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Convergence of a continuous BGK model for initial boundary-value problems for conservation laws *

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Abstract

We consider a scalar conservation law in the quarter plane. This equation is approximated in a continuous kinetic Bhatnagar-Gross-Krook (BGK) model. The convergence of the model towards the unique entropy solution is established in the space of functions of bounded variation, using kinetic entropy inequalities, without special restriction on the flux nor on the equilibrium problem's data. As an application, we establish the hydrodynamic limit for a 2×2 relaxation system with general data. Also we construct a new family of convergent continuous BGK models with simple maxwellians different from the χ models.

1 Introduction

We consider the initial boundary-value problem, for a one-dimensional scalar conservation law,

$$\partial_t u + \partial_x F(u) = 0, \tag{1.1}$$

for $(x,t) \in \mathbb{R} \times (0,T)$ and F a smooth flux function with the initial condition

$$u(x,0) = u^0(x), \tag{1.2}$$

for $x \in \mathbb{R}^+$. The boundary condition

$$u(0,t) = u_b(t) \quad \text{for } t \geq 0,$$

can not be assumed in the proper sense because this is not quite simply true. Our boundary condition will be formulated as a compatibility [2]

$$\sup\{\operatorname{sgn}(u(0,t) - u_b(t))(F(u(0,t)) - F(k))\} = 0, \tag{1.3}$$

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for $t \in [0,T]$, where $\mathrm{sgn}(u)$ is the sign of u and where the sup is taken over k lying between $u(0,t)$ and $u_b(t)$. We recall that $u_b(t)$ is the prescribed boundary condition and that (1.3) means that $u(0,t) = u_b(t)$ whenever the flow is incoming, i.e. $F'(u(0,t)) > 0$.

We will look at (1.1)-(1.3) as an equilibrium for the scalar Bhatnagar-Gross-Krook (BGK) model with (eventually) infinite set of velocities,

$$\partial_t f + a(\xi)\partial_x f = \frac{M_f - f}{\epsilon}, \tag{1.4}$$

where $(x,t) \in \mathbb{R} \times (0,T)$, ξ is in a measure space Ξ with measure $d\xi$, $f(x,t,\xi)$ is the unknown depending also on ϵ, $a(\xi)$ is the velocity and where:

$$M_f(x,t,\xi) = M(u^\epsilon(x,t),\xi), \quad u^\epsilon(x,t) = \int f(x,t,\xi)\,d\xi,$$

are the maxwellian or equilibrium state, and the first momentum or density respectively. In the next section, we will add some conditions on the maxwellian $M : \mathbb{R}\times\Xi \to \mathbb{R}$ so that (a subsequence of) u^ϵ converges to u, the unique entropy solution of (1.1)-(1.3), and f approaches from $M(u,\xi)$ when ϵ goes to zero.

This model will be supplemented by the initial and boundary data

$$f(x,0,\xi) = M(u^0(x),\xi), \tag{1.5}$$

for $(x,\xi) \in \mathbb{R}^+ \times \Xi$ and

$$f(0,t,\xi) = M(u_b(t),\xi) \quad \text{if} \quad a(\xi) > 0, \tag{1.6}$$

for $t \in [0,T]$. When $\epsilon \to 0$, $f(x,t,\xi)$ is intended to be near $M(u(x,t),\xi)$, so we naturally assumed the initial-boundary data at equilibrium. Let us also recall about the boundary condition that, when $\Xi = \{a_1,\dots,a_N\}$ is finite, one must add l linear boundary conditions of the form

$$E(f_1(0,t),\dots,f_N(0,t))^t = G(t),$$

where E is a $l \times N$ matrix, G is a l-component given function and l is the number of positive velocities a_i [10]. (1.6) is a good way to express this fact in our circumstances as well as it is nothing but (1.3) for the scalar transport equation (1.4) for fixed ξ at the equilibrium $M_f = f$.

Our main task in this work is to show that the model (1.4)-(1.6) describes the problem (1.1)-(1.3) when ϵ goes to zero. This will be made in a bounded variation (BV) framework, for general flux F and general BV-initial-boundary data u^0 and u_b while comparing with [25] and [30] respectively.

Our work appears in the more general setting of the relaxation which was deeply studied last years in its theoretical and numerical aspects . We can cote in the Cauchy problem case [1, 5, 6, 8, 12, 14, 17, 19, 27, 28, 29] and one can see [20] and references therein for more information.

For relaxation with boundary condition, there is an important $BV \cap L^\infty$ analysis in [30], especially when the initial boundary data are some small perturbations of a constant non-transonic state. This rather restrictive conditions will be removed in Example 7.1 where we show in fact that the 2×2 system of [30] describes the equilibrium law (1.1) for general BV data.

In studying boundary value problem, the early relaxation stability conditions for Cauchy problems may fail to imply the existence of the hydrodynamic limit, boundary layers can appear and there are cases where the equilibrium system must be supplemented by proper boundary conditions to determine the uniqueness in the limiting process. Such questions are treated in [16, 23, 24, 31].

Relaxation schemes for conservation laws in the quarter plan can be found in [3, 7].

Concerning kinetic BGK models with continuous cite of velocities with the maxwellian χ, the Cauchy problem is studied in [26], see also [4] and reference therein.

A weak entropy study of the initial boundary problem can be found in [25] where the authors established the hydrodynamic limit in several space dimensions but with a restriction on the flux which must be convex, concave, non-increasing or non-decreasing. Our technique in recovering boundary entropy condition allows us to remove this restriction.

