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Stability properties of positive solutions to partial differential equations with delay *

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Abstract

We investigate the stability of positive stationary solutions of semilinear initial-boundary value problems with delay and convex or concave nonlinearity. If the nonlinearity is monotone, then in the convex case $f(0) \leq 0$ implies instability and in the concave case $f(0) \geq 0$ implies stability. Special cases are shown where the monotonicity assumption can be weakened or omitted.

1 Introduction

In this paper, we study the stability of positive stationary solution of the semilinear partial differential equation with delay

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(u_t(x)), \quad t > 0, \ x \in \Omega \tag{1.1}$$

with the Dirichlet boundary condition

$$u(t,x) = 0, \quad t \geq 0, \ x \in \partial\Omega \tag{1.2}$$

and the initial condition

$$u(\theta,x) = \phi(\theta,x), \quad (\theta,x) \in [-r,0] \times \Omega, \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, Δ denotes the Laplacian, $f : C([-r,0],\mathbb{R}) \to \mathbb{R}$ is a convex or concave C^2 function and $u_t(x) \in C([-r,0],\mathbb{R})$ is defined as $u_t(x)(\theta) := u(t+\theta,x)$ for $\theta \in [-r,0]$.

This problem without delay was studied by several authors. We present here a generalization of their results. Shivaji and his co-authors have altogether

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proved that if $F'' > 0$ and $F(0) \leq 0$, then every nontrivial nonnegative stationary solution of the non-delay problem

$$\partial_t u = \Delta u + F(u) \tag{1.4}$$
$$u|_{\partial\Omega} = 0 \tag{1.5}$$

is unstable. They first considered the monotone case (i.e. $F' > 0$) in [1]. The non-monotone case was first proved by Tertikas [15] using sub- and supersolutions. This proof was simplified by Maya and Shivaji [9], reducing the problem to the monotone one via the decomposition of f to a monotone and a linear function.

In [6] a more direct proof was given. Moreover, that proof implied at the same time stability in the concave counterpart ($F'' < 0$ and $F(0) \geq 0$) and the result was proved also in the case of a general elliptic operator and nonlinearity with explicit dependence on x. The main result of [6] is summarized in the following theorem.

Theorem 1.1 *(i) If $F'' > 0$ and $F(0) \leq 0$, then every nontrivial nonnegative stationary solution of (1.4)-(1.5) is unstable.*

(ii) If $F'' < 0$ and $F(0) \geq 0$, then every nontrivial nonnegative stationary solution of (1.4)-(1.5) is stable.

In some sense the sign condition for $F(0)$ is also necessary for the above stability property. Namely, in the case of opposite sign and for balls $\Omega = B_R$ there are typically two stationary solutions of (1.4)-(1.5), and the small one has the opposite stability as in the theorem. This change of stability correlates with the loss of uniqueness. Namely, under the conditions of Theorem 1.1 there is uniqueness whenever the number of stationary solutions is known. This is proved for the concave case on an arbitrary domain, see [2, 3, 4] and the references there; for the convex case generally in one dimension [5, 8] and for special nonlinearities also in several dimensions [7, 10, 12, 13, 14, 16].

In this paper the above Theorem is generalized to problem (1.1)-(1.3). The main results are formulated in Theorems 2, 3 and 4. In Section 2 we consider general delays where we will have an extra technical assumption, namely that f is monotone in the sense that its Fréchet derivative $Df \geq 0$ or $Df \leq 0$. For discrete delays, in Section 3, this assumption will be weakened, in particular in the special case of no delay the monotonicity assumption is not needed.

To formulate our results let us introduce the embedding $\bar{\cdot} : \mathbb{R} \to C([-r, 0], \mathbb{R})$ as

$$\overline{a}(\theta) = a \quad \text{for all } \theta \in [-r, 0], \quad a \in \mathbb{R}\,.$$

The C^2 function $U : \Omega \to \mathbb{R}$ is called a positive stationary solution of (1.1)-(1.3) if

$$\Delta U(x) + f(\overline{U(x)}) = 0, \quad x \in \Omega, \tag{1.6}$$

$$U(x) = 0, \quad x \in \partial\Omega, \qquad U(x) > 0, \quad x \in \Omega, \tag{1.7}$$

The linearisation of (1.1)-(1.3) around U reads

$$\frac{\partial w(t,x)}{\partial t} = \Delta w(t,x) + Df(\overline{U(x)}) \cdot [w_t(x)], \quad t > 0, \ x \in \Omega, \tag{1.8}$$

where $Df(\overline{U(x)})$ is the Fréchet derivative of f calculated at $\overline{U(x)}$. The principle of linearized stability says that the stability of the zero solution determines the local stability of the stationary solution U. By putting $w(t,x) = e^{\lambda t}v(x)$ into (1.8) we arrive at the characteristic equation

$$\Delta v(x) + Df(\overline{U(x)}) \cdot [e^{\lambda \cdot} v(x)] = \lambda v(x), \quad x \in \Omega, \tag{1.9}$$

where $e^{\lambda \cdot} v(x) \in C([-r,0], \mathbb{R})$ is defined as $e^{\lambda \cdot} v(x)(\theta) := e^{\lambda\theta} v(x)$ for $\theta \in [-r,0]$.

