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Orders of solutions of an n-th order linear differential equation with entire coefficients *

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Abstract

We study the solutions of the differential equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where the coefficients are entire functions. We find conditions on the coefficients so that every solution that is not identically zero has infinite order.

1 Introduction

For $n \geq 2$, we consider the linear differential equation

$$f^{(n)} + A_{n-1}(z)f^{(n-1)} + \cdots + A_1(z)f' + A_0(z)f = 0\,, \tag{1.1}$$

where $A_0(z), \dots, A_{n-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. Let $\rho(f)$ denote the order of the growth of an entire function f as defined in [4]:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\big(\log\big(\max_{|z|=r}|f(z)|\big)\big)}{\log r}\,.$$

The value $T(r,f) = \log(\max_{|z|=r}|f(z)|)$ is known as the Nevanlinna characteristic of f [4]. It is well known that all solutions of (1.1) are entire functions and when some of the coefficients of (1.1) are transcendental, (1.1) has at least one solution with order $\rho(f) = \infty$. The question which arises is:

> What conditions on $A_0(z), \dots, A_{n-1}(z)$ will guarantee that every solution $f \not\equiv 0$ has infinite order?

In this paper we prove two results concerning this question.

When $A_0(z), \dots, A_{n-1}(z)$ are polynomials with $A_0(z) \not\equiv 0$, every solution of (1.1) is an entire function with finite rational order; see for example [3], [5, pp. 199-209], [6, pp. 106-108], and [7, pp. 65-67].

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In the study of the differential equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.2}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions, Gundersen proved the following results.

Theorem 1.1 ([1, p. 418]) *Let $A(z)$ and $B(z) \not\equiv 0$ be entire functions such that for real constants α, β, θ_1, θ_2 with $\alpha > 0$, $\beta > 0$, and $\theta_1 < \theta_2$, we have*

$$|B(z)| \geq \exp\{(1 + o(1))\alpha|z|^{\beta}\} \tag{1.3}$$

and

$$|A(z)| \leq \exp\{o(1)|z|^{\beta}\} \tag{1.4}$$

as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $f \not\equiv 0$ of (1.2) has infinite order.

Theorem 1.2 ([1, p. 419]) *Let $\{\Phi_k\}$ and $\{\theta_k\}$ be two finite collections of real numbers satisfying $\Phi_1 < \theta_1 < \Phi_2 < \theta_2 < \cdots < \Phi_n < \theta_n < \Phi_{n+1}$, where $\Phi_{n+1} = \Phi_1 + 2\pi$, and set*

$$\mu = \max_{1 \leq k \leq n} (\Phi_{k+1} - \theta_k). \tag{1.5}$$

Suppose that $A(z)$ and $B(z)$ are entire functions such that for some constant $\alpha \geq 0$,

$$|A(z)| = O(|z|^{\alpha}) \tag{1.6}$$

as $z \to \infty$ with $\Phi_k \leq \arg z \leq \theta_k$ for $k = 1, \ldots, n$ and where $B(z)$ is transcendental with $\rho(B) < \frac{\pi}{\mu}$. Then every solution $f \not\equiv 0$ of (1.2) has infinite order.

2 Statement and proof of results

In this paper we prove the following two theorems:

Theorem 2.1 *Let $A_0(z), \ldots, A_{n-1}(z)$, $A_0(z) \not\equiv 0$ be entire functions such that for real constants α, β, μ, θ_1, θ_2, where $0 \leq \beta < \alpha$, $\mu > 0$ and $\theta_1 < \theta_2$ we have*

$$|A_0(z)| \geq e^{\alpha|z|^{\mu}} \tag{2.1}$$

and

$$|A_k(z)| \leq e^{\beta|z|^{\mu}}, \quad k = 1, \ldots, n-1 \tag{2.2}$$

as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $f \not\equiv 0$ of (1.1) has infinite order.

Theorem 2.2 *Let* $\{\Phi_k\}$ *and* $\{\theta_k\}$ *be two finite collections of real numbers satisfying* $\Phi_1 < \theta_1 < \Phi_2 < \theta_2 < \cdots < \Phi_m < \theta_m < \Phi_{m+1}$ *where* $\Phi_{m+1} = \Phi_1 + 2\pi$, *and set*

$$\mu = \max_{1 \le k \le m} (\Phi_{k+1} - \theta_k). \tag{2.3}$$

Suppose that $A_0(z), \dots, A_{n-1}(z)$ *are entire functions such that for some constant* $\alpha \ge 0$,

$$|A_j(z)| = O(|z|^\alpha), \quad j = 1, \dots, n-1 \tag{2.4}$$

as $z \to \infty$ *with* $\Phi_k \le \arg z \le \theta_k$ *for* $k = 1, \dots, m$ *and where* $A_0(z)$ *is transcendental with* $\rho(A_0) < \pi/\mu$. *Then every solution* $f \not\equiv 0$ *of (1.1) has infinite order.*

Next, we provide a lemma that is used in the proofs of our theorems.