Other works deal with BGK model in the quarter plan with finite cite of velocities. In [22], the authors treat BGK model with two velocities . An extension of their techniques to more than two velocities is in [18]. But the extension of this techniques to continuous BGK model case seems difficult, especially in bounding the variation in space variable x. We overcome this difficulty by using both [25] and [22] ideas.

Let us recall that boundary conditions carry on supplementary complications in such approximation problems. We must not only impose correct conditions for the well-posedness of the conservation law (1.1) and the model (1.4), but we must also try to avoid the apparition of boundary layers. We chose the condition (1.3) emanating from parabolic viscosity approach of the approximated conservation law (1.3). There is another approach to exhibit correct boundary conditions by solving Riemann problems. These two formulations are equivalent for linear systems and scalar conservation laws [9, 13]. Concerning the BGK model, we chose the simplest way to write boundary conditions by respecting the ideas giving well-posedness of linear systems with finite cite of velocities as in [10, 15], on the one hand and by foreseeing the equilibrium phenomenon on the boundary on the other hand.

We will see throughout this paper that the monotony and the momentum equations of the maxwellian M still give the BV compactness and the stability respectively, exactly as in the Cauchy problem case. We also imitate the Cauchy situation in using an infinite set of kinetic entropy inequalities [4, 21, 26].

The paper is organized as follows. In the next section we specify some general and basic facts about equilibrium law, maxwellian and kinetic entropies. We study the well-posedness of the BGK model in section 3. Sections 4 and 5 are devoted respectively to L^∞ and BV stability estimates. In section 6 we prove that our kinetic BGK model describes the initial conservation law (1.1) by L^1 compactness in BV and using kinetic entropy H-functions with careful treatment of the calculus near the boundary. The last section contains two examples, namely the convergence of the relaxation 2×2 system of [30] with general initial-boundary data and a continuous BGK model with maxwellian distinct from the χ one.

2 General setting

Let us specify the meaning of (weak entropy) solutions of (1.1)-(1.3) and some needful technical assumptions.

Definition 2.1 Let $u^0 \in L^1(\mathbb{R}^+)$ and $u_b \in L^1(0,T)$. We say that a function $u \in BV(\mathbb{R}^+ \times (0,T))$ is a solution of (1.1)-(1.3) if for all $k \in \mathbb{R}$ and all nonnegative test function $\phi \in C_c^1(\mathbb{R}^+ \times [0,T))$ we have

$$\begin{aligned}&\int |u-k|\partial_t\phi + \operatorname{sgn}(u-k)(F(u)-F(k))\partial_x\phi\,dx\,dt\\ &+\int |u^0(x)-k|\phi(x,0)\,dx + \int \operatorname{sgn}(u_b(t)-k)(F(u(0,t)-F(k)).\phi(0,t)\,dt\\ &\geq 0.\end{aligned}$$

Here, and throughout this paper, BV stands for the space of the functions of bounded variation, $u(0,t)$ for the trace of the function u on the boundary $x=0$ and $u(x,0)$ for the trace of u on $t=0$. Such traces are well defined whenever u is of bounded variation (see [2]). Moreover, until opposite indication, we write

$$\int q\,dm_1\dots dm_n$$

instead of

$$\int_\Omega q(x_1,\dots,x_n)\,dm_1\dots dm_n = \int_{\omega_1}\dots\int_{\omega_n} q(x_1,\dots,x_n)dm_1(x_1)\dots dm_n(x_n),$$

where the measure dm_i is defined on the space ω_i, $\Omega = \omega_1 \times \dots \times \omega_n$ and $q \in L^1(\Omega)$. In the same way, an integration on a subspace ω of Ω will be written as

$$\int_\omega q\,dm_1\dots dm_n.$$

It is well known that the initial boundary value problem (1.1)-(1.3) admits a unique solution described in definition 2.1, see [2, 3].

Concerning the BGK model, we used [4] to construct ours. We will not go back on Bouchut's technical details, but we just recall axioms for the equilibrium state $M : \mathbb{R} \times \Xi \to \mathbb{R}$ and for kinetic entropies.

We postulate that $M = M(u,\xi)$ is smooth and monotone in $u \in \mathbb{R}$ for all $\xi \in \Xi$ and satisfies habitual moment equations, that is:

$$M(.,\xi) \text{ is nondecreasing for all } \xi, \tag{2.1}$$
$$\textstyle\int M(u,\xi)\,d\xi = u \quad \text{for all } u \in \mathbb{R}, \tag{2.2}$$
$$\textstyle\int a(\xi)M(u,\xi)\,d\xi = F(u) \quad \text{for all } u \in \mathbb{R}\quad . \tag{2.3}$$

For technical reasons, we impose $a \in L^1(\Xi)$ and we prevent $t = 0$ and $x = 0$ to be characteristics in (1.4). This can be written as:

$$a(\xi) \in [-a_\infty, 0[\cup]0, a_\infty], \text{ for all } \xi \in \Xi,$$

whit a positive real a_∞.