Denote the dominant characteristic root of (1.9) by Λ, i.e. there is a function V satisfying the Dirichlet boundary condition such that

$$\Delta V(x) + Df(\overline{U(x)}) \cdot [e^{\Lambda \cdot} V(x)] = \Lambda V(x), \quad x \in \Omega, \tag{1.10}$$

and for all other solutions (λ, v) of (1.9) we have $\mathrm{Re}(\lambda) \le \mathrm{Re}(\Lambda)$. It is known, see [17], that the zero solution of (1.8) is stable, resp. unstable, if $\mathrm{Re}(\Lambda) < 0$, resp. $\mathrm{Re}(\Lambda) > 0$. This fact together with the principle of the linearized stability yields that U is locally stable, resp. unstable, if $\mathrm{Re}(\Lambda) < 0$, resp. $\mathrm{Re}(\Lambda) > 0$.

For the rest of this article, we assume

(H) $\Lambda \in \mathbb{R}$ and $V(x) > 0, \ x \in \Omega$.

Note that positive semigroups generated by the linearized equation (1.8) satisfy (H), see [11].

2 Case of a general delay

In this section we assume that f is monotone in the following sense. The function f will be called increasing (decreasing) if $Df(\bar{a}) \cdot [\phi] \ge 0$ ($Df(\bar{a}) \cdot [\phi] \le 0$) for all $a \ge 0$ and $\phi \in C([-r,0], \mathbb{R})$, $\phi \ge 0$. We will call a C^2 function $h : \mathbb{R} \to \mathbb{R}$ strictly convex (concave) if $h'' \ge 0$ ($h'' \le 0$) and h'' is not identically zero in any open interval.

We note that for decreasing functions the positive stationary solution is always stable. This can be easily seen by multiplying (1.10) with V and integrating on Ω. Namely, the l.h.s. is negative, therefore $\Lambda < 0$, i.e. U is stable. Further in this Section we consider the case, when f is increasing. We have the following theorem.

Theorem 2.1 *Let us assume (H) and that f is increasing.*

(i) If $f(\bar{0}) \le 0$ and $a \mapsto f(\bar{a})$ is strictly convex, then every positive stationary solution of (1.1)-(1.3) is unstable.

(ii) If $f(\bar{0}) \ge 0$ and $a \mapsto f(\bar{a})$ is strictly concave, then every positive stationary solution of (1.1)-(1.3) is stable.

Proof. Set $g(a) := Df(\bar{a}) \cdot [\bar{a}] - f(\bar{a})$ and $h(a) = f(\bar{a})$ for $a \geq 0$. Then it is easy to see that $g(a) = ah'(a) - h(a)$ and in case (i) $g(0) \geq 0$ and $g'(a) > 0$, hence $g(a) > 0$ for all $a > 0$. In case (ii) $g(0) \leq 0$ and $g'(a) < 0$, hence $g(a) < 0$ for all $a > 0$.

Multiply (1.6) by $V(x)$ and (1.10) by $U(x)$. After subtraction and integration over Ω we arrive at

$$\int_\Omega U(x)\Delta V(x) - V(x)\Delta U(x) + U(x)Df(\overline{U(x)}) \cdot [e^{\Lambda \cdot} V(x)] - V(x)f(\bar{U}(x))dx = \Lambda \int_\Omega V(x)U(x)dx.$$

By the symmetric Green formula $\int_\Omega U(x)\Delta V(x) - V(x)\Delta U(x)dx = 0$ and we have that

$$\int_\Omega V(x)g(U(x)) + Df(\overline{U(x)}) \cdot [e^{\Lambda \cdot} - \bar{1}]U(x)V(x)dx = \Lambda \int_\Omega U(x)V(x)dx. \quad (2.1)$$

In case (i) we show that $\Lambda > 0$, hence U is unstable. Let us assume the contrary, $\Lambda \leq 0$. Then the right-hand side of (2.1) is non-positive, but the left-hand side is positive since $g(U(x)) > 0$ and $e^{\Lambda \cdot} - \bar{1} \geq 0$. In case (ii) the inequality $\Lambda < 0$ is to be shown. Let us assume $\Lambda \geq 0$. Then the right-hand side of (2.1) is nonnegative, but the left-hand side is negative since $g(U(x)) < 0$ and $e^{\Lambda \cdot} - \bar{1} \leq 0$.

