F(z)(\tau) d\tau, \tag{13}$$

where

$$\chi_\tau(t) = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{if } t \neq \tau. \end{cases}$$

for $t \geq 3R$. From (10), there is $c > 0$ such that $|u(t; 3R, \varphi)| \leq c|\varphi|_\infty$, for all $t \geq 3R$. Then

$$|z(t)| \leq c|\varphi|_\infty + c \int_{3R}^{t} |F(z)(\tau)| d\tau.$$

Let C be the set of the functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that $\varphi(t)$ is continuous for $t \geq 3R$ and $\varphi(t) = 0$ for $t < 3R$. Let $\beta(t) = ch(t)$. By (H2), $\beta \in L^1$. Let η be a positive number such that $\eta < 1$ and let $\theta : [3R, +\infty[\to \mathbb{R}$ be the function

$$\theta(t) = \sum_{n=1}^{+\infty} g_n(t), \quad \text{with} \quad g_n(t) = \begin{cases} 1 & \text{for } t \in [3Rn, 3Rn + \delta_n], \\ 0 & \text{otherwise} \end{cases}$$

and the δ_n are positive numbers such that for all $n \in \mathbb{N}$,

$$\exp\big(\int_{3R}^{3Rn+\delta_n} \beta(\tau) d\tau \big) - 1 < \big(1 - \frac{1}{2^n}\big)\eta \,. \tag{14}$$

Let

$$\|z\|_\eta = \sup_{t \geq 3R} \exp\big(- \int_{3R}^{t} \beta(\tau)\theta(\tau) d\tau \big) |z(t)|.$$

Since the supremum norm $|\cdot|_\infty$ and the norm $\|\cdot\|_\eta$ are equivalent, $C = (C, \|\cdot\|_\eta)$ is a Banach space. Let $\|z\|_t = \sup_{s \leq t} |z(s)|$. Since $|z(t)| \leq |\varphi|_\infty$ for $t \leq 2R$, for $c \geq 1$ and $\tilde{t} \leq t \leq 3R$, we obtain

$$|z(\tilde{t})| \leq c|\varphi|_\infty + \int_{3R}^{t} \beta(\tau) \|z\|_\tau d\tau \,.$$

By Gronwall's inequality,

$$\|z\|_t \le c|\varphi|_\infty \exp\Big(\int_{3R}^t \beta(\tau)d\tau\Big).$$

Since $\beta \in L^1$, $\|z\|_t$ is bounded. By (13) we obtain that all the solutions of (5) are convergent.

To prove the second part of this lemma we define the operator $\mathcal{A} : C \to C$ such that

$$(\mathcal{A}z)(t) = \xi - \int_t^{+\infty} u(t;\tau,\chi_\tau)\Big[h(\tau)\int_{\tau-r(\tau)}^{\tau} k(s)z(s-r(s))ds\Big]d\tau \quad \text{for } t \ge 3R,$$

and $(\mathcal{A}z)(t) = 0$, for $t < 3R$. By the variation of constants formula we have that any fixed point of this operator is a solution of (5) for $t \in [3R, +\infty[$. Note that:

i) $\mathcal{A}(C) \subseteq C$,

ii) $\mathcal{A}$ has a fixed point $z = z_\xi$ in C,

iii) $(\mathcal{A}z)(t) \to \xi$ as $t \to +\infty$.

By (14), for all $z_1, z_2 \in C$ and $t \ge 3R$, we have

$$\exp\big(-\int_{3R}^t \beta(\tau)\theta(\tau)d\tau\big)\|(\mathcal{A}z_1)(t) - (\mathcal{A}z_2)(t)\| \le \eta\|z_1 - z_2\|_\eta\,. \qquad (15)$$

Taking the supremum in the left-hand side of this equation,

$$\|\mathcal{A}z_1 - \mathcal{A}z_2\|_\eta \le \eta\|z_1 - z_2\|_\eta,$$

for $t \ge 3R$ and all $z_1, z_2 \in C$. Since $0 < \eta < 1$, we have that $\mathcal{A}$ has a fixed point $z_\xi \in C$. So, ii) is proved. Now, by taking $z_2 = 0$ in (15),

$$(\mathcal{A}z)(t) \to \xi \qquad (16)$$

for all $z \in C$ as $t \to +\infty$. So, i) and iii) are satisfied. Therefore, the second part of this lemma is proved. □