Our infinite set of convex entropies will be the Kruzkov's one. Such an entropy is written:

$$\eta_k(u) = |u - k|,$$

and its associated flux is:

$$G_k(u) = \operatorname{sgn}(u-k)(F(u) - F(k)).$$

Consider now, for any $k \in \mathbb{R}$, the kinetic entropy given by:

$$H_k(f, \xi) = |f - M(k, \xi)|,$$

for $f \in \mathbb{R}$ and $\xi \in \Xi$. This kinetic entropy is of course convex in f and one can easily check, using (2.1)-(2.3), that:

$$\int H_k(M(u,\xi),\xi)\, d\xi = \eta_k(u), \tag{2.4}$$

$$\int a(\xi) H_k(M(u,\xi),\xi)\, d\xi = G_k(u), \tag{2.5}$$

$$\int H_k(M(u^f,\xi),\xi)\, d\xi \le \int H_k(f(\xi),\xi)\, d\xi, \tag{2.6}$$

for all $u \in \mathbb{R}$, for $f : \Xi \to \mathbb{R}$ and for

$$u^f = \int f(\xi)\, d\xi.$$

These properties will allow us to obtain the Lax entropy inequalities in the hydrodynamic limit. Indeed, multiplying (1.4) by $\operatorname{sgn}(f(x,t,\xi) - M(k,\xi))$ and using the convexity of $H_k(.,\xi)$, yields:

$$\partial_t H_k(f,\xi) + a(\xi)\partial_x H_k(f,\xi) \le \frac{H_k(M_f,\xi) - H_k(f,\xi)}{\epsilon}.$$

Then integrating with respect to ξ and using (2.6), we obtain

$$\partial_t \int H_k(f(x,t,\xi),\xi)\, d\xi + \partial_x \int a(\xi) H_k(f(x,t,\xi),\xi)\, d\xi \le 0. \tag{2.7}$$

Suppose that u^ϵ converges to $u \in BV$, and remember that $f = f^\epsilon$ is devoted to be close to $M(u,\xi)$ at equilibrium. So, let $\epsilon \to 0$ in (2.7) and use (2.4)-(2.5) to end up with

$$\partial_t \eta_k(u) + \partial_x G(u) \le 0.$$

That is u is a Lax-entropy solution of (1.1) disregarding the boundary condition. But this is not sufficient to give uniqueness in our framework. Later, we will deeply develop (2.7) to reach the boundary entropy inequality (1.3).

3 The BGK model

Let us show first that the kinetic problem defined by (1.4)-(1.6) is well-posed in $L^\infty((0,T);L^1(\mathbb{R}^+\times\Xi))$. To do this, we rewrite (1.1) in an equivalent integral form by using Duhamel's principle; and use a Banach fixed point argument. Because of the boundary data, the quarter plan is divided into two zones for positive $a(\xi)$. We are brought to consider the sets:

$$\begin{aligned} Q_- &= \{(x,t,\xi)\in\mathbb{R}^+\times(0,T)\times\Xi;\ x<a(\xi)t\},\\ Q_+ &= \{(x,t,\xi)\in\mathbb{R}^+\times(0,T)\times\Xi;\ x\geq a(\xi)t\}. \end{aligned}$$

The integral form of the model is

$$\begin{aligned} f(x,t,\xi) &= f(x-a(\xi)t,0,\xi)e^{-t/\epsilon} \qquad (3.1)\\ &\quad +1/\epsilon\int_0^t M(u^\epsilon(x+(s-t)a(\xi),s),\xi)e^{(s-t)/\epsilon}\,ds \quad \text{in } Q_+\\ f(x,t,\xi) &= f(0,t-\frac{x}{a(\xi)},\xi)e^{-\frac{x}{\epsilon a(\xi)}} \qquad (3.2)\\ &\quad +1/\epsilon\int_{t-\frac{x}{a(\xi)}}^t M(u^\epsilon(x+(s-t)a(\xi),s),\xi)e^{(s-t)/\epsilon}\,ds \quad \text{in } Q_-. \end{aligned}$$

Theorem 3.1 *If M satisfies (2.1)-(2.2), $u^0\in L^1(\mathbb{R}^+)$ and $u_b\in L^1(0,T)$ then the BGK model (1.4)-(1.6) has a unique solution $f(x,t,\xi)\in L^\infty((0,T);L^1(\mathbb{R}^+\times\Xi))$ given by (3.1)-(3.2). This solution depends continuously on u^0 and u_b.*

Proof. We look for such a solution in $L^\infty((0,T);L^1(\mathbb{R}^+\times\Xi))$ for fixed and arbitrary positive time T. Let f and g be given by (3.1)-(3.2) and emanating from the initial-boundary data (u^0,u_b) and (v^0,v_b) respectively. We have:

$$\begin{aligned} \int|f-g|(x,t,\xi)\,dx\,d\xi &\leq e^{-t/\epsilon}\int_{x\geq at}|f(x-at,0,\xi)-g(x-at,0,\xi)|\,dx\,d\xi\\ &\quad +\int_{x<at}|f(0,t-\frac{x}{a},\xi)-g(0,t-\frac{x}{a},\xi)|e^{-\frac{x}{\epsilon a}}\,dx\,d\xi\\ &\quad +1/\epsilon\int_{0<s<t}|M(u^\epsilon,\xi)-M(v^\epsilon,\xi)|e^{(s-t)/\epsilon}\,dx\,d\xi\,ds, \end{aligned}$$

where

$$a=a(\xi),\quad u^\epsilon=u^\epsilon(x+(s-t)a(\xi),s),\quad v^\epsilon=v^\epsilon(x+(s-t)a(\xi),s).$$

