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Nontrivial periodic solutions of asymptotically linear Hamiltonian systems *

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Abstract

We study the existence of periodic solutions for some asymptotically linear Hamiltonian systems. By using the Saddle Point Theorem and Conley index theory, we obtain new results under asymptotically linear conditions.

1 Introduction

We consider the Hamiltonian system

$$\dot{z} = JH'(t,z) \tag{1.1}$$

where $H \in C^2([0,1] \times \mathbb{R}^{2N}, \mathbb{R})$ is a 1-periodic function in t, and $H'(t,z)$ denotes the gradient of $H(t,z)$ with respect to the z variable. Here N is a positive integer and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard $2N \times 2N$ symplectic matrix. We denote by (x,y) and $|x|$ the usual inner product and norm in $\mathbb{R}^{2N}$ respectively. The function H satisfies the following conditions.

(H1) There exist $s \in (1,\infty)$ and $a_1, a_2 > 0$ such that

$$|H''(t,z)| \le a_1|z|^s + a_2, \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}.$$

(H2) $H'(t,z) = B_\infty(t)z + o(|z|)$ as $|z| \to \infty$ uniformly in t;

(H3) $H'(t,z) = B_0(t)z + o(|z|)$ as $|z| \to 0$ uniformly in t where $B_0(t)$ and $B_\infty(t)$ are $2N \times 2N$ symmetric matrices, continuous and 1-periodic in t.

The system (1.1) is called asymptotically linear because of (H2). Obviously, (H3) implies that 0 is a "trivial" solution of (1.1). We are interested in nontrivial 1-periodic solutions of (1.1).

The existence of periodic solutions of (1.1) has been studied by many authors. If $B_\infty(t)$ is non-degenerate, i.e. 1 is not a Floquet multiplier of the linear system

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$\dot{y} = JB_\infty(t)y$, one can see the results in [1, 2, 6, 7, 12, 13, 14]. If $B_\infty(t)$ is degenerate, some resonance conditions are needed to control the behavior of

$$G_\infty(t,z) = H(t,z) - \frac{1}{2}(B_\infty(t)z, z).$$

When $|G'_\infty(t,z)|$ is bounded, under the Landesman-Lazer type condition or strong resonance condition, (1.1) was studied by [3, 20] for the case that $B_\infty(t)$ is constant and by [4, 8] for the case that $B_\infty(t)$ is continuous and 1-periodic in t. When $|G'_\infty(t,z)|$ is not bounded, [9, 18, 19] studied the case that $B_\infty(t)$ is "finitely degenerate" [9].

In this paper we shall study the case that $|G'_\infty(t,z)|$ is not bounded and $B_\infty(t)$ is continuous and 1-periodic in t. We assume the following conditions for $G_\infty(t,z)$.

(H4$^\pm$) There exist $c_1, c_2 > 0$ such that

$$\pm[2G_\infty(t,z) - (G'_\infty(t,z), z)] \geq c_1|z| - c_2, \quad \forall (t,z) \in [0,1] \times \mathbb{R}^{2N};$$
$$G_\infty(t,z) \to \pm\infty \quad as \quad |z| \to +\infty.$$

(H5$^\pm$) There exist $1 \leq \alpha < 2$, $0 < \beta < \alpha/2$, and $M_1, M_2, L > 0$ such that

$$|G'_\infty(t,z)| \leq M_1|z|^\beta, \quad \pm G_\infty(t,z) \geq M_2|z|^\alpha, \quad \forall |z| \geq L.$$

(H6$^\pm$) There exist $1 \leq \alpha < 2$, $0 < \beta < \alpha/2$, and $M_1, M_2, L > 0$ such that

$$|G'_\infty(t,z)| \leq M_1|z|^\beta, \quad \pm(G'_\infty(t,z), z) \geq M_2|z|^\alpha, \quad \forall |z| \geq L.$$

According to [6, 13, 14], for a given continuous 1-periodic and symmetric matrix function $B(t)$, one can assign a pair of integers $(i,n) \in \mathbb{Z} \times \{0, \cdots, 2N\}$ to it, which is called the Maslov-type index of $B(t)$. We denote by (i_0, n_0) and (i_∞, n_∞) the Maslov-type indices of $B_0(t)$ and $B_\infty(t)$ respectively. Our main result reads as follows.

Theorem 1.1 *Suppose that H satisfies $(H1)-(H3)$. Then (1.1) possesses a nontrivial 1-periodic solution if one of the following cases occurs:*

(i) (H4$^+$) *and* $i_\infty + n_\infty \notin [i_0, i_0 + n_0]$.

(ii) (H4$^-$) *and* $i_\infty \notin [i_0, i_0 + n_0]$.

(iii) (H5$^+$) *and* $i_\infty + n_\infty \notin [i_0, i_0 + n_0]$.

(iv) (H5$^-$) *and* $i_\infty \notin [i_0, i_0 + n_0]$.

(v) (H6$^+$) *and* $i_\infty + n_\infty \notin [i_0, i_0 + n_0]$.

(vi) (H6$^-$) *and* $i_\infty \notin [i_0, i_0 + n_0]$.

