Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 66, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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Existence of solutions to a superlinear p-Laplacian equation *

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Abstract

Using Morse theory, we establish the existence of solutions to the equation $-\Delta_p u = f(x,u)$ with Dirichlet boundary conditions. We assume that $\int_0^s f(x,t)\,dt$ lies between the first two eigenvalues of the p-Laplacian.

1 Introduction

Consider the Dirichlet problem for the p-Laplacian ($p > 1$),

$$\begin{aligned} -\Delta_p u &= f(x,u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $-\Delta_p u$ is the p-Laplacian: $-\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth; that is,

F1) The inequality $|f(x,u)| \le C(1 + |u|^{q-1})$ holds for all $u \in \mathbb{R}$, $x \in \Omega$, and for some positive constant C, where $1 \le q < \frac{Np}{N-p}$ if $N \ge p+1$, and $1 \le q < \infty$ if $1 \le N < p$.

It is well known that weak solutions $u \in W_0^{1,p}(\Omega)$ of (1.1) are the critical points of the C^1 functional

$$\Phi(u) = \frac{1}{p}\int |\nabla u|^p\,dx - \int F(x,u)\,dx\,,$$

where $F(x,s) = \displaystyle\int_0^s f(x,t)\,dt$.

Let λ_1 and λ_2 be the first and the second eigenvalues of $-\Delta_p$ on $W_0^{1p}(\Omega)$. It is known that $\lambda_1 > 0$ is a simple eigenvalue, and that $\sigma(-\Delta_p) \cap (\lambda_1, \lambda_2) = \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$, (cf. [2]).

We shall assume the following conditions:

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* *Mathematics Subject Classifications:* 49J35, 35J65, 35B34.
Key words: p-Laplacian, critical group.

Submitted August 21, 2001. Published October 11, 2001.

F2) There exist $r > 0$, $\bar{\lambda} \in (\lambda_1, \lambda_2)$ such that $|u| \le r$ implies

$$\lambda_1 |u|^p \le pF(x,u) \le \bar{\lambda}|u|^p,$$

F3) There exist $\theta > p$, $M > 0$ such that $|u| \ge M$ implies

$$0 < \theta F(x,u) \le uf(x,u).$$

Now, we are ready to state our main result.

Theorem 1.1 *Assume (F1), (F2), and (F3). Then (1.1) has a nontrivial weak solution in $W_0^{1,p}(\Omega)$.*

There are many papers devoted to the existence of solutions of (1.1); see for example [1, 4, 5]. In these papers, the main tool is the minmax argument. However, it seems difficult to use the minmax argument in our situation. Thus we will use a different approach: Morse theory [3]. To the best of our knowledge, [7] is the only work using Morse theory to obtain the solvability of p-Laplacian equations. Our work is motivated by [7].

2 Proof of main theorem

In this section we give the proof of Theorem 1.1. Let E denote the Sobolev space $W_0^{1,p}(\Omega)$, and $\|.\|$ denote the norm in E. For Φ a continuously Fréchet differentiable map from E to $\mathbb{R}$, let $\Phi'(u)$ denote its Fréchet derivative.

As stated in Section 1, weak solutions $u \in W_0^{1,p}(\Omega)$ of (1.1) are the critical points of the C^1 functional

$$\Phi(u) = \frac{1}{p}\int |\nabla u|^p \, dx - \int F(x,u)\, dx\,.$$

We will try to find a nontrivial critical point of the functional Φ. First we state the following lemmas.

Lemma 2.1 *Under conditions (F1) and (F3), the functional Φ satisfies the Palais-Smale condition.*

Proof Assume $(u_n) \subset E$, $|\Phi(u_n)| \le B$ for some $B \in \mathbb{R}$, and $\Phi'(u_n) \to 0$. Let $d := \sup_n \Phi(u_n)$. Then by (F3) we have

$$\begin{aligned}
\theta d + \|u_n\| &\ge \theta\Phi(u_n) + \langle \Phi'(u_n), u_n\rangle \\
&= (\frac{\theta}{p} - 1)\|u_n\|^p - \int_{|u_n|\ge M} [\theta F(x,u_n) - f(x,u_n)u_n] \\
&\quad - \int_{|u_n|\le M} [\theta F(x,u_n) - f(x,u_n)u_n] \\
&\ge (\frac{\theta}{p} - 1)\|u_n\|^p - \int_{|u_n|\le M} [\theta F(x,u_n) - f(x,u_n)u_n] \\
&\ge (\frac{\theta}{p} - 1)\|u_n\|^p - D, \quad \text{for some } D \in \mathbb{R}.
\end{aligned}$$