Let us use simple changes of variables, (2.1) and (2.2) to get:

$$\begin{aligned} \int|f-g|(x,t,\xi)\,dx\,d\xi &\leq e^{-t/\epsilon}|f^0-g^0|_{L^1(\mathbb{R}^+\times\Xi)}+a_\infty|f_b-g_b|_{L^1((0,T)\times\Xi)}\\ &\quad +1/\epsilon\int_{0<s<t}|u^\epsilon(x,s)-v^\epsilon(x,s)|e^{(s-t)/\epsilon}\,dx\,ds, \end{aligned}$$

where

$$f^0(x,\xi) = f(x,0,\xi), \quad g^0(x,t) = g(x,0,\xi),$$
$$f_b(t,\xi) = f(0,t,\xi), \quad g_b(t,\xi) = g(0,t,\xi).$$

Now, replace u^ϵ and v^ϵ by their integral form to have:

$$\begin{aligned}\int |f-g|(x,t,\xi)\,dx\,d\xi \quad &\leq \quad e^{-t/\epsilon}|f^0 - g^0|_{L^1(\mathbb{R}^+\times\Xi)} + a_\infty |f_b - g_b|_{L^1((0,T)\times\Xi)} \\ &\quad + (1-e^{-t/\epsilon}) \sup_{0\leq s\leq t} |f(x,s,\xi) - g(x,s,\xi)|_{L^1(\mathbb{R}^+\times\Xi)}.\end{aligned}$$

This inequality allows us to construct, via (3.1)-(3.2), a sequence (f_m) converging to the required solution by fixed point techniques. It shows in addition that:

$$\sup_{0\leq t\leq T} |f-g|_{L^1(\mathbb{R}^+\times\Xi)} \leq |u^0 - v^0|_{L^1(\mathbb{R}^+)} + e a_\infty |u_b - v_b|_{L^1(0,T)},$$

that is the solution depends continuously on the initial-boundary data. This last inequality will be revisited in a different way by permuting f and g in proposition 4.3 below.

4 L^∞ estimates

In this section we shall present some a priori estimates yielding a maximum principle for the BGK model. Let us begin by recalling a useful lemma:

Lemma 4.1 *Let f and g be two weak solutions to the following liner equations*

$$\partial_t f + a\partial_x f = m(x,t),$$
$$\partial_t f + a\partial_x f = n(x,t),$$

for $(x,t) \in \mathbb{R}\times(0,T)$. Then we have:

$$\partial_t [f-g]_+ + a\partial_x [f-g]_+ \leq H(f-g)(m(x,t) - n(x,t)),$$

where H is the usual Heaviside function.

Proof. Use Kruzkov's techniques.

Our main tool in this section will be the following lemma.

Lemma 4.2 *Under assumptions (2.1)-(2.2), if f and g are two solutions of the model (1.4), emanating from two initial-boundary data, then*

$$\partial_t \int [f-g]_+\,d\xi + \partial_x \int a(\xi)[f-g]_+\,d\xi \leq 0.$$

Proof. Write (1.4) respectively for f and g then apply lemma 4.1 to get:

$$\partial_t [f-g]_+ + a(\xi)\partial_x [f-g]_+ \le 1/\epsilon H(f-g)(M_f - M_g - f + g).$$

After integration on ξ, we find

$$\partial_t \int [f-g]_+ \, d\xi + \partial_x \int a(\xi)[f-g]_+ \, d\xi \le 1/\epsilon I,$$

with

$$I = \int H(f-g)(M(u^\epsilon,\xi) - M(v^\epsilon,\xi) - f + g)\, d\xi,$$

where u^ϵ is the first momentum of f and so is v^ϵ for g. Notice that if $u^\epsilon \le v^\epsilon$, then I will be trivially non-positive. If $u^\epsilon \ge v^\epsilon$ then

$$\begin{aligned} I &\le \int M(u^\epsilon,\xi) - M(v^\epsilon,\xi)\, d\xi - \int f - g \, d\xi + \int_{f\le g} f - g\, d\xi \\ &= \int_{f \le g} f - g \, d\xi \le 0. \end{aligned}$$

Now, we establish a comparison result:

Proposition 4.3 *Suppose we have (2.1)-(2.2). Let f and g be two solutions of the model (1.4), emanating respectively from (u^0, u_b) and (v^0, v_b) as L^1-initial-boundary data. Fix $L \ge 0$ and $T \ge 0$. We have:*

$$\int_0^L \int [f(x,T,\xi) - g(x,T,\xi)]_+ \, d\xi \, dx \le \int_0^{L+a_\infty T} [u^0 - v^0]_+ \, dx + a_\infty \int_0^T [u_b - v_b]_+ \, dt.$$

Proof. To be clear, we give a formal proof which is valid for smooth solutions. We integrate the inequality in lemma 4.2 in the domain $D = \{(x,t) \in \mathbb{R}^+ \times (0,T) : 0 < x < a_\infty(T-t) + L\}$ and use (1.5)-(1.6) to have:

$$\int_0^L \int [f(x,T,\xi) - g(x,T,\xi)]_+ \, d\xi \, dx \le I \tag{4.1}$$

with

$$\begin{aligned} I &= \int_0^{L+a_\infty T} \int [M(u^0,\xi) - M(v^0,\xi)]_+ \, d\xi \, dx \\ &\quad + \int_0^T \int a_\infty [M(u_b,\xi) - M(v_b,\xi)]_+ \, d\xi \, dt. \end{aligned}$$

Using successively (2.1) and (2.2) yields

$$\begin{aligned} I &= \int_{[0,L+a_\infty T]\cap\{u^0 \ge v^0\}} \int M(u^0,\xi) - M(v^0,\xi)\, d\xi \, dx \\ &\quad + \int_{(0,T)\cap\{u_b \ge v_b\}} \int a_\infty M(u_b,\xi) - M(v_b,\xi)\, d\xi \, dt \\ &= \int_0^{L+a_\infty T} [u^0 - v^0]_+ \, dx + a_\infty \int_0^T [u_b - v_b]_+ \, dt. \end{aligned}$$

In order to be rigorous, when the solutions are not smooth and when all the inequalities in the above lemmas are in the distributional sense, we must use a sequence of test functions to approximate the characteristic function of the domain D to get (4.1).