Remark 1.2 (1) If

$$H(t,z) = \frac{7|z|^2}{2\ln(e+|z|^2)}, \tag{1.2}$$

by Theorem 1.1(i) the system (1.1) possesses a nontrivial 1-periodic solution. If

$$H(t,z) = \frac{1}{2}|z|^2 - \frac{|z|^2}{\ln(e+|z|^2)}, \tag{1.3}$$

by Theorem 1.1(ii) the system (1.1) possesses a nontrivial 1-periodic solution. These examples can not be solved by earlier results, for example those contained in references. More examples are given in Section 3.
(2) Our result should be compared with those in [9, 18, 19]. First, we do not require that $B_\infty(t)$ be constant or "finitely degenerate". Secondly, the conditions (H6$^\pm$) with $\beta = \alpha - 1$ are required in [9, 18, 19]. This means that the results in [9, 18, 19] can not be applied to some cases such as (1.2), (1.3), or $G_\infty(t,z) \sim |z|^\alpha \ln(1+|z|^2)$ at infinity. But these cases are covered by Theorem 1.1 . Notice that $\beta = \alpha - 1 < \alpha/2$. Therefore Theorem 1.1(v)&(vi) generalizes [9, Theorem 1.1], [18, Theorem 1.2], and [19, Theorem 1.2].
(3) The condition $(H5^\pm)$ is rather close to a condition in the paper [21] by Szulkin and Zou. The author thanks the referee for pointing out this.

The proof of our results is given in Section 2. By using the the Galerkin approximation method [8, 12], Saddle point theorem [5, 15, 16], and Morse index estimates [10, 11, 17], we shall prove Theorem 1.1(i)-(iv). For Theorem 1.1(v)-(vi), we follow the idea in [9] and use Conley index theory [6] to get our conclusions.

2 Periodic solutions of Hamiltonian systems

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, $E = W^{1/2,2}(S^1, \mathbb{R}^{2N})$. Recall that E is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and E consists of those $z(t)$ in $L^2(S^1,\mathbb{R}^{2N})$ whose Fourier series

$$z(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(2\pi nt) + b_n\sin(2\pi nt))$$

satisfies

$$\|z\|^2 = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2) < \infty,$$

where $a_j, b_j \in \mathbb{R}^{2N}$. For a given continuous 1-periodic and symmetric matrix function $B(t)$, we define two selfadjoint operators $A, B \in \mathcal{L}(E)$ by extending the bilinear forms

$$\langle Ax, y\rangle = \int_0^1 (-J\dot{x}, y)\,dt, \quad \langle Bx, y\rangle = \int_0^1 (B(t)x, y)\,dt \tag{2.1}$$

on E. Then B is compact [13]. We define

$$f(z) = \frac{1}{2}\langle Az, z\rangle - \int_0^1 H(t, z)\,dt \tag{2.2}$$

on E. It is well know that $f \in C^2(E, \mathbb{R})$ whenever H satisfies (H1). Looking for the solutions of (1.1) is equivalent to looking for the critical points of f [3, 7].

For $B(t)$, by [6, 13, 14] we can define its Maslov-type index as a pair of integers $(i(B), n(B)) \in \mathbb{Z} \times \{0, \cdots, 2N\}$. Using the Floquet theory, we have

$$n(B) = \dim\ker(A - B). \tag{2.3}$$

Let $B_\infty(t)$ be the matrix function in (H2) with the Maslov-type index (i_∞, n_∞), and B_∞ be the operator, defined by (2.1), corresponding to $B_\infty(t)$. Then by (2.3) we have

$$n_\infty = \dim\ker(A - B_\infty).$$

Let $\cdots \le \lambda_2' \le \lambda_1' < 0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $A - B_\infty$, and Let $\{e_j'\}$ and $\{e_j\}$ be the eigenvectors of $A - B_\infty$ corresponding to $\{\lambda_j'\}$ and $\{\lambda_j\}$ respectively. For $m \ge 0$, set

$$E_0 = \ker(A - B_\infty),$$

$$E_m = E_0 \oplus \operatorname{span}\{e_1, \cdots, e_m\} \oplus \operatorname{span}\{e_1', \cdots, e_m'\}$$

and let P_m be the orthogonal projection from E to E_m. Then $\{P_m\}$ is an approximation scheme with respect to the operator $A - B_\infty$, i.e.

$$\begin{gathered} (A - B_\infty)P_m = P_m(A - B_\infty), \\ P_m x \to x \quad as \quad m \to \infty, \quad \forall x \in E. \end{gathered}$$

In the following we denote $T^\# = (T_{ImT})^{-1}$, and we also denote by $M^+(\cdot)$, $M^-(\cdot)$ and $M^0(\cdot)$ the positive definite, negative definite and null subspaces of the selfadjoint linear operator defining it, respectively. The following result was proved in [8]

Theorem 2.1 ([8]) *For any continuous 1-periodic and symmetric matrix function $B(t)$ with the Maslov-type index (i_0, n_0), there exists an $m^* > 0$ such that for $m \ge m^*$ we have*

$$\begin{gathered} \dim M_d^+(P_m(A - B)P_m) = m + i_\infty - i_0 + n_\infty - n_0 \\ \dim M_d^-(P_m(A - B)P_m) = m - i_\infty + i_0 \\ \dim M_d^0(P_m(A - B)P_m) = n_0 \end{gathered} \tag{2.7}$$

where $d = \frac{1}{4}\|(A - B)^\#\|^{-1}$, $M_d^+(\cdot)$, $M_d^-(\cdot)$ and $M_d^0(\cdot)$ denote the eigenspaces corresponding to the eigenvalue λ belonging to $[d, +\infty), (-\infty, -d]$ and $(-d, d)$ respectively.

