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Existence of solutions for quasilinear degenerate elliptic equations *

Y. Akdim, E. Azroul, & A. Benkirane

Abstract

In this paper, we study the existence of solutions for quasilinear degenerate elliptic equations of the form $A(u) + g(x, u, \nabla u) = h$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega, w)$ to its dual. On the nonlinear term $g(x, s, \xi)$, we assume growth conditions on ξ, not on s, and a sign condition on s.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, p be a real number with $1 < p < \infty$, and $w = \{w_i(x)\}$ $0 \leq i \leq N$ be a vector of weight functions on Ω; i.e. each $w_i(x)$ is a measurable a.e. strictly positive function on Ω, satisfying some integrability conditions (see section 2). Let $X = W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector w. Assume:

(A0) The norm

$$|||u|||_X = \Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}$$

is equivalent to the usual norm on X; see (2.2) below.

(A1) There exists a weight function $\sigma(x)$ on Ω and a parameter q, $1 < q < \infty$, such that the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx \Big)^{1/q} \leq c \Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover, the imbedding $X \hookrightarrow L^q(\Omega, \sigma)$ is compact.

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Let A be the nonlinear operator from X into the dual X^* defined as

$$Au = -\operatorname{div}(a(x,u,\nabla u)), \tag{1.1}$$

where $a(x,s,\xi) = \{a_i(x,s,\xi)\}$, $1 \le i \le N$: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function.

(A2) We assume that

$$|a_i(x,s,\xi)| \le c_1 w_i^{1/p}(x)[k(x) + \sigma^{1/p'}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}],$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, all $i = 1,\dots,N$, some function $k(x) \in L^{p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) and some constant $c_1 > 0$. Here σ and q are as in (A1).

(A3) For a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and some constant $c_0 > 0$, we assume that

$$a(x,s,\xi).\xi \ge c_0 \sum_{i=1}^{N} w_i(x)|\xi_i|^p\,.$$

Recently, Drabek, Kufner and Mustonen [5] proved that under the hypotheses (A0–A3) and certain monotonicity conditions, the Dirichlet problem associated with the equation $Au = h$, $h \in X^*$ has at least one solution u in $W_0^{1,p}(\Omega, w)$. See also [1], where A is of the form $-\operatorname{div}(a(x,u,\nabla u)) + a_0(x,u,\nabla u)$.

The purpose in this paper, is to prove the same result for the general nonlinear elliptic equation

$$Au + g(x,u,\nabla u) = h, h \in X^*$$

where g is a nonlinear lower-order term having natural growth (of order p) with respect to $|\nabla u|$. Regarding $|u|$, we do not assume any growth restrictions. However, we assume the "sign condition"

$$g(x,s,\xi).s \ge 0\,.$$

More precisely, we prove in theorem 3.1 an existence result for the problem

$$\begin{gathered} Au + g(x,u,\nabla u) = h \quad \text{in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega,w), \quad g(x,u,\nabla u) \in L^1(\Omega), \quad g(x,u,\nabla u)u \in L^1(\Omega). \end{gathered} \tag{1.2}$$

It turns out that for a solution u of this system, the term $g(x,u,\nabla u)$ is in $L^1(\Omega)$. However, for a general $v \in W_0^{1,p}(\Omega,w)$, $g(x,v,\nabla v)$ can be very singular (see for example [3] where $w \equiv 1$).

Let us point out that more work in this direction can be found in [7] where the authors have studied the existence of bounded solutions for the degenerate elliptic equation

$$Au - c_0|u|^{p-2}u = h(x,u,\nabla u),$$

with some more general degeneracy, under some additional assumptions on h and $a(x,s,\xi)$. When $w \equiv 1$ (the non weighted case) existence results for the problem (1.2) have been shown in [3].

The present paper is organized as follows: In section 2, we give some preliminaries and we prove some technical lemmas concerning convergence in weighted Sobolev spaces. In section 3, we state our general result which will be proved in section 4. Section 5 is devoted to an example which illustrates our abstract hypotheses. Note that, in the proof of our main result, many ideas have been adapted from Bensoussan et al. [3].

2 Preliminaries

Weighted Sobolev spaces. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, let $1 < p < \infty$, and let $w = \{w_i(x)\}$, $0 \leq i \leq N$ be a vector of weight functions; i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that for $0 \leq i \leq N$,

$$w_i \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad w_i^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega)\,. \tag{2.1}$$

We define the weighted space with weight γ on Ω as

$$L^p(\Omega,\gamma) = \{u = u(x) : \ u\gamma^{1/p} \in L^p(\Omega)\}\,.$$

In this space, we define the norm

$$\|u\|_{p,\gamma} = \Big(\int_\Omega |u(x)|^p \gamma(x)\,dx \Big)^{1/p}.$$

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \text{ for all } i = 1,\dots,N\,.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \Big(\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx \Big)^{1/p}. \tag{2.2}$$

To deal with the Dirichlet problem, we use the space

$$X = W_0^{1,p}(\Omega,w)$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega,w)$ and $(X,\|.\|_{1,p,w})$ is a reflexive Banach space. We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}$ $i = 0,\dots,N$, and p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$. For more details, we refer the reader to [6].

Definition. Let X be a reflexive Banach space. An operator B from X to the dual X^* satisfies property (M) if for any sequence $(u_n) \subset X$ satisfying $u_n \rightharpoonup u$ in X weakly, $B(u_n) \rightharpoonup \chi$ in X^* weakly and $\limsup_{n\to\infty}\langle Bu_n, u_n\rangle \le \langle \chi, u\rangle$ then one has $\chi = B(u)$.

Now we state the following assumption.

(H1) The expression

$$|||u|||_X = (\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx)^{1/p} \tag{2.3}$$

is a norm defined on X and is equivalent to the norm (2.2).

Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space. There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{2.4}$$

with $q' = \frac{q}{q-1}$ and such that the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\,dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \tag{2.5}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$X \hookrightarrow L^q(\Omega, \sigma), \tag{2.6}$$

determined by the inequality (2.5) is compact.

Now we state and prove the following technical lemmas which are needed later.