Thus (u_n) is bounded in E. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in E. Now because of condition (F1), a standard argument shows that $u_n \to u$ in E and the proof is complete. $\diamondsuit$

Let $V = \operatorname{span} \phi_1$ be the one-dimensional eigenspace associated to λ_1, where $\phi_1 > 0$ in Ω and $\|\phi_1\| = 1$. Taking a subspace $W \subset E$ complementing V, that is $E = V \oplus W$. Obviously the genus of $W \backslash 0$ satisfies $\gamma(W \backslash 0) \geq 2$. Therefore, by the variational characterization of λ_2, for $\forall u \in W$,

$$\int |\nabla u|^p \geq \lambda_2 \int |u|^p.$$

Lemma 2.2 *Under Assumption (F2), the functional Φ has a local linking at the origin with respect to $E = V \oplus W$. That is, there exists $\rho > 0$, such that*

$$\begin{aligned} \Phi(u) \leq 0, &\quad u \in V,\ \|u\| \leq \rho, \\ \Phi(u) > 0, &\quad u \in W,\ 0 < \|u\| \leq \rho. \end{aligned}$$

The proof of this lemma can be found in [7, Lemma 3.3].

For a C^1-functional $\Phi : E \to \mathbb{R}$ and u an isolate critical point of Φ, $\Phi(u) = c$, we define the critical group of Φ at u as

$$C_q(\Phi, u) := H_q(\Phi_c, \Phi_c \backslash \{u\}).$$

Where $H_q(X, Y)$ is the q-th homology group of the topological pair (X, Y) over the ring $\mathbb{Z}$.

Since $\dim V = 1 < +\infty$, from Lemma 2.2 and Theorem 2.1 in [6], we have

Lemma 2.3 *Under assumption (F2), 0 is a critical point of Φ and $C_1(\Phi, 0) \neq 0$.*

To find a nontrivial critical point of Φ, we investigate the behavior of Φ near infinity.

Lemma 2.4 *Under Assumption (F3), there exists a constant $A > 0$ such that*

$$\Phi_a \simeq S^\infty, \quad \textit{for } a < -A,$$

where S^∞ is the unit sphere in E.

Proof Integrating on the inequality of (F2), we obtain a constant $C_1 > 0$ such that

$$F(x, t) \geq C_1 |t|^\theta, \qquad \text{for } |t| \geq M.$$

Thus, for $u \in S^\infty$, we have $\Phi(tu) \to -\infty$, as $t \to +\infty$. Set

$$A := \left(1 + \frac{1}{p}\right) M|\Omega| \max_{\bar{\Omega} \times [-M, M]} |f(x, u)| + 1\,.$$

Using (F3) we obtain

$$\begin{aligned}
&\int F(x,v) - \frac{1}{p}\int vf(x,v)\\
= \;& \int_{|v|\ge M} F(x,v) + \int_{|v|\le M} F(x,v) - \frac{1}{p}\int_{|v|\ge M} vf(x,v) - \frac{1}{p}\int_{|v|\le M} vf(x,v)\\
\le \;& \Big(\frac{1}{\theta}-\frac{1}{p}\Big)\int_{|v|\ge M} vf(x,v) + \int_{|v|\le M} F(x,v) - \frac{1}{p}\int_{|v|\le M} vf(x,v)\\
\le \;& \Big(\frac{1}{\theta}-\frac{1}{p}\Big)\int_{|v|\ge M} vf(x,v) + \Big(1+\frac{1}{p}\Big) M\,|\Omega| \max_{\bar\Omega\times[-M,M]} |f(x,u)|\\
\le \;& \Big(\frac{1}{\theta}-\frac{1}{p}\Big)\int_{|v|\ge M} vf(x,v) + A - 1.
\end{aligned}$$