To prove Theorem 1.1 we need the following definition and saddle point theorem which were given in [10].

Definition 2.2 ([10]) Let E be a C^2-Riemannian manifold, D be a closed subset of E. A family $\mathcal{F}(\alpha)$ is said to be a homological family of dimension q with boundary D if, for some nontrivial class $\alpha \in H_q(E, D)$, the family $\mathcal{F}(\alpha)$ is defined by

$$\mathcal{F}(\alpha) = \{G \subset E : \alpha \quad \text{is in the image of } i_* : H_q(G, D) \to H_q(E, D)\},$$

where i_* is the homomorphism induced by the immersion $i : G \to E$.

Theorem 2.3 ([10]) *As in Definition 2.2, for given E, D and α, let $\mathcal{F}(\alpha)$ be a homological family of dimension q with boundary D. Suppose that $f \in C^2(E, R)$ satisfies (PS) condition. Set*

$$c \equiv c(f, \mathcal{F}(\alpha)) = \inf_{G\in\mathcal{F}(\alpha)} \sup_{w\in G} f(w)$$

If $\sup_{w\in D} f(w) < c$ and f' is Fredholm on

$$\mathcal{K}_c = \{x \in E : f'(x) = 0, f(x) = c\},$$

then there exists $x \in \mathcal{K}_c$ such that the Morse indices $m^-(x)$ and $m^0(x)$ of the functional f at x satisfy

$$q - m^0(x) \le m^-(x) \le q.$$

Let f be defined as (2.2) and f_m be the restriction of f to the space E_m. We say that f satisfies the $(PS)^*_c$ condition for $c \in \mathbb{R}$, if any sequence $\{x_m\}$ such that $x_m \in E_m$, $f'_m(x_m) \to 0$ and $f_m(x_m) \to c$ possesses a subsequence convergent in E [12].

Lemma 2.4 *Under the conditions of Theorem 1.1, f satisfies the $(PS)^*_c$ condition for any $c \in \mathbb{R}$.*

Proof. For any given $c \in \mathbb{R}$, let $\{z_m\}$ be the $(PS)^*_c$ sequence, i.e., for $z_m \in E_m$,

$$f'(z_m) \to 0, \quad f_m(z_m) \to c. \tag{2.8}$$

We want to show that $\{z_m\}$ is bounded in E. Then by standard arguments [12], $\{z_m\}$ possesses a subsequence convergent in E.

Suppose $\{z_m\}$ is not bounded and $\|z_m\| \to +\infty$ as $m \to +\infty$. Define

$$g(z) = \int_0^1 G_\infty(t, z)dt, \quad \forall z \in E.$$

Then $f(z) = \frac{1}{2}\langle(A - B_\infty)z, z\rangle - g(z)$, for all $z \in E$. By (H2) we know that

$$\frac{|G'_\infty(t, z)|}{|z|} \to 0 \quad \text{as } |z| \to \infty.$$

This means that, for any $\varepsilon > 0$, there exist $M > 0$ such that

$$|G'_\infty(t,z)|^2 \le \varepsilon |z|^2 + M.$$

Therefore,

$$\begin{aligned}|\langle g'(z_m), y\rangle| &= |\int_0^1 (G'_\infty(t,z_m), y)dt| \\ &\le \int_0^1 |G'_\infty(t,z_m)||y|dt \le (\int_0^1 |G'_\infty(t,z_m)|^2)^{1/2}\|y\|_{L^2} \\ &\le (\varepsilon\|z_m\|_{L^2}^2 + M)^{1/2}\|y\|_{L^2} \le (\varepsilon\|z_m\|^2+M)^{1/2}\|y\|.\end{aligned}$$

This implies

$$\lim_{m\to\infty} \frac{\|g'(z_m)\|}{\|z_m\|} \le \varepsilon, \text{ for any } \varepsilon > 0,$$

i.e.

$$\frac{\|g'(z_m)\|}{\|z_m\|} \to 0 \quad \text{as} \quad m \to +\infty. \tag{2.9}$$

Write

$$\begin{aligned} z_m &= z_m^+ + z_m^- + z_m^0 \\ &\in M^+(P_m(A-B_\infty)P_m) \oplus M^-(P_m(A-B_\infty)P_m) \oplus M_m^0(A-B_\infty)P_m).\end{aligned}$$

Then

$$\begin{aligned}\langle f'_m(z_m), z_m^+\rangle &= \frac{1}{2}\langle (A-B_\infty)z_m^+, z_m^+\rangle - \langle g'(z_m), z_m^+\rangle \\ &\ge C_1\|z_m^+\|^2 - \|g'(z_m)\|\|z_m^+\|.\end{aligned}$$

By (2.8) and (2.9), we have

$$\frac{\|z_m^+\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \tag{2.10}$$

Similarly, we have

$$\frac{\|z_m^-\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \tag{2.11}$$

Case(i): ($H4^+$) holds.

$$\begin{aligned}\langle f'_m(z_m), z_m\rangle - 2f_m(z_m) &= \int_0^1 [2G_\infty(t,z_m) - (G'_\infty(t,z_m), z_m)]dt \\ &\ge C_1\int_0^1 |z_m|dt - C_2 \\ &\ge C_1 \int_0^1 |z_m^0|dt - \int_0^1 C_1(|z_m^+| + |z_m^-|)dt - C_2 \\ &\ge C_3\|z_m^0\| - C_4(\|z_m^+\| + \|z_m^-\| + 1).\end{aligned}$$

Here we used the fact that $M^0(P_m(A-B_\infty)P_m) = \ker(A-B_\infty)$ is finite dimensional. By (2.8), (2.10) and (2.11), we have

$$\frac{\|z_m^0\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \tag{2.12}$$

But this implies the following contradiction,

$$1 = \frac{\|z_m\|}{\|z_m\|} \le \frac{\|z_m^0\| + \|z_m^-\| + \|z_m^+\|}{\|z_m\|} \to 0 \quad \text{as } m \to +\infty. \tag{2.13}$$

Therefore $\{z_m\}$ must be bounded, and f satisfies $(PS)_c^*$ condition under $(H4^+)$.