Lemma 2.1 *Let $g \in L^r(\Omega,\gamma)$ and let $g_n \in L^r(\Omega,\gamma)$, with $\|g_n\|_{r,\gamma} \le c$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^r(\Omega,\gamma)$, where $\rightharpoonup$ denotes weak convergence and γ is a weight function on Ω.*

Proof. Since $g_n\gamma^{1/r}$ is bounded in $L^r(\Omega)$ and $g_n(x)\gamma^{1/r}(x) \to g(x)\gamma^{1/r}(x)$, a.e. in Ω, then by [11, Lemma 3.2], we have

$$g_n\gamma^{1/r} \rightharpoonup g\gamma^{1/r} \ \text{ in } L^r(\Omega).$$

Moreover for all $\varphi \in L^{r'}(\Omega, \gamma^{1-r'})$, we have $\varphi\gamma^{-\frac{1}{r}} \in L^{r'}(\Omega)$. Then

$$\int_\Omega g_n\varphi\,dx \to \int_\Omega g\varphi\,dx, \quad \text{i.e. } g_n \rightharpoonup g \text{ in } L^r(\Omega,\gamma).$$

Lemma 2.2 *Assume that (H1) holds. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Remark. The previous lemma is a generalization of the corresponding in [8, pp. 151-152], where $w \equiv 1$ and $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$, and of the corresponding in [2], where $w_0 \equiv w_1 \equiv \cdots \equiv w_N$ is some weight function, $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. Also note that the previous lemma implies that functions in $W_0^{1,p}(\Omega, w)$ can be truncated.

Proof of Lemma 2.2 First, note that the proof of the second part of Lemma 2.2 is identical to the corresponding in non weighted case in [8]. Consider firstly the case $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. Let $u \in W_0^{1,p}(\Omega, w)$. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$, there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,p}(\Omega, w)$. Passing to a subsequence, we can assume that,

$$\begin{gathered} u_n \to u \textit{ a.e. } \text{in } \Omega \\ \nabla u_n \to \nabla u \textit{ a.e. } \text{in } \Omega. \end{gathered}$$

Then

$$F(u_n) \to F(u) \quad \text{a.e. in } \Omega. \tag{2.7}$$

On the other hand, from the relation $|F(u_n)|^p w_0 \leq \|F'\|_\infty |u_n|^p w_0$ and

$$|\frac{\partial F(u_n)}{\partial x_i}|^p w_i = |F'(u_n)\frac{\partial u_n}{\partial x_i}|^p w_i \leq M|\frac{\partial u_n}{\partial x_i}|^p w_i,$$

we deduce that the function $F(u_n)$ remains bounded in $W_0^{1,p}(\Omega, w)$. Thus, going to a further subsequence, we obtain

$$F(u_n) \rightharpoonup v \text{ in } W_0^{1,p}(\Omega, w). \tag{2.8}$$

Thanks to (2.7),(2.8) and (2.6) we conclude that

$$v = F(u) \in W_0^{1,p}(\Omega, w).$$

We now turn our attention to the general case. Taking convolutions with mollifiers ρ_n in $\mathbb{R}$, we have $F_n = F * \rho_n$, $F_n \in C^1(\mathbb{R})$ and $F_n' \in L^\infty(\mathbb{R})$. Then by the first case we have $F_n(u) \in W_0^{1,p}(\Omega, w)$. Since $F_n \to F$ uniformly in every compact, we have $F_n(u) \to F(u)$ a.e. in Ω. On the other hand, $(F_n(u))$ is bounded in $W_0^{1,p}(\Omega, w)$, then for a subsequence $F_n(u) \rightharpoonup \bar{v}$ in $W_0^{1,p}(\Omega, w)$ and a.e. in Ω (due to (2.6)), then

$$\bar{v} = F(u) \in W_0^{1,p}(\Omega, w).$$

The following lemmas follow from the previous lemma.

Lemma 2.3 *Assume that (H1) holds. Let $u \in W_0^{1,p}(\Omega, w)$, and let $T_k(u)$, $k \in \mathbb{R}^+$, be the usual truncation then $T_k(u) \in W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$T_k(u) \to u \textit{ strongly in } W_0^{1,p}(\Omega, w).$$

Lemma 2.4 *Assume that (H1) holds. Let* $u \in W_0^{1,p}(\Omega, w)$*, then* $u^+ = \max(u, 0)$ *and* $u^- = \max(-u, 0)$ *lie in* $W_0^{1,p}(\Omega, w)$*. Moreover, we have*

$$\frac{\partial(u^+)}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i}, & \text{if } u > 0 \\ 0, & \text{if } u \le 0 \end{cases}$$

$$\frac{\partial(u^-)}{\partial x_i} = \begin{cases} 0, & \text{if } u \ge 0 \\ -\frac{\partial u}{\partial x_i}, & \text{if } u < 0. \end{cases}$$

Lemma 2.5 *Assume that (H1) holds. Let* (u_n) *be a sequence of* $W_0^{1,p}(\Omega, w)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega, w)$*. Then,* $u_n^+ \rightharpoonup u^+$ *weakly in* $W_0^{1,p}(\Omega, w)$ *and* $u_n^- \rightharpoonup u^-$ *weakly in* $W_0^{1,p}(\Omega, w)$*.*

Proof. Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$ and by (2.8) we have for a subsequence $u_n \to u$ in $L^q(\Omega, \sigma)$ and a.e. in Ω. On the other hand,

$$\begin{aligned} |||u_n|||_X^p &= \sum_{i=1}^N \int_\Omega |\frac{\partial u_n}{\partial x_i}|^p w_i \ge \sum_{i=1}^N \int_{\{u_n \ge 0\}} |\frac{\partial u_n}{\partial x_i}|^p w_i \\ &= \sum_{i=1}^N \int_\Omega |\frac{\partial u_n^+}{\partial x_i}|^p w_i = |||u_n^+|||_X^p \,. \end{aligned}$$

Then (u_n^+) is bounded in $W_0^{1,p}(\Omega, w)$ hence by (2.6), $u_n^+ \rightharpoonup u^+$ in $W_0^{1,p}(\Omega, w)$. Similarly, we prove that $u_n^- \rightharpoonup u^-$ in $W_0^{1,p}(\Omega, w)$.

3 Main result

Let A be the nonlinear operator from $W_0^{1,p}(\Omega, w)$ into the dual $W^{-1,p'}(\Omega, w^*)$ defined as

$$Au = -\operatorname{div}(a(x, u, \nabla u))\,,$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-function satisfying the following assumptions:

(H2) For $i = 1, \dots, N$,

$$|a_i(x, s, \xi)| \le \beta w_i^{1/p}(x)[k(x) + \sigma^{1/p'}|s|^{\frac{q}{p'}} + \sum_{j=1}^N w_j^{1/p'}(x)|\xi_j|^{p-1}]\,, \quad (3.1)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \quad \text{for all } \xi \ne \eta \in \mathbb{R}^N\,, \quad (3.2)$$

$$a(x, s, \xi).\xi \ge \alpha \sum_{i=1}^N w_i |\xi_i|^p, \quad (3.3)$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are positive constants.