3 Case of discrete delays

Let $F : \mathbb{R}^{k+1} \to \mathbb{R}$ be a C^2 function, $r_1, r_2, \dots r_k > 0$ and $r = \max\{r_1, \dots, r_k\}$. In this section we study the case when

$$f(\phi) = F(\phi(-r_1), \phi(-r_2), \dots \phi(-r_k), \phi(0)) \quad \text{for } \phi \in C([-r, 0], \mathbb{R}). \quad (3.1)$$

Here we also use the notation $\bar{a} = (a, a, \dots, a) \in \mathbb{R}^{k+1}$ for $a \in \mathbb{R}$. Now we have the following stability result.

Theorem 3.1 *Let f be given by (3.1) and let us assume (H) and that $\partial_l F(\bar{a}) \geq 0$ for $l = 1, 2, \dots, k$, $a \geq 0$.*

(i) If $F(\bar{0}) \leq 0$ and $a \mapsto F(\bar{a})$ is strictly convex, then every positive stationary solution of (1.1)-(1.3) is unstable.

(ii) If $F(\bar{0}) \geq 0$ and $a \mapsto F(\bar{a})$ is strictly concave, then every positive stationary solution of (1.1)-(1.3) is stable.

Proof. We can follow the same method as in the proof of Theorem 2.1. Now,

$$Df(\bar{a}) \cdot [e^{\Lambda\cdot}] = \partial_1 F(\bar{a})e^{-\Lambda r_1} + \ldots + \partial_k F(\bar{a})e^{-\Lambda r_k} + \partial_{k+1}F(\bar{a}) \ .$$

and (2.1) reads

$$\begin{aligned}\int_\Omega V(x)\Big[g(U(x)) + U(x)(\partial_1 F(\overline{U(x)})(e^{-\Lambda r_1} - 1) + \ldots \\ +\partial_k F(\overline{U(x)})(e^{-\Lambda r_k} - 1))\Big]dx \quad &= \quad \Lambda \int_\Omega U(x)V(x)dx.\end{aligned}$$

Then the proof can be completed similarly as that of Theorem 2.1.

In the case $k = 1$, $\partial_2 F \equiv 0$ the sign condition for the partial derivative is not needed. This is the content of the following theorem.

Theorem 3.2 *Let $k = 1$, f be given by (3.1) and let us assume (H) and that $\partial_2 F \equiv 0$.*

(I) If $F(\bar{0}) \leq 0$ and $a \mapsto F(\bar{a})$ is strictly convex, then every positive stationary solution of (1.1)-(1.3) is unstable.

(II) If $F(\bar{0}) \geq 0$ and $a \mapsto F(\bar{a})$ is strictly concave, then every positive stationary solution of (1.1)-(1.3) is stable.

Proof. Note that in this case $F(x, y) \equiv F(x)$, $f(\phi) = F(\phi(-r))$, the characteristic equation has the form

$$\Delta V(x) + F'(U(x))V(x)e^{-\Lambda r} = \Lambda V(x), \tag{3.2}$$

while $g(a) = aF'(a) - F(a)$. First we prove (I). There are two cases.

Case 1. There exist a unique $\alpha > 0$ such that $F(\alpha) = 0$.

Case 2. $F(\beta) > 0$ for all $\beta > 0$.

Let us define $\Omega' \subset \Omega$ by $\Omega' := \{x \in \Omega \ : \ U(x) > \alpha\}$ in case 1, and $\Omega' := \Omega$ in case 2. It is easy to see that $\Omega' \neq \emptyset$ and that the following facts hold:

(i) $F(U(x)) = 0, \quad x \in \partial\Omega'$,

(ii) $F(U(x)) > 0, \quad x \in \Omega'$,

(iii) $\langle \operatorname{grad} U, \nu\rangle < 0, \quad x \in \partial\Omega'$, where ν is the outer normal vector and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$,

(iv) Either $F'(\alpha) > 0$ (case 1) or $F'(0) \geq 0$ (case 2).