The main consequence of proposition 4.3 is the maximum principle which we express, under notations and assumptions of this proposition:

Corollary 4.4 *If $u^0 \leq v^0$ and $u_b \leq v_b$ then $f \leq g$. Especially, if u^0 and u_b are bounded, then*

$$M(\min\{\inf u^0, \inf u_b\}, \xi) \leq f \leq M(\max\{\sup u^0, \sup u_b\}, \xi)$$

which is integrated on ξ as

$$\min\{\inf u^0, \inf u_b\} \leq u^\epsilon \leq \max\{\sup u^0, \sup u_b\},$$

for the density u^ϵ.

Proof. The first part is obvious and leads to the second one by using the particular constant solutions:

$$f = M(\min\{\inf u^0, \inf u_b\}, \xi) \quad \text{and} \quad f = M(\max\{\sup u^0, \sup u_b\}, \xi).$$

Let us close this section with the following statement.

Theorem 4.5 *If M satisfies (2.1)-(2.2), $u^0 \in L^1 \cap L^\infty$ and $u_b \in L^\infty(0,T)$, then the model (1.4)-(1.6) admits a unique solution f in $L^\infty((0,T); L^1(\mathbb{R}^+ \times \Xi))$ for each T and ϵ. For fixed T, these solutions and their associated first momentums u^ϵ are bounded, independently of ϵ, in $L^\infty(\mathbb{R}^+ \times (0,T) \times \Xi)$ and in $L^\infty(\mathbb{R}^+ \times (0,T))$ respectively. Moreover, when u^0 and u_b are smooth with $u^0(0) = u_b(0)$, then $f(.,.,\xi)$ is smooth outside of the characteristic $\{x = a(\xi)t\}$.*

The proof of this theorem follows from theorem 3.1, corollary 4.4 and standard regularity arguments.

Remark 4.1 Our next task is to establish some BV estimates. We shall deal only with the case of smooth data, since the proposition 4.3 yields a L^1 contraction result permitting to approach BV solutions by smooth ones and to extend these estimates to the BV case. Equally, if we don't specify anything, all the subsequent proofs are written for smooth case, letting to the reader the care to recover the general results by using the L^1 contraction.

5 BV estimates

Following proposition 4.3 and remark 4.1, we suppose throughout this section, except in theorem 5.3, that u^0 and u_b are smooth. The solutions of the BGK model and their associated densities will be smooth to.

Lemma 5.1 *If M satisfies (2.1)-(2.2) and if u^0 and u_b are of bounded variation, then the solution f of the model satisfies, for every t*

$$\int |\partial_t f(x,t,\xi)|\,d\xi\,dx \le a_\infty(TV(u^0)+TV(u_b)),$$

that is to say that its density satisfies

$$\int |\partial_t u^\epsilon(x,t)|\,dx \le a_\infty(TV(u^0)+TV(u_b)).$$

Proof. Derive (1.4) with respect to t, multiply it by $sg(\partial_t f)$ and integrate in ξ to have:

$$\partial_t \int |\partial_t f|\,d\xi + \partial_x \int a(\xi)|\partial_t f|\,d\xi = \int sg(\partial_t f) M_u(u^\epsilon,\xi)\,d\xi \int \partial_t f\,d\xi - \int |\partial_t f|\,d\xi.$$

However, the hypothesis on M gives

$$|\int sg(\partial_t f) M_u(u^\epsilon,\xi)\,d\xi| \le \int M_u(u^\epsilon,\xi)\,d\xi = 1,$$

from where

$$\partial_t \int |\partial_t f|\,d\xi + \partial_x \int a(\xi)|\partial_t f|\,d\xi \le 0.$$

Integrating in the domain $D = \{(x,s)\in\mathbb{R}^+\times(0,t) : 0 < x < a_\infty(t-s)+L\}$, we obtain,

$$\begin{aligned}
&\int_0^L \int |\partial_t f|\,d\xi(x,t)\,dx \\
&\le \int_0^{L+a_\infty t} \int |\partial_t f|\,d\xi(x,0)dx + \int_0^t \int_{a(\xi)>0} a(\xi)|\partial_t f|\,d\xi(0,s)ds.
\end{aligned}$$

But we can easily see that

$$\partial_t f(x,0,\xi) = -a(\xi)\partial_x M(u^0(x),\xi)$$

and $\partial_t f(0,t,\xi) = \partial_t M(u_b(t),\xi)$ when $a(\xi) > 0$. We conclude by using (2.1)-(2.2).

Lemma 5.2 *If M satisfies (2.1)-(2.2) and if u^0 and u_b are of bounded variation, then the solution f of the model satisfies, for every t*

$$\int |\partial_x f(x,t,\xi)|\,d\xi\,dx \le K,$$

where K is a constant independent of ϵ. That is to say that its density satisfies

$$\int |\partial_x u^\epsilon(x,t)|\,dx \le K,$$

for every t.