(H3) $g(x,s,\xi)$ is a Carathéodory function satisfying

$$g(x,s,\xi)s \geq 0\,, \tag{3.4}$$

$$|g(x,s,\xi)| \leq b(|s|)(\sum_{i=1}^{N} w_i|\xi_i|^p + c(x)), \tag{3.5}$$

where $b:\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x)$ is positive function which in $L^1(\Omega)$.

For the nonlinear Dirichlet boundary-value problem (1.2), we state our main result as follows.

Theorem 3.1 *Under assumptions (H1)-(H3) and $h \in W^{-1,p'}(\Omega, w^*)$, there exists a solution of (1.2).*

Remarks. (1) Theorem 3.1, generalizes to weighted case the analogous statement in [3].
(2) The assumption (2.4) appear to be necessary only for proving the boundedness of g in $W_0^{1,p}(\Omega,w)$. Thus, when $g \equiv 0$, we do not need assumption (2.4).
(3) If we assume that $w_0(x) \equiv 1$ and that there exists $\nu \in]\frac{N}{P},\infty[\cap[\frac{1}{P-1},\infty[$ such that $w_i^{-\nu} \in L^1(\Omega)$ for all $i=1,\dots,N$, (which is an integrability condition, stronger than (2.1)), then

$$|||u|||_X = \Big(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega,w)$ and equivalent to (2.2). Also we have that

$$W_0^{1,p}(\Omega,w) \hookrightarrow L^q(\Omega)$$

for $1 \leq q < p_1^*$, $p\nu < N(\nu+1)$, and $q \geq 1$ is arbitrary for $p\nu \geq N(\nu+1)$ where $p_1 = \frac{p\nu}{\nu+1}$. Where $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see [6]). Thus the hypotheses (H1) is verified (for $\sigma \equiv 1$).

For Theorem 3.1, we needed the following lemma.

Lemma 3.2 *Assume that (H1) and (H2) are satisfied, and let (u_n) be a sequence in $W_0^{1,p}(\Omega,w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\,dx \to 0. \tag{3.6}$$

Then, $u_n \to u$ in $W_0^{1,p}(\Omega,w)$.

Proof. Let $D_n = [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)$. Then by (3.2), D_n is a positive function and by (3.6) $D_n \to 0$ in $L^1(\Omega)$. Extracting a subsequence still denoted by u_n, and using (2.6), we can write

$$\begin{cases} u_n \to u & \text{a.e. in } \Omega \\ D_n \to 0 & \text{a.e. in } \Omega. \end{cases}$$

Then, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \backslash B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $|k(x)| < \infty$, $w_i(x) > 0$ and $u_n(x) \to u(x)$, $D_n(x) \to 0$. We set $\xi_n = \nabla u_n(x)$, $\xi = \nabla u(x)$. Then

$$\begin{aligned} D_n(x) =& [a(x,u_n,\xi_n) - a(x,u_n,\xi)](\xi_n - \xi) \\ \geq& \alpha \sum_{i=1}^N w_i |\xi_n^i|^p + \alpha \sum_{i=1}^N w_i |\xi^i|^p \\ &- \sum_{i=1}^N \beta w_i^{1/p} [k(x) + \sigma^{1/p'} |u_n|^{\frac{q}{p'}} + \sum_{j=1}^N w_j^{1/p'} |\xi_n^j|^{p-1}] |\xi^i| \\ &- \sum_{i=1}^N \beta w_i^{1/p} [k(x) + \sigma^{1/p'} |u_n|^{\frac{q}{p'}} + \sum_{j=1}^N w_j^{1/p'} |\xi^j|^{p-1}] |\xi_n^i| \\ \geq& \alpha \sum_{i=1}^N w_i |\xi_n^i|^p - c_x \Big[1 + \sum_{j=1}^N w_j^{1/p'} |\xi_n^j|^{p-1} + \sum_{i=1}^N w_i^{1/p} |\xi_n^i| \Big] \end{aligned} \tag{3.7}$$

where c_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$ where M_x is some positive constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n; indeed (3.7) becomes,

$$D_n(x) \geq \sum_{i=1}^N |\xi_n^i|^p \Big(\alpha w_i - \frac{c_x}{N|\xi_n^i|^p} - \frac{c_x w_i^{1/p'}}{|\xi_n^i|} - \frac{c_x w_i^{1/p}}{|\xi_n^i|^{p-1}} \Big).$$

If $|\xi_n| \to \infty$ (for a subsequence) there exists at least one i_0 such that $|\xi_n^{i_0}| \to \infty$, which implies that $D_n(x) \to \infty$ which gives a contradiction.

Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by the continuity of a with respect to the two last variables we obtain

$$(a(x,u(x),\xi^*) - a(x,u(x),\xi))(\xi^* - \xi) = 0.$$

In view of (3.2) we have $\xi^* = \xi$. The uniqueness of the cluster point implies

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega.$$

Since the sequence $a(x,u_n,\nabla u_n)$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, w_i^*)$ and $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, Lemma 2.1 implies

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{in } \prod_{i=1}^N L^{p'}(\Omega, w_i^*) \text{ and a.e. in } \Omega.$$

We set $\bar{y}_n = a(x, u_n, \nabla u_n)\nabla u_n$ and $\bar{y} = a(x, u, \nabla u)\nabla u$. As in [4, Lemma 5] we can write

$$\bar{y}_n \to \bar{y} \quad \text{in } L^1(\Omega).$$

By (3.3) we have

$$\alpha \sum_{i=1}^{N} w_i |\frac{\partial u_n}{\partial x_i}|^p \leq a(x, u_n, \nabla u_n)\nabla u_n\,.$$

Let $z_n = \sum_{i=1}^N w_i|\frac{\partial u_n}{\partial x_i}|^p$, $z = \sum_{i=1}^N w_i|\frac{\partial u}{\partial x_i}|^p$, $y_n = \frac{\bar{y}_n}{\alpha}$ and $y = \frac{\bar{y}}{\alpha}$. Then, by Fatou's theorem we obtain

$$\int_\Omega 2y\,dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\,dx$$

i.e. $0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|\,dx$ then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|\,dx \leq \limsup_{n\to\infty} \int_\Omega |z_n - z|\,dx \leq 0,$$

this implies,

$$\nabla u_n \to \nabla u \ \text{ in } \prod_{i=1}^{N} L^p(\Omega, w_i),$$

which with (2.3) completes the present proof.