Example 2. Consider the homogeneous integral-differential equation

$$tz'(t) = \int_{t-1}^t (\sin(s^2)[z(t) - z(s-1)] + [e^{1/s} - 1]z(s-1))ds, \qquad (17)$$

where $t \ge 1$. Then, by the above lemma, all the solutions of (17) are convergent. Moreover, given $\xi \in \mathbb{C}$, there is a solution of $z = z(t)$ of (17) such that $z(t) \to \xi$ as $t \to +\infty$.

3 Main Result

Theorem 2 *Assume that the functions λ and r satisfy (H). Then for every solution x of (3), there exists $c \in \mathbb{C}$ such that*

$$x(t) = \exp\Big(\int_{3R}^{t}[\lambda(\tau) - \tilde{b}(\tau)]d\tau\Big)\Big(c + O(\int_{t}^{+\infty}\mu(\tau)d\tau)\Big), \tag{18}$$

where

$$\mu(t) = |\lambda(t)| \int_{t-r(t)}^{t} |\lambda(s)| \left| e^{-\int_{s-r(s)}^{t} \lambda(\xi)d\xi} \left(e^{-\int_{s-r(s)}^{t} \tilde{b}(\xi)d\xi} - 1\right) \right| ds$$

and $\tilde{b}(t) = \lambda(t)\int_{t-r(t)}^{t} \lambda(s) e^{-\int_{s-r(s)}^{t}\lambda(\xi)d\xi}ds$ defined for $t \geq 3R$ as $t \to +\infty$. Conversely, given $\xi \in \mathbb{C}$, there is $x(t) = x_\xi(t)$, solution of (3) defined for $t \geq 3R$ which satisfies (18) with $c = \xi$.

Proof. We write (3) as

$$x'(t) = \lambda(t)x(t) - \lambda(t)(x(t) - x(t-r(t))), \quad t \geq 0\,.$$

Iterating the above equation and applying the uniqueness theorem for functional differential equations, we obtain that (3) is equivalent to

$$x'(t) = \lambda(t)x(t) - \lambda(t)\int_{t-r(t)}^{t} \lambda(s)x(s-r(s))ds, \tag{19}$$

for $t \geq 3R$. In this equation we make the change of variables

$$x(t) = \exp\left(\int_{3R}^{t}[\lambda(s) - \tilde{b}(s)]\right) z(t)$$

and we obtain

$$z'(t) = \tilde{b}(t)z(t) - \lambda(t)\int_{t-r(t)}^{t} \lambda(s)e^{-\int_{s-r(s)}^{t}[\lambda(\xi)-\tilde{b}(\xi)]} z(s-r(s))ds$$

which is equivalent to

$$\begin{aligned} z'(t) \;&=\; \tilde{b}(t)z(t) - \lambda(t)\int_{t-r(t)}^{t} \lambda(s)e^{-\int_{s-r(s)}^{t}\lambda(\xi)d\xi} z(s-r(s))ds \\ &\quad -\lambda(t)\int_{t-r(t)}^{t} \lambda(s)e^{-\int_{s-r(s)}^{t}\lambda(\xi)d\xi}\left(e^{-\int_{s-r(s)}^{t}\tilde{b}(\xi)d\xi} - 1\right) z(s-r(s))ds. \end{aligned}$$