Proof. Let us reconsider (3.1)-(3.2) to write

$$\int |\partial_x f(x,t,\xi)|\, d\xi\, dx = I + J$$

with

$$\begin{aligned} I &= \int_{x<at} |\partial_x f|\, dx\, d\xi \le I_1 + I_2 + I_3 + I_4 \\ J &= \int_{x>at} |\partial_x f|\, dx\, d\xi \le J_1 + J_2 \end{aligned}$$

where $a = a(\xi)$ and

$$\begin{aligned} I_1 &= \int_{x<at} |M(u_b(t-\frac{x}{a}),\xi)| \frac{1}{\epsilon a} e^{-\frac{x}{\epsilon a}}\, dx\, d\xi, \\ I_2 &= \int_{x<at} \frac{1}{a} |u_b{}'(t-\frac{x}{a})| M_u(u_b(t-\frac{x}{a}),\xi)\, dx\, d\xi, \\ I_3 &= \int_{x<at} \int_{t-\frac{x}{a}<s<t} |\partial_x M(u^\epsilon(x+a(s-t),s),\xi)| 1/\epsilon e^{\frac{s-t}{\epsilon}}\, ds\, dx\, d\xi, \\ I_4 &= \int_{x<at} |M(u^\epsilon(0,t-\frac{x}{a}),\xi)| \frac{1}{\epsilon a} e^{-\frac{x}{\epsilon a}}\, dx\, d\xi, \\ J_1 &= \int_{at<x} |u^{0\prime}(x-at)| M_u(u^0(x-at),\xi)\, dx\, d\xi, \\ J_2 &= \int_{at<x} \int_{0<s<t} |\partial_x M(u^\epsilon(x+a(s-t),s),\xi)| 1/\epsilon e^{\frac{s-t}{\epsilon}}\, ds\, dx\, d\xi. \end{aligned}$$

Use simple change of variables and (2.1)-(2.2) to get successively:

$$\begin{aligned} I_1 &\le \int_{0<x<t} (|M(u_b(t-x),\xi) - M(0,\xi)| + |M(0,\xi)|) 1/\epsilon e^{-\frac{x}{\epsilon}}\, d\xi\, dx \\ &\le |u_b|_\infty + \int |M(0,\xi)|\, d\xi \\ I_2 &\le \int_{0<x<t} |u_b{}'(t-x)| M_u(u_b((t-x),\xi)\, d\xi\, dx \\ &\le TV(u_b) \\ I_4 &\le \int_{0<x<t} (|M(u^\epsilon(0,t-x),\xi) - M(0,\xi)| + |M(0,\xi)|) \frac{1}{\epsilon} e^{-\frac{x}{\epsilon}}\, d\xi\, dx \\ &\le |u^\epsilon|_\infty + \int |M(0,\xi)|\, d\xi \\ J_1 &\le \int_{0<x} |u^{0\prime}(x)| M_u(u^0(x),\xi)\, d\xi\, dx \\ &\le TV(u^0) \\ I_3 + J_2 &\le \int_{0<s<t} \int |\partial_x u^\epsilon(x,s)| M_u(u^\epsilon(x,s),\xi) 1/\epsilon e^{\frac{s-t}{\epsilon}}\, d\xi\, dx\, ds \end{aligned}$$

$$\leq \int_{0<s<t} \int |\partial_x f(x,s,\xi)| 1/\epsilon e^{\frac{s-t}{\epsilon}} \, d\xi \, dx \, ds$$

Let us summarize these inequalities in the following one

$$\int |\partial_x f(x,t,\xi)| \, d\xi \, dx \leq K + \int_{0<s<t} \int |\partial_x f(x,s,\xi)| 1/\epsilon e^{\frac{s-t}{\epsilon}} \, d\xi \, dx \, ds,$$

for a generic constant K independent of ϵ. Use now Granwall's lemma to end this proof.

We can sum up our main BV estimates, in the general BV-data case, as follows:

Theorem 5.3 *If M satisfies (2.1)-(2.2), and if u^0 and u_b are of bounded variation, then there exits K independent of ϵ, such that*

$$|u^\epsilon|_{BV(\mathbb{R}^+ \times (0,T))} \leq K.$$

Proof. This result is straightforward according to lemmas 5.1 and 5.2 in the case of smooth data. Nevertheless, by virtue of the L^1 contraction result essentially expressed in proposition 4.3, weak solutions with any BV initial boundary data, may be readily be constructed as L^1 limits of classical solutions. This completes the proof.

6 Convergence result

In this section, we prove that the sequence of the densities u^ϵ converges, as $\epsilon \to 0$, to the unique entropy solution of the initial-boundary value problem (1.1)-(1.3) specified in definition 2.1. Beside the stability results of the two previous sections, we have to show that the solutions of the BGK model satisfy kinetic entropy inequalities. But, when we will want to pass to the limit on ϵ, we will have need of

Lemma 6.1 *Under the assumptions of theorem 5.3, the distance to the equilibrium is controlled by*

$$\int |M_f - f| \, d\xi \, dx \, dt \leq K\epsilon,$$

with K independent of ϵ.

Proof. Derive (1.4) with respect to t, use Duhamel's principle for $M_f - f$ and (2.1)-(2.2) to obtain

$$\int |M_f - f| \, d\xi \leq 2a_\infty \int_{0<s<t} |\partial_x f(x,s,\xi)| e^{(s-t)/\epsilon} \, d\xi \, ds,$$

which achieves this proof with lemma 5.2.