4 Proof of Theorem 3.1

Step (1) The approximate problem. Let

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon|g(x, s, \xi)|}$$

and consider the equation

$$\begin{gathered} A(u_\varepsilon) + g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = h \\ u_\varepsilon \in W_0^{1,p}(\Omega, w) \end{gathered} \tag{4.1}$$

We define the operator $G_\varepsilon :\ X \to X^*$ by

$$\langle G_\varepsilon u, v\rangle = \int_\Omega g_\varepsilon(x, u, \nabla u)v\,dx.$$

Thanks to Hölder's inequality, for all $v \in X$ and $\varphi \in X$,

$$\begin{aligned} |\int_\Omega g_\varepsilon(x, v, \nabla v)\varphi\,dx| &\leq \Big(\int_\Omega |g_\varepsilon(x, v, \nabla v)|^{q'} \sigma^{-\frac{q'}{q}}\,dx\Big)^{1/q'} \Big(\int_\Omega |\varphi|^q \sigma\,dx\Big)^{1/q} \\ &\leq \frac{1}{\varepsilon}\Big(\int_\Omega \sigma^{1-q'}\,dx\Big)^{1/q'} \|\varphi\|_{q,\sigma} \leq c_\varepsilon |||\varphi||| \end{aligned} \tag{4.2}$$

For the above inequality, we have used (2.4) and (2.6).

Lemma 4.1 *The operator $A + G_\varepsilon \ : X \to X^*$ is bounded, coercive, hemicontinous ,and satisfies property (M).*

In view of Lemma 4.1, Problem (4.1) has a solution by a classical result [10, Theorem 2.1 and Remark 2.1]. Since g_ε verifies the sign condition and using (3.3), we obtain

$$\alpha \sum_{i=1}^{N} \int_\Omega w_i |\frac{\partial u_\varepsilon}{\partial x_i}|^p \leq \langle h, u_\varepsilon \rangle$$

i.e. $\alpha |||u_\varepsilon|||^p \leq c \|h\|_{X^*} |||u_\varepsilon|||$. Then

$$|||u_\varepsilon||| \leq \beta_0, \tag{4.3}$$

where β_0 is some positive constant. Hence, we can extract a subsequence still denoted by u_ε such that,

$$u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, w) \text{ and a.e. in } \Omega.$$

Step (2) Convergence of the positive part of u_ε. We shall prove that

$$u_\varepsilon^+ \to u^+ \text{ in } W_0^{1,p}(\Omega, w) \quad \text{strongly.}$$

Let $k > 0$. Define $u_k^+ = u^+ \wedge k = \min\{u^+, k\}$. We shall fix k, and use the notation

$$z_\varepsilon = u_\varepsilon^+ - u_k^+.$$

Assertion:

$$\limsup_{\varepsilon \to 0} \int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u_k^+)] \nabla (u_\varepsilon^+ - u_k^+)^+ \, dx \leq R_k \tag{4.4}$$

where $R_k \to 0$ as $k \to +\infty$. Indeed, by Lemmas 2.3 and 2.4, we have $z_\varepsilon \in W_0^{1,p}(\Omega, w)$ and $z_\varepsilon^+ \in W_0^{1,p}(\Omega, w)$. Multiplying (4.1) by z_ε^+ we obtain

$$\langle Au_\varepsilon, z_\varepsilon^+ \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) z_\varepsilon^+ \, dx = \langle h, z_\varepsilon^+ \rangle.$$

If $z_\varepsilon^+ > 0$, we have $u_\varepsilon > 0$ and from (3.4) $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \geq 0$, then $\langle Au_\varepsilon, z_\varepsilon^+ \rangle \leq \langle h, z_\varepsilon^+ \rangle$ i.e.

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla z_\varepsilon^+ \, dx \leq \langle h, z_\varepsilon^+ \rangle.$$

Since $u_\varepsilon = u_\varepsilon^+$ in $\{x \in \Omega : z_\varepsilon^+ > 0\}$ then

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon^+) \nabla z_\varepsilon^+ \, dx \leq \langle h, z_\varepsilon^+ \rangle.$$

Which implies

$$\begin{aligned} &\int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u_k^+)] \nabla (u_\varepsilon^+ - u_k^+)^+ \, dx \\ &\qquad \leq - \int_\Omega a(x, u_\varepsilon, \nabla u_k^+)] \nabla (u_\varepsilon^+ - u_k^+)^+ + \langle h, z_\varepsilon^+ \rangle. \end{aligned} \tag{4.5}$$

As $\varepsilon \to 0$, we have $z_\varepsilon^+ \to (u^+ - u_k^+)^+$ a.e. in Ω. However z_ε^+ is bounded in $W_0^{1,p}(\Omega, w)$; hence

$$z_\varepsilon^+ \rightharpoonup (u^+ - u_k^+)^+ \quad \text{in } W_0^{1,p}(\Omega, w).$$

Since $a(x, u_\varepsilon, \nabla u_k^+) \to a(x, u, \nabla u_k^+)$ in $\prod_{i=1}^N L^{p'}(\Omega, w_i^*)$, by passing to the limit in ε in (4.5), we obtain (4.4) with

$$R_k = -\int_\Omega a(x, u, \nabla u_k^+)]\nabla(u^+ - u_k^+)^+ + \langle h, (u^+ - u_k^+)^+\rangle.$$

Because $(u^+ - u_k^+)^+ \to 0$ in $W_0^{1,p}(\Omega, w)$ as $k \to \infty$, we have $R_k \to 0$ as $k \to \infty$.
Assertion:

$$-\liminf_{\varepsilon\to 0}\int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u_k^+)]\nabla(u_\varepsilon^+ - u_k^+)^- \, dx \le 0. \tag{4.6}$$

Indeed, we shall use the test function $v_\varepsilon = \varphi_\lambda(z_\varepsilon^-)$ with $\varphi_\lambda(s) = se^{\lambda s^2}$ in (4.1). We have $0 \le z_\varepsilon^- \le k$, i.e. $z_\varepsilon^- \in L^\infty(\Omega)$ and since $z_\varepsilon^- \in W_0^{1,p}(\Omega, w)$, hence by Lemma 2.2, we have $v_\varepsilon \in W_0^{1,p}(\Omega, w)$. Multiplying (4.1) by v_ε we obtain