Case (ii): (H4$^-$) holds. Similar to case (i), we have

$$\begin{aligned} 2f_m(z_m) - \langle f_m'(z_m), z_m\rangle &= \int_0^1 [G_\infty'(t,z_m), z_m) - 2G_\infty(t,z_m)]dt \\ &\ge C_1\int_0^1 |z_m| - C_2 \ge C_3\|z_m^0\| - C_4\|z_m^+\| + \|z_m^-\| + 1). \end{aligned}$$

This implies (2.12) and (2.13). Thus $\{z_m\}$ must be bounded, and f satisfies $(PS)_c^*$ condition under (H4$^-$).

Notice that we assume $\|z_m\| \to +\infty$ as $m \to +\infty$. Then by (2.10) and (2.11), there exist $m_0 > 0$ such that for $m \ge m_0$

$$\|z_m^0\| \ge \|z_m^+ + z_m^-\|. \tag{2.14}$$

Moreover, if $|G_\infty'(t,z_m)| \le M_1|z|^\beta$ for $|z| \ge L$, we will show that for m large enough

$$\|z_m^+ + z_m^-\| \le \varepsilon_0\|z_m^0\|^\beta, \tag{2.15}$$

where $\varepsilon_0 > 0$ is a constant independent of m. In fact, we have

$$|G_\infty'(t,z)|^2 \le M_1^2|z|^{2\beta} + M_2;$$

$$\begin{aligned} |\langle g'(z_m), y\rangle| &\le \int_0^1 |G_\infty'(t,z_m)||y|dt \le (\int_0^1 |G_\infty'(t,z_m)|^2dt)^{1/2}\|y\|_{L^2} \\ &\le (M_1^2\|z_m\|_{L^{2\beta}}^{2\beta} + M_2)^{1/2}\|y\|_{L^2} \le (M_1^2\|z_m\|^{2\beta} + M_2)^{1/2}\|y\|. \end{aligned}$$

This implies that for m large enough

$$\frac{\|g'(z_m)\|}{\|(z_m)\|^\beta} \le M_3. \tag{2.16}$$

By (2.8), (2.14) and (2.16), for m large enough, we have

$$\begin{aligned} 0 \leftarrow \|f_m'(z_m)\| &= \|\langle A - B_\infty)z_m - P_m g'(z_m)\| \\ &\ge \varepsilon_1\|z_m^+ + z_m^-\| - M_3\|z_m\|^\beta \\ &\ge \varepsilon_1\|z_m^+ + z_m^-\| - M_3 2^\beta\|z_m^0\|^\beta. \end{aligned}$$

This implies that, for m large enough, (2.15) holds.

Case (iii): (H5$^+$) holds. By (2.2) and (2.15), for m large enough,

$$\begin{aligned} g(z_m) &= \frac{1}{2}\langle (A-B_\infty)(z_m^+ + z_m^-), z_m^+ + z_m^-\rangle - f(z_m) \\ &\le C_1\|z_m^+ + z_m^-\|^2 + C_0 \le C_2\|z_m^0\|^{2\beta} + C_0. \end{aligned} \tag{2.17}$$

On the other hand, by (H5$^+$),

$$g(z_m) = \int_0^1 G_\infty(t,z_m)dt \ge \int_0^1 M_2|z_m|^\alpha dt - M_3 \ge M_4\|z_m^0\|^\alpha - M_3. \tag{2.18}$$

Notice that $\alpha > 2\beta$, we get a contradiction from (2.17) and (2.18). Therefore $\{z_m\}$ is bounded, and f satisfies $(PS)_c^*$ condition under $(H5^+)$. Here in (2.18) we used the following claim.

Claim: For m large enough, there exists $\varepsilon_2 > 0$ such that

$$\int_0^1 |z_m|^\alpha dt \ge \varepsilon_2\|z_m^0\|^\alpha. \tag{2.19}$$

In fact, for $\alpha > 1$, by (2.15) and the fact $\beta < 1$, we have

$$\begin{aligned} \int_0^1 (z_m, z_m^0)dt &\le \Big(\int_0^1 |z_m|^\alpha dt\Big)^{1/\alpha}\Big(\int_0^1 |z_m^0|^{\frac{\alpha}{\alpha-1}}dt\Big)^{\frac{\alpha-1}{\alpha}} \\ &\le C_\alpha\Big(\int_0^1 |z_m|^\alpha dt\Big)^{1/\alpha}\|z_m^0\|; \end{aligned}$$

$$\begin{aligned} \int_0^1 (z_m, z_m^0)dt &= \int_0^1 (z_m^0, z_m^0)dt + \int_0^1 (z_m^+ + z_m^-, z_m^0)dt \\ &\ge \int_0^1 (z_m^0)^2 dt - \varepsilon_3\|z_m^+ + z_m^-\|\|z_m^0\| \\ &\ge \varepsilon_4\|z_m^0\|^2 - \varepsilon_5\|z_m^0\|^{1+\beta} \ge \varepsilon_6\|z_m^0\|^2, \end{aligned}$$

for m large enough. This implies (2.19) for $\alpha > 1$.