For $a < -A$ and

$$\Phi(tu) = \frac{|t|^p}{p} - \int F(x,tu) \le a, \quad (u\in S^\infty),$$

we have

$$\begin{aligned}
\frac{d}{dt}\Phi(tu) &= \langle \Phi'(tu), u\rangle = |t|^{p-2}t - \int uf(x,tu)\\
&\le \frac{p}{t}\Big\{\int F(x,tu) - \frac{1}{p}\int tuf(x,tu) + a\Big\}\\
&\le \frac{p}{t}\Big\{(\frac{1}{\theta}-\frac{1}{p})\int_{|tu|\ge M} tuf(x,tu) + A - 1 + a\Big\}\\
&\le \frac{p}{t}\Big\{(\frac{1}{\theta}-\frac{1}{p})\int_{|tu|\ge M} tuf(x,tu) - 1\Big\}\\
&\le \frac{p}{t}\Big\{(\frac{1}{\theta}-\frac{1}{p})C_1\theta\int_{|tu|\ge M} |tu|^\theta - 1\Big\} < 0.
\end{aligned}$$

By the Implicit Function Theorem, there is a unique $T\in C(S^\infty,\mathbb{R})$ such that

$$\Phi(T(u)u) = a, \quad \forall u\in S^\infty.$$

For $u\ne 0$, set $\tilde T(u) = \frac{1}{\|u\|}T(\frac{u}{\|u\|})$. Then $\tilde T\in C(E\backslash 0,\mathbb{R})$ and for all $u\in E\backslash 0$, $\Phi(\tilde T(u)u) = a$. Moreover, if $\Phi(u)=a$, then $\tilde T(u) = 1$.

We define a function $\hat T : E\backslash 0\to\mathbb{R}$ as

$$\hat T(u) := \begin{cases} \tilde T(u), & \text{if } \Phi(u)\ge a,\\ 1, & \text{if } \Phi(u)\le a.\end{cases}$$

Since $\Phi(u) = a$ implies $\tilde T(u) = 1$, we conclude that $\hat T\in C(E\backslash\{0\},\mathbb{R})$.

Finally we set $\eta : [0,1]\times(E\backslash 0)\to E\backslash 0$ as

$$\eta(s,u) = (1-s)u + s\hat T(u)u.$$

It is easy to see that η is a strong deformation retract from $E\backslash 0$ to Φ_a. Thus $\Phi_a \simeq E\backslash 0 \simeq S^\infty$ and present proof is complete. $\diamondsuit$

We also use the following topological result,which was proved by Perera [8].

Lemma 2.5 *Let $Y \subset B \subset A \subset X$ be topological spaces and $q \in \mathbb{Z}$. If*

$$H_q(A,B) \neq 0 \quad \textit{and} \quad H_q(X,Y) = 0$$

then

$$H_{q+1}(X,A) \neq 0 \quad \textit{or} \quad H_{q-1}(B,Y) \neq 0\,.$$

Now we can prove the main theorem.

Proof of Theorem 1.1 By Lemma 2.1, Φ satisfies the Palais-Smale condition. Note that $\Phi(0) = 0$, from [3] Chapter I, Theorem 4.2, there is a $\varepsilon > 0$, such that

$$H_1(\Phi_\varepsilon, \Phi_{-\varepsilon}) = C_1(\Phi, 0) \neq 0.$$

By Lemma 2.4, for $a < -A$ (A is as in the lemma) we have $\Phi_a \simeq S^\infty$. Since $\dim E = +\infty$,

$$H_1(E, \Phi_a) = H_1(E, S^\infty) = 0.$$

So that Lemma 2.5 yields

$$H_2(E, \Phi_\varepsilon) \neq 0 \quad \text{or} \quad H_0(\Phi_{-\varepsilon}, \Phi_a) \neq 0.$$

It follows that Φ has a critical point u for which

$$\Phi(u) > \varepsilon \quad \text{or} \quad -\varepsilon > \Phi(u) > a\,.$$

Therefore, u is a nonzero critical point of Φ, and (1.1) has a nontrivial solution.

Remark Result similar to Lemma 2.4 has been proved (for $p = 2$) in [9] and [3], under the additional conditions

$$f \in C^1(\Omega \times \mathbb{R}, \mathbb{R}), \quad f(x,0) = \frac{\partial f(x,t)}{\partial t}\Big|_{t=0} = 0\,.$$

From these two references, we have obtained the motivation for this paper.

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