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla z_\varepsilon^- \varphi'_\lambda(z_\varepsilon^-)\, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\, dx = \langle h, \varphi_\lambda(z_\varepsilon^-)\rangle.$$

Define

$$E_\varepsilon = \{x \in \Omega : u_\varepsilon^+(x) \le u_k^+(x)\} \quad \text{and} \quad F_\varepsilon = \{x \in \Omega : 0 \le u_\varepsilon(x) \le u_k^+(x)\}.$$

Since $\varphi_\lambda(z_\varepsilon^-) = 0$ in E_ε^c,

$$\int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\, dx = \int_{E_\varepsilon} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\, dx.$$

When $u_\varepsilon \le 0$, we have $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \le 0$ and since $\varphi_\lambda(z_\varepsilon^-) \ge 0$, we obtain

$$\begin{aligned}
&\int_{E_\varepsilon} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\, dx \\
&\le \int_{F_\varepsilon} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\, dx \\
&\le \int_{F_\varepsilon} b(|u_\varepsilon|)[\sum_{i=1}^N w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p + c(x)]\varphi_\lambda(z_\varepsilon^-)\, dx \\
&\le b(k)\int_{F_\varepsilon} [\sum_{i=1}^N w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p + c(x)]\varphi_\lambda(z_\varepsilon^-)\, dx \\
&\le \frac{b(k)}{\alpha}\int_{F_\varepsilon} a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla u_\varepsilon \varphi_\lambda(z_\varepsilon^-)\, dx + b(k)\int_{F_\varepsilon} c(x)\varphi_\lambda(z_\varepsilon^-).
\end{aligned}$$

As in [3, $Theorem 1.1$], we can show that

$$
\begin{aligned}
&-\frac{1}{2}\int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla u_k^+)]\nabla(u_\varepsilon^+ - u_k^+)^- \\
&\le \int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla u_\varepsilon^+)]\nabla u_k^+\varphi'_\lambda(u_k^+)\,dx + \langle -h, \varphi_\lambda(z_\varepsilon^-)\rangle \\
&\quad + \int_\Omega a(x,u_\varepsilon,\nabla u_k^+)\nabla z_\varepsilon^-\varphi'_\lambda(z_\varepsilon^-)\,dx + \frac{b(k)}{\alpha}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon^+)\nabla u_k^+\varphi_\lambda(z_\varepsilon^-)\,dx \\
&\quad + \frac{b(k)}{\alpha}\int_\Omega a(x,u_\varepsilon,\nabla u_k^+)\nabla(u_\varepsilon^+ - u_k^+)\varphi_\lambda(z_\varepsilon^-)\,dx + b(k)\int_\Omega c(x)\varphi_\lambda(z_\varepsilon^-)\,dx,
\end{aligned}
$$

for $\lambda = \frac{b(k)^2}{4\alpha^2}$. For short notation, we rewrite the above inequality as

$$I_{\varepsilon k} \le I_{\varepsilon k}^1 + I_{\varepsilon k}^2 + I_{\varepsilon k}^3 + I_{\varepsilon k}^4 + I_{\varepsilon k}^5.$$

Now, we extract a subsequence that satisfies the following two conditions:

$$a(x,u_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \gamma_1 \quad \text{and} \quad a(x,u_\varepsilon,\nabla u_\varepsilon^+) \rightharpoonup \gamma_2 \quad \text{in } \prod_{i=1}^N L^{p'}(\Omega, w_i^*)\,. \tag{4.7}$$

Lemma 4.2 *For k fixed, as $\varepsilon \to 0$, the following statements hold:*

(a) $I_{\varepsilon k}^1 \to I_k^1 = \int_\Omega [\gamma_1 - \gamma_2]\nabla u_k^+\varphi'_\lambda(u_k^+)\,dx + \langle -h, \varphi_\lambda((u^+ - u_k^+)^-)\rangle$

(b) $I_{\varepsilon k}^2 \to I_k^2 = \int_\Omega a(x,u,\nabla u_k^+)\nabla((u^+ - u_k^+)^-)\varphi'_\lambda((u^+ - u_k^+)^-)$

(c) $I_{\varepsilon k}^3 \to I_k^3 = \frac{b(k)}{\alpha}\int_\Omega \gamma_2\nabla u_k^+\varphi_\lambda((u^+ - u_k^+)^-)\,dx$

(d) $I_{\varepsilon k}^4 \to I_k^4 = \frac{b(k)}{\alpha}\int_\Omega a(x,u,\nabla u_k^+)\nabla(u^+ - u_k^+)\varphi_\lambda((u^+ - u_k^+)^-)\,dx$

(e) $I_{\varepsilon k}^5 \to I_k^5 = b(k)\int_\Omega c(x)\varphi_\lambda((u^+ - u_k^+)^-)\,dx$

In view of Lemma 4.2, $(u^+ - u_k^+)^- = 0$ and $\varphi_\lambda(0) = 0$, we have

$$\limsup_{\varepsilon\to 0} I_{\varepsilon k} \le I_k^1 + I_k^2 + I_k^3 + I_k^4 + I_k^5 = \int_\Omega [\gamma_1(x) - \gamma_2(x)]\nabla u_k^+\varphi'_\lambda(u_k^+)\,dx.$$

Moreover, if $u_\varepsilon < 0$ we have $(u_\varepsilon)_k^+ = 0$, hence,

$$(a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla u_\varepsilon^+))(u_\varepsilon)_k^+ = 0 \quad \text{a.e.}$$

which implies $(\gamma_1(x) - \gamma_2(x))u_k^+ = 0$, and so $\limsup_{\varepsilon\to 0} I_{\varepsilon k} \le 0$; thus, (4.6) follows.

Assertion:

$$u_\varepsilon^+ \to u^+ \quad \text{in } W_0^{1,p}(\Omega, w) \quad \text{strongly.} \tag{4.8}$$

As in [3, theorem 1.1], from (4.4)and (4.6), we have

$$\limsup_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon^+)-a(x,u_\varepsilon,\nabla u^+)]\nabla(u_\varepsilon^+-u^+) \leq R_k+\int_\Omega[\gamma_2(x)-a(x,u,\nabla u_k^+)]\nabla(u_k^+-u^+).$$

Letting $k\to\infty$ and using lemma 3.2 we obtain (4.8).