Now, let us revisit the kinetic entropy framework of section 2. Under the same notations used over their, we have:

Lemma 6.2 *Let (2.1)-(2.2) be satisfied, let u^0 and u_b be of bounded variation and let f be the solution of the BGK model. Then for every $\phi \in C_c^1(\mathbb{R}^+ \times [0,T[)$, with $\phi \geq 0$, and for every $k \in \mathbb{R}$, there holds*

$$\begin{aligned}
&\int H_k(f(x,t,\xi),\xi)(\partial_t\phi(x,t) + a(\xi)\partial_x\phi(x,t))\,d\xi\,dx\,dt \\
&+ \int H_k(M(u^0(x),\xi),\xi)\phi(x,0)\,d\xi\,dx + \int a(\xi)H_k(M(u_b(t),\xi),\xi)\phi(0,t)\,d\xi\,dt \\
&+ \int_{a(\xi)<0} a(\xi)\partial_f H_k(M(u_b(t),\xi),\xi)(f(0,t,\xi) - M(u_b(t),\xi))\phi(0,t)\,d\xi\,dt \\
&\geq 0
\end{aligned}$$

where $\partial_f H_k(f,\xi)$ is a subdifferential of the convex function $H_k(.,\xi)$.

Proof. Let us multiply (1.4) by $\mathrm{sgn}(f - M(k,\xi))\phi(x,t)$, use the convexity of H_k, and integrate over ξ, using (2.6) as in section 2, to recapitulate (2.7) in the distributional sense. Then integrate in the domain $\mathbb{R}^+ \times (0,T)$ to get:

$$\begin{aligned}
\int & H_k(f(x,t,\xi),\xi)(\partial_t\phi(x,t) + a(\xi)\partial_x\phi(x,t))\,d\xi\,dx\,dt \\
&+ \int H_k(M(u^0(x),\xi),\xi)\phi(x,0)\,d\xi\,dx \\
&+ \int_{a(\xi)>0} a(\xi)H_k(M(u_b(t),\xi),\xi)\phi(0,t)\,d\xi\,dt \\
&+ \int_{a(\xi)<0} a(\xi)H_k(f(0,t,\xi),\xi)\phi(0,t)\,d\xi\,dt \;\geq\; 0.
\end{aligned}$$

Make so that the integral on Ξ appears in the third line and use the definition of the subdifferential of a convex function to conclude.

These calculations are valid for general convex entropies H, in our Kruskove's entropies H_k case, we have:

Corollary 6.3 *Under the assumptions of lemma 6.2 and (2.3), we have:*

$$\begin{aligned}
&\int H_k(f,\xi)(\partial_t\phi + a\partial_x\phi)\,d\xi\,dx\,dt + \int H_k(M^0,\xi)\phi(x,0)\,d\xi\,dx \\
&\quad\geq \quad \int \mathrm{sgn}(u_b - k)[F(k) - \int_{a>0} aM_b\,d\xi - \int_{a<0} af(0,t,\xi)\,d\xi]\phi(0,t)\,dt.
\end{aligned}$$

with $a = a(\xi)$, $M^0 = M(u^0(x),\xi)$ and $M_b = M(u_b(t),\xi)$.

Proof. Use $\mathrm{sgn}(f - M(k,\xi)) \in \partial H_k(f,\xi)$ and (2.3).

We are now able to demonstrate our main theorem

Theorem 6.4 *Let the maxwellian M satisfies (2.1)-(2.3). If u^0 and u_b are of bounded variation, then the sequence of first momentums u^ϵ arising from the solutions f of the BGK model (1.4)-(1.6) converges to the unique entropy solution u of the initial-boundary value problem (1.1)-(1.3) described in definition 2.1.*

Proof. The momentums u^ϵ are uniformly bounded in $L^\infty(\mathbb{R}^+ \times (0,T))$ by theorem 4.5 as they are uniformly bounded in $BV(\mathbb{R}^+ \times (0,T))$ by theorem 5.3. We can then extract a subsequence, which we denote also by u^ϵ, converging in L^1 and almost every where to $u \in L^\infty \cap BV(\mathbb{R}^+ \times (0,T))$. In addition,

$$|\int_{a(\xi)<0} a(\xi) f(0,t,\xi)\, d\xi| \le K \int |a(\xi)|\, d\xi,$$

that is we can extract from

$$v^\epsilon = \int_{a>0} aM_b\, d\xi + \int_{a<0} af(0,t,\xi)\, d\xi$$

a subsequence, indexed also by ϵ, converging in the weak* L^∞-topology toward $h \in L^\infty(0,T)$. Therefore, we pass to the limit on ϵ in corollary 6.3, up to a subsequence, using lemma 6.1 and (2.4)-(2.5) to obtain

$$\begin{aligned} &\int |u-k|\partial_t\phi + \operatorname{sgn}(u-k)(F(u)-F(k))\partial_x\phi\, dx\, dt \\ &\quad + \int |u^0-k|\phi(x,0)\, dx \;\ge\; \int \operatorname{sgn}(u_b-k)(F(k)-h(t))\phi(0,t)\, dt. \end{aligned} \tag{6.1}$$

Choosing $\phi = \rho_\delta(x)\psi(t)$ with $\psi \in C_c(]0,T[)$ and

$$\rho_\delta(0) = 1, \quad \rho_\delta(x) = 0 \text{ if } x \ge \delta, \quad 0 \le \rho_\delta \le 1,$$

and tending δ toward zero in (6.1) yields

$$sg(u(0,t)-k)(F(k)-F(u(0,t)) \ge \operatorname{sgn}(u_b(t)-k)(F(k)-h(t)),$$

for all $k \in \mathbb{R}$. Choose now

$$k < \min\{\inf(u^0), \inf(u_b)\} \;\text{ and }\; k > \max\{\sup(u^0), \sup(u_b)\}$$

to get $h(t) = F(u(0,t))$, which ends this proof.