For $\alpha = 1$, since $z_m^0 \in \ker(A - B_\infty)$, we know that z_m^0 satisfies the linear system

$$\dot z = JB_\infty(t)z.$$

This implies that $z_m^0(t) \ne 0, \quad \forall t \in [0,1]$. Therefore

$$c_1\|z_m^0\| \le |z_m^0(t)| \le c_2\|z_m^0\|, \quad \forall t \in [0,1],$$

where $c_1, c_2 > 0$ are constants independent of m [4]. Now we have

$$\begin{aligned} \int_0^1 (z_m, z_m^0)dt &\le \int_0^1 |z_m||z_m^0|dt \le (\int_0^1 |z_m|dt)\|z_m^0(t)\|_\infty \\ &\le c_2\|z_m^0\|(\int_0^1 |z_m|dt). \end{aligned}$$

Combining this with the proved fact

$$\int_0^1 (z_m, z_m^0)dt \geq \varepsilon_6 \|z_m^0\|^2,$$

we get (2.19) for $\alpha = 1$.

Case(iv): (H5$^-$) holds. Similar to case(iii), we have

$$-\int_0^1 G_\infty(t, z_m)dt \leq |f_m(z_m)| + |\frac{1}{2}\langle (A - B_\infty)z_m, z_m\rangle| \leq C_2\|z_m^0\|^{2\beta} + C_0;$$
$$-\int_0^1 G_\infty(t, z_m)dt \geq \int_0^1 (M_2|z_m|^\alpha - M_3)dt \geq M_4\|z_m^0\|^\alpha - M_3.$$

We get a contradiction because of $\alpha > 2\beta$. Thus $\{z_m\}$ is bounded, and f satisfies $(PS)_c^*$ condition under (H5$^-$).

Case(v): (H6$^+$) holds. For m large enough, by (2.15) and the claim in Case (iii), we have

$$\begin{aligned}\int_0^1 &(G'_\infty(t, z_m), z_m)dt \\ &\leq | - \langle f'_m(z_m), z_m\rangle + \langle (A - B_\infty)(z_m^+ + z_m^-), (z_m^+ + z_m^-)\rangle| \\ &\leq \|z_m\| + \varepsilon_6\|z_m^+ + z_m^-\|^2 \leq \|z_m^0\| + \varepsilon_0\|z_m^0\|^\beta + \varepsilon_7\|z_m^0\|^{2\beta};\end{aligned} \tag{2.20}$$

$$\int_0^1 (G'_\infty(t, z_m), z_m)dt \geq M_2 \int_0^1 |z_m|^\alpha dt - M_3 \geq M_4\|z_m^0\|^\alpha - M_3. \tag{2.21}$$

We get a contradiction from $\alpha > 2\beta$, (2.20) and (2.21). Thus $\{z_m\}$ is bounded, and f satisfies $(PS)_c^*$ condition under $(H6^+)$.

Case(vi): (H6$^-$) holds. Similar to case(v), we have

$$-\int_0^1 (G'_\infty(t, z_m), z_m)dt \leq \|z_m^0\| + \varepsilon_0\|z_m^0\|^\beta + \varepsilon_7\|z_m^0\|^{2\beta};$$
$$-\int_0^1 (G'_\infty(t, z_m), z_m)dt \geq M_4\|z_m^0\|^\alpha - M_3.$$

Then $\alpha > 2\beta$ implies that $\{z_m\}$ must be bounded, and f satisfies $(PS)_c^*$ condition under (H6$^-$). □

Proof of Theorem 1.1 Case(i) & (iii): By a direct computation, (H4$^+$) and (H5$^+$) imply that

$$G_\infty(t, z) \to +\infty \quad \text{as} \quad |z| \to \infty. \tag{2.22}$$

By (H2), for any $\varepsilon > 0$, there exists $M > 0$ such that

$$|G_\infty(t,z)| \le \varepsilon |z|^2 + M. \tag{2.23}$$

For $m > 0$, by using the same arguments as in the proof of Lemma 2.4, we know that f_m satisfies (PS) condition. Let

$$X_m = M^-(P_m(A-B_\infty)P_m) \oplus M^0(P_m(A-B_\infty)P_m),$$
$$Y_m = M^+(P_m(A-B_\infty)P_m).$$

By (2.23), for all $z^+ \in Y_m$, we have

$$f_m(z^+) = \frac{1}{2}\langle (A-B_\infty)z^+, z^+\rangle - \int_0^1 G_\infty(t,z^+)dt \ge C_1\|z^+\|^2 - \varepsilon\|z^+\|^2 - M.$$

We can choose $0 < \varepsilon \le C_1/2$. Then there is a $\delta > 0$ such that

$$f_m(z^+) \ge -\delta > -\infty, \quad \forall z^+ \in Y_m. \tag{2.24}$$

By (2.22), there exist $M_0 > 0$, such that $G_\infty(t,z) \ge -M_0$, for all $z \in \mathbb{R}^{2N}$. This implies that, for all $z^- \oplus z^0 \in X_m$,

$$\begin{aligned} f_m(z^- \oplus z^0) &= \frac{1}{2}\langle (A-B_\infty)z^-, z^-\rangle - \int_0^1 G_\infty(t, z^- + z^0)dt \\ &\le -C_1\|z^-\|^2 + M_0 \le -2\delta, \end{aligned}$$

if $\|z^-\| \ge L = \sqrt{\frac{2\delta + M_0}{C_1}}$.