Step (3) Convergence of the negative part of u_ε. As in the preceding step, we shall prove that

$$u_\varepsilon^- \to u^- \quad \text{in } W_0^{1,p}(\Omega,w) \quad \text{strongly.} \tag{4.9}$$

Assertion:

$$\limsup_{\varepsilon\to 0}\int_\Omega -[a(x,u_\varepsilon,-\nabla u_\varepsilon^-)-a(x,u_\varepsilon,-\nabla u_k^-)]\nabla(u_\varepsilon^- - u_k^-)^+\,dx \leq \tilde{R}_k, \tag{4.10}$$

where $\tilde{R}_k\to 0$ as $k\to+\infty$. Indeed, when we define $u_k^- = u^-\wedge k$, $y_\varepsilon = u_\varepsilon^- - u_k^-$, and multiply (4.1) by y_ε^+, we obtain

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla y_\varepsilon^+\,dx+\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)y_\varepsilon^+\,dx=\langle h,y_\varepsilon^+\rangle.$$

Since $y_\varepsilon^+>0$ implies $u_\varepsilon<0$, from (3.4) we have $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\leq 0$. Hence $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)y_\varepsilon^+\leq 0$ a.e. in Ω. Then

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla y_\varepsilon^+\,dx\geq\langle h,y_\varepsilon^+\rangle.$$

Since $u_\varepsilon=-u_\varepsilon^-$ on the set $\{x\in\Omega: y_\varepsilon^+>0\}$, we can write

$$\int_\Omega a(x,u_\varepsilon,-\nabla u_\varepsilon^-)\nabla y_\varepsilon^+\,dx\geq\langle h,y_\varepsilon^+\rangle,$$

which implies

$$-\int_\Omega[a(x,u_\varepsilon,-\nabla u_\varepsilon^-)-a(x,u_\varepsilon,-\nabla u_k^-)]\nabla(u_\varepsilon^- - u_k^-)^+\,dx \leq \int_\Omega a(x,u_\varepsilon,-\nabla u_k^-)\nabla(u_\varepsilon^- - u_k^-)^+ - \langle h,y_\varepsilon^+\rangle.$$

As $\varepsilon\to 0$ we have $y_\varepsilon^+\to(u^- - u_k^-)^+$ a.e. in Ω. Since y_ε^+ is bounded in $W_0^{1,p}(\Omega,w)$, $y_\varepsilon^+\rightharpoonup(u^- - u_k^-)^+$ in $W_0^{1,p}(\Omega,w)$ (for k fixed). Passing to the limit in ε we obtain (4.10) with

$$\tilde{R}_k=\int_\Omega a(x,u,-\nabla u_k^-)\nabla(u^- - u_k^-)^+ - \langle h,(u^- - u_k^-)^+\rangle.$$

Because $(u^- - u_k^-)^+\to 0$ in $W_0^{1,p}(\Omega,w)$ as $k\to\infty$ we obtain that $\tilde{R}_k\to 0$ as $k\to\infty$.

Assertion:

$$\limsup_{\varepsilon\to 0}\int_\Omega [a(x,u_\varepsilon,-\nabla u_\varepsilon^-)-a(x,u_\varepsilon,-\nabla u_k^-)]\nabla(u_\varepsilon^- - u_k^-)^-\,dx\le 0\,. \tag{4.11}$$

This can be done as in (4.6) by considering a test function $v_\varepsilon=\varphi_\lambda(y_\varepsilon^-)$. Finally combining (4.10) and (4.11), we deduce as in (4.8) the assertion (4.9).
Step (4) Convergence of u_ε. From (4.8) and (4.9), we deduce that for a subsequence,

$$u_\varepsilon\to u \quad \text{in } W_0^{1,p}(\Omega,w) \quad \text{and a.e. in } \Omega \tag{4.12}$$

$$\nabla u_\varepsilon\to\nabla u \quad \text{a.e. in } \Omega, \tag{4.13}$$

which implies

$$\begin{gathered} g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\to g(x,u,\nabla u) \quad \text{a.e. in } \Omega\\ g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\to g(x,u,\nabla u)u \quad \text{a.e. in } \Omega. \end{gathered} \tag{4.14}$$

On the other hand, multiplying (4.1) by u_ε and using (3.3), (3.4), (4.2), (4.3) we obtain

$$0\le\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx\le\tilde\beta, \tag{4.15}$$

where $\tilde\beta$ is some positive constant. For any measurable subset E of Ω and any $m>0$, we have

$$\int_E |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx=\int_{E\cap X_m^\varepsilon}|g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx+\int_{E\cap Y_m^\varepsilon}|g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx$$

where

$$X_m^\varepsilon=\{x\in\Omega:|u_\varepsilon(x)|\le m\},\quad Y_m^\varepsilon=\{x\in\Omega:|u_\varepsilon(x)|>m\} \tag{4.16}$$

From this and (3.5),(4.15),(4.16), we have

$$\begin{aligned}\int_E |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx &\le\int_{E\cap X_m^\varepsilon}|g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx+\frac{1}{m}\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx\\ &\le b(m)\int_E(\sum_{i=1}^N w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p+c(x))+\tilde\beta\frac{1}{m}.\end{aligned}$$

Since the sequence (∇u_ε) converges strongly in $\prod_{i=1}^N L^p(\Omega,w_i)$, then above inequality implies the equi-integrability of $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$. Thanks to (4.14) and Vitali's theorem,

$$g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{4.17}$$

From (4.12) and (4.17) we can pass to the limit in

$$\langle Au_\varepsilon,v\rangle+\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)v=\langle h,v\rangle$$

and we obtain

$$\langle Au, v\rangle + \int_\Omega g(x,u,\nabla u)v = \langle h, v\rangle \quad \forall v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{4.18}$$

Moreover, since $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon \ge 0$ a.e. in Ω, by (4.14), (4.15) and Fatou's lemma, we have $g(x,u,\nabla u)u \in L^1(\Omega)$. It remains to show that,

$$\langle Au, u\rangle + \int_\Omega g(x,u,\nabla u)u = \langle h, u\rangle.$$

Put $v = u_k$ in (4.18) where u_k is the truncation of u. Then

$$\langle Au - h, u_k\rangle \to \langle Au - h, u\rangle$$

and

$$g(x,u,\nabla u)u_k \to g(x,u,\nabla u)u \text{ in } L^1(\Omega)\,.$$

Using Lebesgue's dominated convergence theorem, since

$$|g(x,u,\nabla u)u_k| \le |g(x,u,\nabla u)||u| \in L^1(\Omega)$$

we conclude that $g(x,u,\nabla u)u_k \to g(x,u,\nabla u)u$ a.e. in Ω.