Then,

$$z'(t) = c(t)\int_{t-r(t)}^{t} b(s)[z(t) - z(s-r(s))]ds + F(z)(t), \tag{20}$$

where $c(t) = \lambda(t)e^{-\int_{3R}^t \lambda(\xi)d\xi}$, $b(t) = \lambda(t)e^{-\int_{t-r(t)}^{3R} \lambda(\xi)d\xi}$ and

$$F(z)(t) = -\lambda(t)\int_{t-r(t)}^{t} \lambda(s)e^{-\int_{s-r(s)}^t \lambda(\xi)d\xi}(e^{-\int_{s-r(s)}^t \tilde{b}(\xi)d\xi} - 1)z(s-r(s))ds,$$

for $t \geq 3R$. By (H), $\exp\big(-\int_{t-R}^t \tilde{b}(\xi)d\xi\big) - 1 \in L^{\frac{p}{p-1}}$. So,

$$|\lambda(t)|\int_{t-r(t)}^{t} |\lambda(s)|\,\Big|e^{-\int_{s-r(s)}^t \lambda(\xi)d\xi}\Big(e^{-\int_{s-r(s)}^t \tilde{b}(\xi)d\xi} - 1\Big)\Big|\,ds \in L^1.$$

Then (20) satisfies the same hypotheses as (5) in Lemma 2. Thus, we obtain the conclusion of this theorem. □

Example 3. Consider equation (3) with $\lambda(t) = \Gamma(t)$ and $r(t) = \frac{1}{\Gamma(t)}$ for $t \geq 0$, where

$$\Gamma(t) = \int_0^{+\infty} s^{t-1}e^{-s}ds$$

is the Euler's Gamma Function. Then (3) takes the form

$$x'(t) = \Gamma(t)x(t - \frac{1}{\Gamma(t)}), \tag{21}$$

for $t \geq 0$. By the above theorem, it is not difficult to check that all the solutions of (21) are of the form

$$x(t) = (c + o(1))\exp\Big(\sum_{k=1}^{n-1}[k! - k!(k-1)!e^{-(k-2)(k-2)!}]\Big),$$

for $n \leq t < n+1$ as $n \to +\infty$.

4 Consequences

Corollary 1 *Assume that r is bounded and $r \in L^p$ for some $p : 1 \leq p \leq 2$. Then for each solution x of (1), there exists $c \in \mathbb{C}$ such that*

$$x(t) = \exp\Big(at - a^2\int_{3R}^t r(\tau)d\tau - a^3\int_{3R}^t \Big[\int_{\tau-r(\tau)}^t r(s)ds\Big]d\tau\Big)(c + o(1)). \tag{22}$$

Conversely, given $\xi \in \mathbb{C}$ there is a solution $x(t) = x_\xi(t)$ of (1), defined for $t \geq 3R$, satisfying (22) with $c = \xi$.

Proof. If we take $\lambda(t) = a$ in Theorem 2, we obtain

$$|\lambda(t)|\int_{t-r(t)}^t |\lambda(s)|e^{-Re\int_{s-r(s)}^t \lambda}ds = |a|^2\int_{t-r(t)}^t e^{-Re(a)(t-s+r(s))}ds \leq c_1 r(t),$$

for some positive constant c_1. Therefore, (1) satisfies (H). By Theorem 1, we prove this corollary by taking

$$\begin{aligned} b(t) &= \lambda(t)\int_{t-r(t)}^{t}\lambda(s)e^{-\int_{s-r(s)}^{t}\lambda(\xi)d\xi}ds \\ &= a^2\int_{t-r(t)}^{t}e^{-a(t-s+r(s))}ds \\ &= a^2r(t)-a^3\int_{t-r(t)}^{t}r(\tau)d\tau - I_1(t), \end{aligned}$$

where

$$I_1(t)=a^2\int_{t-r(t)}^{t}[e^{-a(t-s+r(s))}-1-r(s)]ds.$$

Since r is bounded,

$$\begin{aligned} |I_1(t)| &\le |a|^2\int_{t-r(t)}^{t}((e^{Re(a)(s-t)}-1)e^{-Re(a)r(s)}+[e^{-Re(a)r(s)}-1-r(s)])ds \\ &\le c_1\int_{t-r(t)}^{t}(t-s)ds+c_3\int_{t-r(t)}^{t}r(s)^2ds \\ &\le c_2\frac{r(t)^2}{2}+c_3\int_{t-r(t)}^{t}r(s)^2ds \end{aligned}$$

for some positive constants c_2 and c_3. Since $r\in L^p$ with $1\le p\le 2$ and r is bounded, we have that $r\in L^2$. By the last inequality, $I_1\in L^1$ which completes the present proof. $\square$

Next, we obtain a version of Theorem 1 using weaker hypotheses.