Since $M^0(P_m(A-B_\infty)P_m) = M^0(A-B_\infty)$ is a finite dimensional space, by (2.22) we have that

$$\int_0^1 G_\infty(t, z^- + z^0)dt \to +\infty \quad \text{as } \|z^0\| \to +\infty \quad \text{uniformly for } \|z^-\| \le L\,.$$

Thus there exists $L_1 > 0$ such that for $\|z^0\| \ge L_1$ and $\|z^-\| \le L$

$$f_m(z^- + z^0) \le -\int_0^1 G_\infty(t, z^- + z^0)dt \le -2\delta.$$

Let $Q_m = \{z^- \oplus z^0 \in X_m : \quad \|z^- + z^0\| \le L + L_1\}$. Then we have

$$f_m(z) \le -2\delta, \quad \forall z \in \partial Q_m. \tag{2.25}$$

Let $S = Y_m$. Then ∂Q_m and S homologically link [5]. Let $D = \partial Q_m$ and $\alpha = [Q_m] \in H_k(E_m, D)$ with $k = \dim(X_m)$. Then α is nontrivial and $\mathcal{F}(\alpha)$ defined by Definition 2.2 is a homological family of dimension k with boundary D [5, p. 84]. By Theorem 2.3, (2.24) and (2.25), there exists a critical point x_m of f_m such that the Morse indices $m^-(x_m)$ and $m^0(x_m)$ of f_m at x_m satisfies

$$\dim X_m - m^0(x_m) \le m^-(x_m) \le \dim X_m; \tag{2.26}$$
$$-\delta \le f_m(x_m) = c_m = c(f_m, \mathcal{F}(\alpha)). \tag{2.27}$$

Since $Q_m \in \mathcal{F}(\alpha)$, by (2.23) we have

$$\begin{aligned} -\delta \le c_m &\le \sup_{z^- + z^0 \in Q_m} f_m(z^- + z^0) \\ &\le \frac{1}{2}\|(A - B_\infty)\|(L + L_1)^2 + \varepsilon(L + L_1)^2 + M = M_2, \end{aligned}$$

where δ and M_2 are constants independent of m. Hence passing to a subsequence we have

$$c_m \to c, \quad -\delta \le c \le M_2.$$

Since f satisfies $(PS)_c^*$ condition, passing to a subsequence, there exist $x^* \in E$ such that

$$x_m \to x^*, \quad f(x^*) = c, \quad f'(x^*) = 0\,. \tag{2.28}$$

By standard arguments, x^* is a classical solution of (1.1).

Let $B^*(t) = H''(t, x^*(t))$ and B^* be the operator, defined by (2.1), corresponding to $B^*(t)$. Let (i^*, n^*) be the Maslov-type index of $B^*(t)$. It is easy to show that

$$\|f''(z) - (A - B^*)\| \to 0 \quad \text{as} \quad \|z - x^*\| \to 0.$$

Let $d = \frac{1}{4}\|(A - B^*)^\#\|^{-1}$. Then there exists $r_0 > 0$ such that

$$\|f''(z) - (A - B^*)\| < \frac{1}{2}d, \quad \forall z \in V_{r_0} = \{z \in E : \|z - x^*\| \le r_0\}.$$

This implies that

$$\dim M^\pm(f''_m(z)) \ge \dim M^\pm_d(P_m(A - B^*)P_m), \quad \forall z \in V_{r_0} \cap E_m. \tag{2.29}$$

By (2.26), (2.28), (2.29) and Theorem 2.1, there exist $m_1 > m^*$ such that for $m \ge m_1$,

$$m + n_\infty = \dim(X_m) \ge m^-(x_m) \ge \dim M^-_d(P_m(A - B^*)P_m) = m - i_\infty + i^*;$$

$$\begin{aligned} m + n_\infty &= \dim(X_m) \le m^-(x_m) + m^0(x_m) \\ &\le \dim[M^-_d(P_m(A - B^*)P_m) \oplus M^0_d(P_m(A - B^*)P_m)] \\ &= m - i_\infty + i^* + n^*. \end{aligned}$$

This implies that $i_\infty + n_\infty \in [i^*, i^* + n^*]$, which means that $x^* \neq 0$, i.e., x^* is a nontrivial 1-periodic solution of the system (1.1).