Proof of Lemma 4.1 We set $B_\varepsilon = A + G_\varepsilon$. Using (3.1) and Hölder's inequality we can show that A is bounded [5]. Thanks to (4.2) we have B_ε bounded. The coercivity follows from (3.3) and (3.4). To show that B_ε is hemicontinous, let $t \to t_0$ and prove that

$$\langle B_\varepsilon(u+tv), \tilde w\rangle \to \langle B_\varepsilon(u+t_0v), \tilde w\rangle \text{ as } t \to t_0 \quad \text{for all } u, v, \tilde w \in X.$$

Since for a.e. $x \in \Omega$, $a_i(x, u+tv, \nabla(u+tv)) \to a_i(x, u+t_0v, \nabla(u+t_0v))$ as $t \to t_0$, thanks to the growth condition (3.1), Lemma 2.1 implies

$$a_i(x,u+tv,\nabla(u+tv)) \rightharpoonup a_i(x,u+t_0v,\nabla(u+t_0v)) \quad \text{in } L^{p'}(\Omega, w_i^{1-p'}) \quad \text{as } t \to t_0\,.$$

Finally for all $\tilde w \in X$,

$$\langle A(u+tv), \tilde w\rangle \to \langle A(u+t_0v), \tilde w\rangle \quad \text{as } t \to t_0.$$

On the other hand, $g_\varepsilon(x,u+tv,\nabla(u+tv)) \to g_\varepsilon(x,u+t_0v,\nabla(u+t_0v))$ as $t \to t_0$ for a.e. $x \in \Omega$. Also $(g_\varepsilon(x,u+tv+\nabla(u+tv)))_t$ is bounded in $L^{q'}(\Omega,\sigma^{1-q'})$ because

$$\int_\Omega |g_\varepsilon(x,u+tv,\nabla(u+tv))|^{q'}\sigma^{1-q'} \le (\frac{1}{\varepsilon})^{q'} \int_\Omega \sigma^{1-q'} \le c_\varepsilon,$$

then Lemma 2.1 gives

$$g_\varepsilon(x,u+tv,\nabla(u+tv)) \rightharpoonup g_\varepsilon(x,u+t_0v,\nabla(u+t_0v)) \text{ in } L^{q'}(\Omega,\sigma^{1-q'}) \text{ as } t \to t_0.$$

Since $\tilde{w} \in L^q(\Omega, \sigma)$ for all $\tilde{w} \in X$,

$$\langle G_\varepsilon(u + tv), \tilde{w}\rangle \to \langle G_\varepsilon(u + t_0 v), \tilde{w}\rangle \quad \text{as } t \to t_0.$$

Next we show that B_ε satisfies property (M); i.e. for a sequence u_j in X satisfying: (i) $u_i \rightharpoonup u$ in X, (ii) $B_\varepsilon u_j \rightharpoonup \chi$ in X^*, and (iii) $\limsup_{j\to\infty}\langle B_\varepsilon u_j, u_j - u\rangle \le 0$, we have $\chi = B_\varepsilon u$. Indeed, by Hölder's inequality and (2.6),

$$\begin{aligned}
&\int_\Omega g_\varepsilon(x, u_j, \nabla u_j)(u_j - u)\\
&\le \Big(\int_\Omega |g_\varepsilon(x, u_j, \nabla u_j)|^{q'} \sigma^{-q'/q}\, dx\Big)^{1/q'} \Big(\int_\Omega |u_j - u|^q \sigma\, dx\Big)^{1/q}\\
&\le \frac{1}{\varepsilon}\Big(\int_\Omega \sigma^{\frac{-q'}{q}}\, dx\Big)^{1/q'} \|u_j - u\|_{q,\sigma} \to 0 \quad \text{as } j \to \infty,
\end{aligned}$$

i.e., $\langle G_\varepsilon u_j, u_j - u\rangle \to 0$ as $j \to \infty$. Combining the last convergence with (iii), we obtain

$$\limsup_{j\to\infty}\langle A u_j, u_j - u\rangle \le 0.$$

And by the pseudo-monotonicity of A [5, Prop. 1], we have $Au_j \rightharpoonup Au$ in X^* and $\lim_{j\to\infty}\langle Au_j, u_j - u\rangle = 0$. On the other hand,

$$\begin{aligned}
0 &= \lim_{j\to\infty}\int_\Omega a(x, u_j, \nabla u_j)\nabla(u_j - u)\, dx\\
&= \lim_{j\to\infty}\int_\Omega (a(x, u_j, \nabla u_j) - a(x, u_j, \nabla u))\nabla(u_j - u)\, dx\\
&\quad + \int_\Omega a(x, u_j, \nabla u)\nabla(u_j - u)\, dx\,.
\end{aligned}$$

The last integral in the right hand tends to zero since $a(x, u_j, \nabla u) \to a(x, u, \nabla u)$ in $\prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ as $j \to \infty$; hence, by Lemma 3.2 we have $\nabla u_j \to \nabla u$ a. e. in Ω. Then

$$g_\varepsilon(x, u_j, \nabla u_j) \to g_\varepsilon(x, u, \nabla u) \quad \text{a.e. in } \Omega \quad \text{as } j \to \infty.$$

And since

$$|g_\varepsilon(x, u_j, \nabla u_j)\sigma^{\frac{1-q'}{q'}}| \le \frac{1}{\varepsilon}\sigma^{\frac{1-q'}{q'}} \in L^{q'}(\Omega) \quad \text{(due to (2.4)},$$

by Lebesgue's dominated convergence theorem, we obtain

$$g_\varepsilon(x, u_j, \nabla u_j) \to g_\varepsilon(x, u, \nabla u) \quad \text{in } L^{q'}(\Omega, \sigma^{1-q'}) \quad \text{as } j \to \infty,$$

which with (2.6) imply

$$\int_\Omega g_\varepsilon(x, u_j, \nabla u_j) v\, dx \to \int_\Omega g_\varepsilon(x, u, \nabla u) v\, dx \quad \text{as } j \to \infty, \quad \text{for all } v \in X,$$

i.e., $G_\varepsilon u_j \rightharpoonup G_\varepsilon u$ in X^*. Finally,

$$B_\varepsilon u_j = Au_j + G_\varepsilon u_j \rightharpoonup Au + G_\varepsilon u = B_\varepsilon u = \chi \text{ in } X^*.$$

Proof of Lemma 4.2 Part (a) follows from $\nabla\varphi_\lambda(u_k^+) \in \prod_{i=1}^N L^p(\Omega, w_i)$ and (4.7). Using Lemma 2.1, $\nabla(\varphi_\lambda(z_\varepsilon^-)) \rightharpoonup \nabla(\varphi_\lambda(u^+ - u_k^+)^-)$ in $\prod_{i=1}^N L^p(\Omega, w_i)$; then part (b) follows since $a(x, u_\varepsilon, \nabla u_k^+) \to a(x, u, \nabla u_k^+)$ in $\prod_{i=1}^N L^{p'}(\Omega, w_i^*)$.