7 Examples

Example 7.1: Relaxation. Let us consider the so called relaxation system introduced by Jin and Xin [12] to approximate the conservation law (1.1):

$$\partial_t u^\epsilon + \partial_x v^\epsilon = 0 \quad 0 < x,\ 0 < t < T, \tag{7.1}$$
$$\partial_t v^\epsilon + a^2\partial_x u^\epsilon = 1/\epsilon(F(u^\epsilon) - v^\epsilon) \quad 0 < x,\ 0 < t < T, \tag{7.2}$$
$$u^\epsilon(x,0) = u^0(x), \quad v^\epsilon(x,0) = F(u^0(x)) \quad 0 \le x, \tag{7.3}$$
$$u^\epsilon(0,t) = u_b(t) \quad 0 \le t. \tag{7.4}$$

This system is studied in [30] where it is shown, under restrictive assumption on the initial-boundary data, namely for small perturbation of a non transonic constant state, that (u^ϵ, v^ϵ) converges to $(u, F(u))$, where u is the unique entropy solution of (1.1)-(1.3). Let us slightly change the boundary condition (7.4) as follows:

$$au^\epsilon(0,t) + v^\epsilon(0,t) = au_b(t) + F(u_b(t)) \quad 0 \le t, \tag{7.5}$$

and let

$$f_1 = au + v \quad \text{and} \quad f_2 = au - v$$

be the Riemann invariants corresponding respectively to the characteristics $\pm a$. Then we get the equivalent formulation

$$\begin{aligned}
\partial_t f_1 + a\partial_x f_1 &= 1/\epsilon(M_1(u^\epsilon) - f_1) \quad 0 < x,\ 0 < t < T, &(7.6)\\
\partial_t f_2 - a\partial_x f_2 &= 1/\epsilon(M_2(u^\epsilon) - f_2) \quad 0 < x,\ 0 < t < T, &(7.7)\\
f_1(x,0) &= au^0(x) + F(u^0(x)) \quad 0 \le x, &(7.8)\\
f_2(x,0) &= au^0(x) - F(u^0(x)) \quad 0 \le x, &(7.9)\\
f_1(0,t) &= au_b(t) + F(u_b(t)) \quad 0 \le t, &(7.10)
\end{aligned}$$

where

$$\begin{aligned}
u^\epsilon &= \tfrac{f_1+f_2}{2a}, &(7.11)\\
M_1(u) &= au + F(u) \quad \text{for all } u \in \mathbb{R}, &(7.12)\\
M_2(u) &= au - F(u) \quad \text{for all } u \in \mathbb{R}. &(7.13)
\end{aligned}$$

We can easily show that M_1 and M_2 satisfy (2.2)-(2.3) with

$$\Xi = \{1,2\}, \quad d\xi\{1\} = d\xi\{2\} = \frac{1}{2a}, \quad a_1 = a, \quad a_2 = -a.$$

These maxwellians are monotone under the so called subcharacteristic condition:

$$-a \le F'(u) \le a. \tag{7.14}$$

We can set in the light of preceding sections:

Theorem 7.1 *Let u^0 and u_b be of bounded variation. If F satisfies (7.14), then the relaxed problem (7.1,7.2,7.3,7.5) admits a unique BV-solution (u^ϵ, v^ϵ). These solutions converges to $(u, F(u))$ where u is the unique entropy solution of (1.1)-(1.3).*

Proof. The subcharacteristic condition (7.14) with (7.12)-(7.13) imply (2.1)-(2.3), thus the densities given by (7.11) of the BGK model (7.6)-(7.10) converge towards the entropy solution u of (1.1)-(1.3). So, the solutions of (7.1)-(7.3) and (7.5) go to $(u, F(u))$ when $\epsilon \to 0$.

Let us remark that, if we choose

$$f_1 = \frac{au^\epsilon + v^\epsilon}{2a} \quad \text{and} \quad f_2 = \frac{au^\epsilon - v^\epsilon}{2a},$$

then we recover the BGK model treated in [22] which is also equivalent to the relaxation system and yields, in a similar way, the convergence of (7.1,7.2,7.3,7.5) towards (1.1)-(1.3).

Example 7.2: Discrete and continuous BGK model. Let $(\Xi, d\xi)$ be a probability space and let $a(\xi) \in L^2(\Xi)$ with $a(\xi) \neq 0$ for all $\xi \in \Xi$ such that $\displaystyle\int a\,d\xi = 0$. Then the maxwellian defined by

$$M(u,\xi) = u + \frac{a(\xi)}{|a|^2_{L^2}} F(u),$$

satisfies (2.2)-(2.3) and is non-decreasing in u under the following subcharacteristic condition:

$$-|a|^2_{L^2} \leq a(\xi) F'(u),$$

for all $(\xi, u) \in \Xi \times \mathbb{R}$.

We can construct in this way a class of continuous BGK model with a maxwellian M different from the χ one usually appearing in the literature.

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