Case(ii)&(iv): By (H4$^-$) and (H5$^-$) we have

$$G_\infty(t, z) \to -\infty \quad \text{as} \quad |z| \to \infty.$$

Let $X_m = M^-(P_m(A - B_\infty)P_m)$ and $Y_m = M^+(P_m(A - B_\infty)P_m) \oplus M^0(P_m(A - B_\infty)P_m)$. By using similar arguments as in the proof of (2.24) and (2.25) we have

$$\begin{aligned} f_m(z^+ + z^0) &\ge -\delta_1 > -\infty, \quad \forall z^+ + z^0 \in Y_m; \\ f_m(z) &\le -2\delta_1, \qquad \forall z \in \partial Q_m, \end{aligned}$$

where $Q_m = \{z^- \in X_m : \|z^-\| \le L_2\}$. Here $\delta_1 > 0$ and $L_2 > 0$ are constants independent of m. By using the same arguments, one can prove that (2.26)-(2.29) still hold. By Theorem 2.1 there exist $m_2 > m^*$ such that for $m \ge m_2$

$$m = \dim(X) \ge m^-(x_m) \ge \dim M_d^-(P_m(A-B^*)P_m) = m - i_\infty + i^*;$$
$$m = \dim(X) \le m^-(x_m) + m^0(x_m) \le m - i_\infty + i^* + n^*.$$

Therefore, we have $i_\infty \in [i^*, i^*+n^*]$, which implies $x^* \ne 0$, i.e., x^* is a nontrivial 1-periodic solution of the system (1.1).

Case(v): (H6$^+$) holds. We shall use the same idea as in the proof of [9, Theorem 1.1]. Let $X = \ker(A - B_\infty)$, $Y = Im(A - B_\infty)$, and $P : E \to X$, $Q : E \to Y$ be the orthogonal projections. By the special construction of the Galerkin approximation scheme $\{P_m\}$, we have

$$E_m = X \oplus P_m Y, \quad \ker(P_m(A - B_\infty)P_m) = X, \quad Im(P_m(A - B_\infty)P_m) = P_m Y.$$

For given $m > 0$, since $\dim E_m < +\infty$, we have

$$\int_0^1 |z|^\alpha dt \ge c_m \|z\|^\alpha, \quad \forall z \in E_m, \tag{2.30}$$

where $c_m > 0$ is a constant which depends on m. Let π be the flow of f_m in E_m, generated by

$$\begin{aligned} \dot{y} &= -(A - B_\infty)y + QP_m g'(x + y), \\ \dot{x} &= P(P_m g'(x + y)), \quad \text{for } (x, y) \in X \oplus P_m Y = E_m. \end{aligned}$$

Let

$$V^\pm = \{y^\pm \in M^\pm(P_m(A-B_\infty)P_m) : \|y^\pm\| \le r_Y\}, \quad W = \{x \in X : \|x\| \le r_X\}.$$

We shall show that there are $r_Y > 0$ and $r_X > 0$ such that $D = (V^- \times V^+) \times W$ is an isolating block of π.

By using the some arguments as in the proof of (2.16), (H6$^+$) implies that

$$\|g'(z)\| \le M_3 \|z\|^\beta, \quad \forall z \in E, \quad \|z\| \ge L. \tag{2.31}$$

On the other hand, (2.30) and (H6$^+$) also imply that

$$\langle g'(z_m), z_m\rangle \ge M_2 c_m \|z_m\|^\alpha - M_4, \quad \forall z_m \in E_m. \tag{2.32}$$

For any $x \in \partial W$, $y \in V^- \times V^+$, by (2.30)-(2.32) we have

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}\|x\|^2)|_{t=0} &= \langle x, \dot{x}\rangle|_{t=0} = \langle x, g'(x + y)\rangle|_{t=0} \\ &= \langle x + y, g'(x + y)\rangle|_{t=0} - \langle y, g'(x + y)\rangle|_{t=0} \\ &\ge M_2 c_m \|x + y\|^\alpha - M_4 - M_3 \|x + y\|^\beta \|y\| \\ &\ge \|x + y\|^\beta [M_2 c_m \|x + y\|^{\alpha-\beta} - M_3 \|y\|] - M_4 \\ &\ge r_X^\beta [M_2 c_m r_X^{\alpha-\beta} - M_3 r_Y] - M_4 \\ &\ge r_X^\beta r_Y - M_4 \ge 1 > 0, \end{aligned}$$

provided

$$r_X^{\alpha-\beta}=(\frac{M_3+1}{M_2c_m})r_Y=c'r_Y,\quad \text{and}\quad r_X\geq[c'(M_4+1)]^{1/2}+1. \tag{2.33}$$

For any $y^-\in\partial V^-$, $y^+\in V^+$, $x\in W$, and $y=y^++y^-$, by (2.30)-(2.33) we have

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}\|y^-\|^2)|_{t=0}&=\langle\dot{y}^-,y^-\rangle|_{t=0}\\&=[-\langle(A-B_\infty)y^-,y^-\rangle+\langle QP_mg'(x+y),y^-\rangle]|_{t=0}\\&\geq\rho r_Y^2-M_3\|x+y\|^\beta\|y^-\|\geq\rho r_Y^2-M_3(r_X+2r_Y)^\beta r_Y\\&\geq\rho r_Y^2-M_3[(c'r_Y)^{\frac{1}{\alpha-\beta}}+2r_Y]^\beta r_Y,\end{aligned}\tag{2.34}$$

where

$$\rho=\inf_{\|y^-\|=1}|\langle y^-,(A-B_\infty)y^-\rangle|,\quad\text{and}\quad y^-\in M^-(A-B_\infty).$$

If $\alpha-\beta\geq1$ and $r_Y\geq1$, by (2.34) we have

$$\frac{d}{dt}(\frac{1}{2}\|y^-\|^2)|_{t=0}\geq\rho r_Y^2-M_3[c'^{\frac{1}{\alpha-\beta}}+2]^\beta r_Y^{\beta+1}>0,$$

provided

$$r_Y\geq(\frac{M_3[c'^{\frac{1}{\alpha-\beta}}+2]^\beta+1}{\rho})^{\frac{1}{1-\beta}}+1. \tag{2.35}$$