To prove part (c), we have

$$\frac{\partial u_k^+}{\partial x_i}\varphi_\lambda(z_\varepsilon^-)w_i^{1/p} \to \frac{\partial u_k^+}{\partial x_i}\varphi_\lambda((u^+ - u_k^+)^-)w_i^{1/p} \quad \text{a.e. in } \Omega$$

and

$$|\frac{\partial u_k^+}{\partial x_i}\varphi_\lambda(z_\varepsilon^-)w_i^{1/p}|^p \le \tilde\beta|\frac{\partial u_k^+}{\partial x_i}w_i^{1/p}|^p \in L^1(\Omega),$$

where $\tilde\beta$ is a positive constants. Then, by Lebesgue's dominated convergence theorem we have

$$\frac{\partial u_k^+}{\partial x_i}\varphi_\lambda(z_\varepsilon^-) \to \frac{\partial u_k^+}{\partial x_i}\varphi_\lambda((u^+ - u_k^+)^-) \quad \text{in } L^p(\Omega, w_i),$$

i.e. $\nabla u_k^+\varphi_\lambda(z_\varepsilon^-) \to \nabla u_k^+\varphi_\lambda((u^+ - u_k^+)^-)$ in $\prod_{i=1}^N L^p(\Omega, w_i)$. Then by (4.7) we obtain part (c).

To prove part (d), we have

$$a_i(x, u_\varepsilon, \nabla u_k^+)\varphi_\lambda((u_\varepsilon^+ - u_k^+)^-)w_i^{\frac{1-p'}{p'}} \to a_i(x, u, \nabla u_k^+)\varphi_\lambda((u^+ - u_k^+)^-)w_i^{\frac{1-p'}{p'}}$$

a.e. in Ω, and

$$|a_i(x, u_\varepsilon, \nabla u_k^+)\varphi_\lambda((u_\varepsilon^+ - u_k^+)^-)w_i^{\frac{1-p'}{p'}}|^{p'} \le M|a_i(x, u_\varepsilon, \nabla u_k^+)|^{p'}w_i^{1-p'}.$$

Then the generalized Lebesgue's dominated convergence theorem implies

$$a_i(x, u_\varepsilon, \nabla u_k^+)\varphi_\lambda((u_\varepsilon^+ - u_k^+)^-) \to a_i(x, u, \nabla u_k^+)\varphi_\lambda((u^+ - u_k^+)^-) \quad \text{in } L^{p'}(\Omega, w_i^*).$$

Since $\nabla(u_\varepsilon^+ - u_k^+) \rightharpoonup \nabla(u^+ - u_k^+)$ in $L^p(\Omega, w_i)$ we conclude part (d). Part (e) follows from $|c(x)\varphi_\lambda((u^+ - u_k^+)^-)| \in L^1(\Omega)$ and Lebesgue's dominated convergence theorem.

5 Example

Some ideas of this example come from [5]. Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \ge 1)$, satisfying the cone condition. Let us consider the Carathéodory functions:

$$a_i(x, s, \xi) = w_i|\xi_i|^{p-1}\operatorname{sgn}(\xi_i) \quad \text{for } i = 1, \dots, N$$
$$g(x, s, \xi) = \operatorname{sgn}(s)\sum_{i=1}^N w_i|\xi_i|^p,$$

where $w_i(x)$ are a given weight functions strictly positive almost everywhere in Ω. We shall assume that the weight functions satisfy,

$$w_i(x) = w(x), \quad x \in \Omega, \quad \text{for all } i = 0, \dots, N.$$

Then, we consider the Hardy inequality (2.5) in the form,

$$\Big(\int_\Omega |u(x)|^q \sigma(x)\, dx\Big)^{1/q} \le c\Big(\int_\Omega |\nabla u(x)|^p w\Big)^{1/p}.$$

It is easy to show that the $a_i(x, s, \xi)$ are Carathéodory functions satisfying the growth condition (3.1) and the coercivity (3.3). Also the Carathéodory function $g(x, s, \xi)$ satisfies the conditions (3.4) and (3.5). On the other hand, the monotonicity condition is verified. In fact,

$$\begin{aligned}&\sum_{i=1}^{N}(a_i(x,s,\xi) - a_i(x,s,\hat{\xi}))(\xi_i - \hat{\xi}_i)\\ &\qquad = w(x)\sum_{i=1}^{N}(|\xi_i|^{p-1}\operatorname{sgn}\xi_i - |\hat{\xi}_i|^{p-1}\operatorname{sgn}\hat{\xi}_i)(\xi_i - \hat{\xi}_i) > 0\end{aligned}$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in \mathbb{R}^N$ with $\xi \neq \hat{\xi}$, since $w > 0$ a.e. in Ω. In particular, let us use the special weight functions w and σ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \operatorname{dist}(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\Big(\int_\Omega |u(x)|^q\, d^\mu(x)\, dx\Big)^{1/q} \le c\Big(\int_\Omega |\nabla u(x)|^p\, d^\lambda(x)\, dx\Big)^{1/p}.$$

The corresponding imbedding is compact if: (i) For, $1 < p \le q < \infty$,

$$\lambda < p - 1, \quad \frac{N}{q} - \frac{N}{p} + 1 \ge 0, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0, \tag{5.1}$$

(ii) For $1 \le q < p < \infty$,

$$\lambda < p - 1, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0, \tag{5.2}$$

(iii) For $q > 1$,

$$\mu(q' - 1) < 1\,. \tag{5.3}$$

Remarks.

1. Condition (5.1) or (5.2) are sufficient for the compact imbedding (2.6) to hold; see for example [5, Example 1], [6, Example 1.5], and [12, Theorems 19.17, 19.22].
2. Condition (5.3) is sufficient for (2.4) to hold [9, pp. 40-41].

Finally, the hypotheses of Theorem 3.1 are satisfied. Therefore, (1.2) has at least one solution.

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Y. Akdim (e-mail: y.akdim1@caramail.com)
E. Azroul (e-mail: elazroul@caramail.com)
A. Benkirane (e-mail: abdelmoujib@iam.net.ma)

Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, B.P. 1796 Atlas, Fès, Maroc.