Multiply (1.6) by $F'(U(x))V(x)$ and (3.2) by $F(U(x))$. After subtraction and integration over Ω' we get

$$\begin{aligned}
&\int_{\Omega'} \Delta V(x)F(U(x)) + F(U(x))F'(U(x))e^{-\Lambda r}V(x) \\
&-\Delta U(x)F'(U(x))V(x) - F(U(x))F'(U(x))V(x)dx \\
&\quad = \quad \Lambda \int_{\Omega'} V(x)F(U(x))dx.
\end{aligned} \tag{3.3}$$

By using the Green formula and (i) we have that

$$\begin{aligned}
\int_{\Omega'} \Delta V(x)F(U(x))dx &= \int_{\partial\Omega'} \langle \operatorname{grad} V(x), \nu(x)\rangle F(U(x))d\sigma \\
&\quad - \int_{\Omega'} \langle \operatorname{grad} V(x), \operatorname{grad} F(U(x))\rangle dx \\
&= - \int_{\Omega'} \langle \operatorname{grad} V(x), \operatorname{grad} F(U(x))\rangle dx.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int_{\Omega'} \Delta U(x)F'(U(x))V(x) &= \int_{\partial\Omega'} \langle \operatorname{grad} U(x), \nu(x)\rangle F'(U(x))V(x)d\sigma \\
&\quad - \int_{\Omega'} \langle \operatorname{grad} U(x), \operatorname{grad}(F'(U(x))V(x))\rangle dx.
\end{aligned}$$

Set

$$A := - \int_{\partial\Omega'} \langle \operatorname{grad} U(x), \nu(x)\rangle F'(U(x))V(x)d\sigma.$$

By using (iii) and (iv) we have that $A \geq 0$. Thus (3.3) is simplified to

$$\begin{aligned}
&\int_{\Omega'} -\langle \operatorname{grad} V(x), \operatorname{grad} F(U(x))\rangle + \langle \operatorname{grad} U(x), \operatorname{grad}(F'(U(x))V(x))\rangle \\
&+F(U(x))F'(U(x))(e^{-\Lambda r}V(x) - V(x))dx + A \\
&\quad = \quad \Lambda \int_{\Omega'} V(x)F(U(x))dx,
\end{aligned} \tag{3.4}$$

or

$$\begin{aligned}
&\int_{\Omega'} F(U(x))V(x)F'(U(x))(e^{-\Lambda r} - 1) \\
&+\sum_{i=1}^{n} (\partial_i U(x))^2 V(x)F''(U(x))dx + A \quad = \quad \Lambda \int_{\Omega'} V(x)F(U(x))dx.
\end{aligned} \tag{3.5}$$

Now suppose that $\Lambda \leq 0$. Because of (ii) the right-hand side of (3.5) is non-positive in this case. However, since $e^{-\Lambda r} - 1 \geq 0$ the left-hand side is strictly positive, which is a contradiction. Thus we deduce that $\Lambda > 0$ and the proof of (i) is complete.

Let us turn to the proof of claim (II). Multiply (1.6) by $V(x)$ and (3.2) by $U(x)$. After subtraction and integration over Ω and using the symmetric Green formula we have that

$$\int_\Omega V(x)(U(x)F'(U(x))e^{-\Lambda r} - F(U(x))dx = \Lambda \int_\Omega U(x)V(x)dx$$

or

$$\int_\Omega V(x)e^{-\Lambda r}g(U(x)) + (e^{-\Lambda r} - 1)V(x)F(U(x))dx = \Lambda \int_\Omega U(x)V(x)dx. \quad (3.6)$$

First we claim that $F(U(x)) \geq 0$ for all $x \in \Omega$. Indeed, $F(\beta) < 0$ for all $\beta > 0$ is impossible since U is a positive stationary solution. On the other hand, let us denote the first (and unique) positive root of F by α (if it exists, otherwise our claim is trivial). It is enough to show that $U(x) \leq \alpha$. Denote the maximum point of U by x_0. Then $\Delta U(x_0) \leq 0$. If there exists a point $x \in \Omega$ such that $U(x) > \alpha$ then $U(x_0) > 0$ and thus $F(U(x_0)) < 0$. Finally $0 = \Delta U(x_0) + F(U(x_0)) < 0$ is a contradiction which proves our claim.

Now suppose that $\Lambda \geq 0$. Then the right-hand side of (3.6) is nonnegative. However, since $e^{-\Lambda r} - 1 \leq 0$ the left-hand side of (3.6) is strictly negative which is a contradiction. Thus we deduce that $\Lambda < 0$ and the proof is complete.

Remarks The case of decreasing nonlinearity corresponds to the assumption $\partial_l F(\bar{a}) \leq 0$ for $l = 1, 2, \ldots, k+1$, $a \geq 0$. Thus under these conditions the positive stationary state is always stable. The non-delay case is a special case of Theorem 3.1, namely $\partial_l F \equiv 0$ for $l = 1, 2, \ldots, k$.

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