If $\alpha-\beta<1$ and $r_Y\geq1$, we have

$$\frac{d}{dt}(\frac{1}{2}\|y^-\|^2)|_{t=0}\geq\rho r_Y^2-M_3[c'^{\frac{1}{\alpha-\beta}}+2]^\beta\cdot r_Y^{\frac{\alpha}{\alpha-\beta}}>0,$$

provided

$$r_Y\geq(\frac{M_3[c'^{\frac{1}{\alpha-\beta}}+2]^\beta+1}{\rho})^{\frac{\alpha-\beta}{\alpha-2\beta}}+1. \tag{2.36}$$

Now we can choose $r_X>0$ and $r_Y>0$ such that (2.33)-(2.36) hold. Similarly, for any $y^+\in\partial V^+$, $y^-\in V^-$, $x\in W$, we have

$$\frac{d}{dt}(\frac{1}{2}\|y^+\|^2)|_{t=0}<0.$$

Therefore D is an isolating block of π and

$$D^-=(\partial V^-\times V^+)\times W\cup(V^-\times V^+)\times\partial W.$$

Follow the same arguments as in the proof of Theorem 1.1 in [9], by Conley index theory, f has a critical point $x^*\neq0$, i.e., x^* is a nontrivial 1-periodic solution of the system (1.1).

Case(vi): ($H6^-$) holds. Using the same arguments as in the proof of Case(v), ($H6^-$) implies that

$$\frac{d}{dt}(\frac{1}{2}\|x\|^2)|_{t=0} < 0.$$

Therefore D is an isolating block of π and $D^- = (\partial V^- \times V^+) \times W$. By Conley index theory [9, Theorem 3.3], f has a critical point $x^* \neq 0$, i.e., x^* is a nontrivial 1-periodic solution of the system (1.1). We omit the details.

3 Examples

In this section, we give some examples which can not be solved directly by the results in the references.

Example 3.1: Consider the function given by (1.2), i.e.,

$$H(t,z) = \frac{7|z|^2}{2\ln(e+|z|^2)}, \quad \forall t \in [0,1],\ \forall z \in \mathbb{R}^{2N}.$$

Then $B_0(t) = 7I_{2N}$, $B_\infty(t) = 0$. By a direct computation,

$$(i_0, n_0) = (3N, 0), \quad (i_\infty, n_\infty) = (-N, 2N),$$
$$i_\infty + n_\infty = N \notin [3N, 3N] = [i_0, i_0 + n_0].$$

Moreover, $G_\infty(t,z) = H(t,z)$ satisfies ($H4^+$). By Theorem 1.1(i), the system (1.1) possesses a nontrivial 1-periodic solution.

Example 3.2: Consider the function given by (1.3), i.e.,

$$H(t,z) = \frac{1}{2}|z|^2 - \frac{|z|^2}{\ln(e+|z|^2)}, \quad \forall t \in [0,1],\ \forall z \in \mathbb{R}^{2N}.$$

Then $B_0(t) = -I_{2N}$, $B_\infty(t) = I_{2N}$. By a direct computation

$$(i_0, n_0) = (-N, 0), \quad (i_\infty, n_\infty) = (N, 0), \text{ and } G_\infty(t,z) = -\frac{|z|^2}{\ln(e+|z|^2)}.$$

One can show that ($H4^-$) holds. Theorem 1.1(ii) implies that the system (1.1) has a nontrivial 1-periodic solution.

Example 3.3: Let $H(t,z) \in C^2([0,1] \times \mathbb{R}^{2N}, \mathbb{R})$ such that

$$H(t,z) = \frac{7}{2}|z|^2 \quad \text{for } |z| \leq 1;$$
$$H(t,z) = |z|\ln(1+|z|^2) \quad \text{for } |z| \geq 100.$$

Then $B_0(t) = 7I_{2N}$, $B_\infty(t) = 0$, and $G_\infty(t,z) = H(t,z)$ satisfies ($H5^+$) with $\alpha = 1$, $\beta = \frac{1}{4}$ and L being large enough. By Theorem 1.1(iii), the system (1.1) has a nontrivial 1-periodic solution.

Example 3.4: Let $H(t, z) \in C^2([0, 1] \times \mathbb{R}^{2N}, \mathbb{R})$ such that

$$H(t, z) = \frac{7}{2}|z|^2 \quad \text{for } |z| \le 1;$$
$$H(t, z) = |z|^{\frac{4}{3}} \ln(1 + |z|^2) \quad \text{for } |z| \ge 100.$$

By a direct computation, $G_\infty(t, z) = H(t, z)$ satisfies (H6$^+$). Thus the system (1.1) has a nontrivial 1-periodic solution by Theorem 1.1(v).

Example 3.5: Let $H(t, z) \in C^2([0, 1] \times \mathbb{R}^{2N}, \mathbb{R})$ such that

$$H(t, z) = \frac{7}{2}|z|^2 \quad \text{for } |z| \le 1;$$
$$H(t, z) = -|z|^{\frac{4}{3}} \ln(1 + |z|^2) \quad \text{for } |z| \ge 100.$$

Then (H6$^-$) holds. By Theorem 1.1(vi), the system (1.1) possesses a nontrivial 1-periodic solution.

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