Corollary 2 *Suppose that r is bounded and belongs to L^p with $1\le p\le 2$, and that $\int_{t-r(t)}^{t} r\in L^1$. Then the solutions of (1) satisfy (2). Conversely, given $\xi\in\mathbb{C}$ there is a solution $x(t)=x_\xi(t)$ of (1), defined for $t\ge 3R$ and satisfying (2) with $c=\xi$.*

The study of the L^p perturbations has been exhaustively studied; see for example by Haddock and Sacker [8]. In the next corollary we give an extension of their conjecture for $2\le p\le 4$.

Corollary 3 *Suppose that in (4), $\lambda\in L^p$ with $1\le p\le 4$ and $r>0$. Then, given a solution x of equation (3), there exists $c\in\mathbb{C}$ such that*

$$x(t)=\exp\Big(\int_{3R}^{t}[\lambda(\tau)-\lambda(\tau)\int_{\tau-r}^{\tau}\lambda(s)e^{-\int_{s-r}^{\tau}\lambda(\xi)d\xi}ds]d\tau\Big)\Big(c+o(1)|big), \quad (23)$$

as $t\to+\infty$. Conversely, given $c\in\mathbb{C}$ there is a solution x of (4) such that (23) is satisfied.

Corollary 4 *Suppose that λ and r satisfy (H) with $p = 1$. Then for every solution x of (3) there exists $c \in \mathbb{C}$ such that*

$$x(t) = \exp\Big(\int_{3R}^{t} \lambda(s)ds\Big)\big(c + o(1)\big), \tag{24}$$

as $t \to +\infty$. Conversely, given $c \in \mathbb{C}$ there exists a solution x of (3) such that satisfies (24).

Example 4. Consider the equation

$$x'(t) = \frac{t}{t^2-1}x(t-\frac{1}{t}), \quad t \geq 2. \tag{25}$$

Then (25) satisfies (H) with $p = 1$. We observe that one of the solutions of (25) is $x(t) = t$. By Corollary 1, all the solutions x of (25) have the asymptotic formula

$$x(t) = t(c + o(1))$$

as $t \to +\infty$, where $c \in \mathbb{R}$, i.e, all the solutions of (25) are like $x(t) = t$.

Example 5. Consider the equation

$$x'(t) = t^{\alpha}x(t - t^{-(2\alpha+1)}), \quad t \geq 2,\ \alpha > 1. \tag{26}$$

By Corollary 4, all the solutions x of (26) have the asymptotic formula

$$x(t) = e^{t^{\alpha}/(\alpha+1)}(c + o(1))$$

as $t \to +\infty$, where $c \in \mathbb{C}$.

Corollary 5 *Suppose that in (3), $\lambda(t) \geq 0$ is a not decreasing function and $\lambda(t)^2 r(t) \in L^p$ with $1 \leq p \leq 2$. Then for each solution x of (3) there exists $c \in \mathbb{C}$ such that*

$$x(t) = \exp\Big(\int_{3R}^{t}[\lambda(\tau) - \lambda(\tau)\int_{\tau-r(\tau)}^{\tau} \lambda(s)e^{-\int_{s-r(s)}^{\tau}\lambda(\xi)d\xi}ds]d\tau\Big)(c + o(1)), \tag{27}$$

as $t \to +\infty$. Conversely, for each $c \in \mathbb{C}$ there exists a solution x of (3) such that (27) is satisfied.

Example 6. Consider the equation

$$x'(t) = e^t x(t - \frac{1}{te^{2t}}), \quad t \geq 1. \tag{28}$$

By Corollary 5, all of the solutions x of this equation have the asymptotic formula

$$x(t) = \exp\Big(\exp\big(t - e^{t-e^t}\big)\int_{t-\frac{1}{te^{2t}}}^{t} \exp\big(s + e^{s-\frac{1}{se^{2s}}}\big)ds\Big)\Big(c + o(1)\Big),$$

where $c \in \mathbb{R}$ as $t \to +\infty$.

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