Lemma 2.3 ([2, p. 89]) *Let* w *be a transcendental entire function of finite order* ρ. *Let* $\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers satisfying* $k_i > j_i \ge 0$ *for* $i = 1, \dots, m$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi_0 \in [0, 2\pi) - E$, *then there is a constant* $R_0 = R_0(\psi_0) > 0$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \ge R_0$ *and for all* $(k, j) \in \Gamma$, *we have*

$$\Big|\frac{w^{(k)}(z)}{w^{(j)}(z)}\Big| \le |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Proof of Theorem 2.1

Suppose that $f \not\equiv 0$ is a solution of (1.1) with $\rho(f) < \infty$. Set $\delta = \rho(f)$. Then from Lemma 1, there exists a real constant ψ_0 where $\theta_1 \le \psi_0 \le \theta_2$, such that

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| = o(1)|z|^{k\,\delta}, \quad k = 1, \dots, n \tag{2.5}$$

as $z \to \infty$ with $\arg z = \psi_0$. Then from (2.5) and (1.1), we obtain that

$$|A_0(z)| \le o(1)|z|^\delta |A_1(z)| + \cdots + o(1)|z|^{(n-1)\,\delta}|A_{n-1}(z)| + o(1)|z|^{n\,\delta} \tag{2.6}$$

as $z \to \infty$ with $\arg z = \psi_0$. However this contradicts (2.1) and (2.2). Therefore, every solution $f \not\equiv 0$ of (1.1) has infinite order.

Next we give an example that illustrates Theorem 2.1.

Example 1. Consider the differential equation

$$f'' - (3 + 6e^z)f'' + (2 + 6e^z + 11e^{2z})f' - 6e^{3z}f = 0 \tag{2.7}$$

In this equation, for $z = re^{i\theta}$, $r \to +\infty$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{4}$ we have

$$\begin{gathered}
|A_0(z)| = \ |-6\,e^{3z}| = 6e^{3r\cos\theta} > e^{3\frac{\sqrt{2}}{2}r}, \\
|A_1(z)| = 2 + 6e^z + 11e^{2z}| \le 19e^{2r\cos\theta} \le 19e^{\sqrt{3}\,r} < e^{2r} \\
|A_2(z)| = |-(3 + 6e^z)| \le 9e^{r\cos\theta} \le 9e^{\frac{\sqrt{3}}{2}r} < e^{2r}.
\end{gathered}$$

As we see, conditions (2.1) and (2.2) of Theorem 2.1 are verified. The three linearly independent functions $f_1(z) = e^{e^z}$, $f_2(z) = e^{2e^z}$, $f_3(z) = e^{3e^z}$ are solutions of (2.7) with $\rho(f_1) = \rho(f_2) = \rho(f_3) = \infty$.

Next we give a generalization of Example 1.

Example 2. Consider the differential equation

$$f^{(n)} + P_{n-1}(e^z)f^{(n-1)} + \cdots + P_1(e^z)f' + \beta e^{\alpha z} f = 0\,, \tag{2.8}$$

where $\alpha \in \mathbb{R}$, $\alpha > 0$, $\beta \in \mathbb{C}$, $|\beta| \geq 1$, and $P_1, \ldots, P_{n-1}$ are polynomials. If we take the sector $\theta_1 \leq \arg z \leq \theta_2$, θ_1, $\theta_2 \in]0, \frac{\pi}{2}[$ with θ_1 near enough to θ_2 such that $\max_{1\leq k\leq n-1} \deg(P_k) < \alpha \frac{\cos\theta_2}{\cos\theta_1}$, then conditions (2.1) and (2.2) of Theorem 2.1 are satisfied as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. From Theorem 2.1, it follows that every solution $f \not\equiv 0$ of (2.8) has infinite order.

Proof of Theorem 2.2

Suppose that $f \not\equiv 0$ is a solution of (1.1) where $\rho(f) < \infty$ and we set $\beta = \rho(f)$. From Lemma 1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [\Phi_k, \theta_k) - E$ for some k, then

$$|\frac{f^{(l)}(z)}{f(z)}| = O(|z|^{l\beta}), \quad l = 1, \ldots, n \tag{2.9}$$

as $z \to \infty$ with $\arg z = \psi_0$. From (2.9) ,(2.4) and (1.1), we obtain that

$$|A_0(z)| \leq |\frac{f^{(n)}}{f}| + |A_{n-1}(z)||\frac{f^{(n-1)}}{f}| + \cdots + |A_1(z)||\frac{f'}{f}| = O(|z|^\sigma) \tag{2.10}$$

as $z \to \infty$ with $\arg z = \psi_0$, where $\sigma = \alpha + n\beta$. Let $\varepsilon > 0$ be a small constant that satisfies $\rho(A_0) < \frac{\pi}{\mu+2\varepsilon}$ (this is possible since $\rho(A_0) < \frac{\pi}{\mu}$). By using the Phragmén-Lindelöf theorem on (2.10), it can be deduced that for some integer $s > 0$

$$|A_0(z)| = O(|z|^s) \tag{2.11}$$

as $z \to \infty$ with $\Phi_k + \varepsilon \leq \arg z \leq \theta_k - \varepsilon$ for $k = 1, \ldots, m$.

Now for each k, we have from (2.3) that $\Phi_{k+1} + \varepsilon - (\theta_k - \varepsilon) \leq \mu + 2\varepsilon$, and so $\rho(A_0) < \frac{\pi}{\Phi_{k+1}-\theta_k+2\varepsilon}$. Hence using the Phragmén-Lindelöf theorem on (2.11) we can deduce that $|A_0(z)| = O(|z|^s)$ as $z \to \infty$ in the whole complex plane. This means that $A_0(z)$ is a polynomial which contradicts our hypothesis and completes the proof of Theorem 2.2.

Next we give an example that illustrates Theorem 2.2.

Example 3. If $A_0(z)$ is transcendental with $\rho(A_0) < 2$, then from Theorem 2.2, every solution $f \not\equiv 0$ of the equation

$$f^{(n)} + P_{n-1}(z)f^{(n-1)} + \cdots + P_2(z)f'' + (e^{z^3} + e^{i\,z^3})f' + A_0(z)f = 0\,,$$

where $P_{n-1}, \ldots, P_2$ are polynomials, is of infinite order.

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