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Asymptotic behavior of solutions to wave equations with a memory condition at the boundary *

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Abstract

In this paper, we study the stability of solutions for wave equations whose boundary condition includes a integral that represents the memory effect. We show that the dissipation is strong enough to produce exponential decay of the solution, provided the relaxation function also decays exponentially. When the relaxation function decays polynomially, we show that the solution decays polynomially and with the same rate.

1 Introduction

The main purpose of this work is study the asymptotic behavior of solution of the wave equation with a boundary condition of memory type. For this, we consider the following initial boundary-value problem

$$u_{tt} - \mu(t)u_{xx} = 0 \quad \text{in } (0,1) \times (0,\infty), \tag{1.1}$$

$$u(0,t) = 0, \quad u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)ds = 0, \quad \forall t > 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } (0,1). \tag{1.3}$$

The integral in (1.2) is a boundary condition which includes the memory effect. Here, by u we denote the displacement and by g the relaxation function. By $\mu = \mu(t)$ we represent a function of $W^{1,\infty}_{loc}(0,\infty : \mathbb{R})$, such that $\mu(t) \geq \mu_0 > 0$ and $\mu'(t) \leq 0$ for all $t \geq 0$. We refer to [4] for the physical motivation of this model.

Frictional dissipative boundary condition for the wave equation was studied by several authors, see for example [4, 5, 8, 9, 10, 11, 12, 15, 16] among others. In these works existence of solutions and exponential stabilization were proved for linear and for nonlinear equations. In contrast with the large literature for frictional dissipative, for boundary condition with memory, we have only a few works as for example [2, 3, 7, 13, 14]. Let us explain briefly each of the

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above works. In [2] Ciarletta established theorems of existence, uniqueness and asymptotic stability for a linear model of heat conduction. In this case the memory condition describes a boundary that can absorb heat and due to the hereditary term, can retain part of it. In [3] Fabrizio & Morro considered a linear electromagnetic model with boundary condition of memory type and proved the existence, uniqueness and asymptotic stability of solutions. While in [13] showed the existence of global smooth solution for the one dimensional nonlinear wave equation, provided the initial data (u_0, u_1) is small in the $H^3 \times H^2$-norm, moreover he showed that the solution tends to zero as time goes to infinity. In all the above works was left open the rate of decay. Rivera & Andrade [7] consider a nonlinear one dimensional wave equation with a viscoelastic boundary condition. They proved the existence, uniqueness of global smooth solution, provided the initial data (u_0, u_1) is small in the $H^2 \times H^1$-norm and also that the solution decays uniformly in time (exponentially and algebraically). Finally, in [14] Qin proved a blow up result for the nonlinear one dimensional wave equation with memory boundary condition. Our main result is to show that the solution of system (1.1)-(1.3) decays uniformly in time, with rates depending on the rate of decay of the relaxation function. More precisely, denoting by k the resolvent kernel of g' (the derivative of the relaxation function) we show that the solution decays exponentially to zero provided k decays exponentially to zero. When k decays polynomially, we show that the corresponding solution also decays polynomially to zero with the same rate of decay.

The method used here is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{L}(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\alpha+1}}$$

for some positive constants c_1, c_2, α and γ. To study the existence of solution of (1.1)-(1.3), we introduce the space $V := \{v \in H^1(0,1); v(0) = 0\}$. The notation used in this paper is standard and can be found in Lions's book [6]. In the sequel by c (sometime $c_1, c_2, \ldots$) we denote various positive constants independent of t and on the initial data. The organization of this paper is as follows. In section 2 we establish a existence and regularity result. In section 3 prove the uniform rate of exponential decay. Finally in section 4 we prove the uniform rate of polynomial decay.

2 Existence and Regularity

In this section we shall study the existence and regularity of solutions for the (1.1)-(1.3). We assume that the kernel g is positive and k satisfies:

$$\begin{gathered} 0 < k(t) \leq b_0 e^{-\gamma_0 t}, \\ -b_1 k(t) \leq k'(t) \leq -b_2 k(t), \\ -b_3 k'(t) \leq k''(t) \leq -b_4 k'(t) \end{gathered} \tag{2.1}$$

for some positive constants b_i, $i = 0, 1, \dots, 4$, and γ_0. To facilitate our analysis, we introduce the following binary operators:

$$\begin{aligned}(f\Box\varphi)(t) &= \int_0^t f(t-s)|\varphi(t)-\varphi(s)|^2 ds,\\ (f*\varphi)(t) &= \int_0^t f(t-s)\varphi(s)ds,\end{aligned}$$

where $*$ is the convolution product. Differentiating (1.2) we arrive to the Volterra integral equation

$$\mu(t)u_x(1,t) + \frac{1}{g(0)} g' * \mu(t)u_x(1,t) = -\frac{1}{g(0)} u_t(1,t).$$

Using the Volterra inverse operator, we obtain

$$\mu(t)u_x(1,t) = -\frac{1}{g(0)}\{u_t(1,t) + k * u_t(1,t)\}.$$

With $\tau = \frac{1}{g(0)}$ and using the above identity, we write

$$\mu(t)u_x(1,t) = -\tau\{u_t(1,t) + k(0)u(1,t) - k(t)u_0(1) + k' * u(1,t)\}. \tag{2.2}$$

The following lemma state an important property of the convolution operator.

Lemma 2.1 *For $f, \varphi \in C^1([0,\infty[: \mathbb{R})$ we have*

$$\begin{aligned}\int_0^t f(t-s)\varphi(s)ds\varphi_t &= -\frac{1}{2}f(t)|\varphi(t)|^2 + \frac{1}{2}f'\Box\varphi\\ &\quad -\frac{1}{2}\frac{d}{dt}\left[f\Box\varphi - (\int_0^t f(s)ds)|\varphi|^2\right].\end{aligned}$$

The proof of this lemma follows by differentiating the term $f\Box\varphi$.

We summarize the well-posedness of (1.1)-(1.3) in the following theorem.

Theorem 2.2 *Let $(u_0, u_1) \in V \times L^2(0,1)$, then there exists only one solution to the (1.1)-(1.3) satisfying*

$$u \in C([0,T] : V) \cap C^1([0,T] : L^2(0,1)).$$

Moreover, if $(u_0, u_1) \in H^2(0,1) \cap V \times V$ satisfies the compatibility condition

$$\mu(0)u_{0,x}(1) = -\tau u_1(1)\,, \tag{2.3}$$

then $u \in C([0,T] : H^2(0,1) \cap V) \cap C^1([0,T] : V)$.

This theorem can be showed using the standard Galerkin method, for this reason we omit it here.

3 Exponential Decay

In this section we show that the solution of (1.1)-(1.3) decays exponentially. Our point of departure will be to establish some inequalities for the solution of system (1.1)-(1.3). For this end, we introduce the functional

$$F(t) = \frac{1}{2}\int_0^1 |u_t|^2 + \mu(t)|u_x|^2 dx + \frac{\tau}{2}(k(t)|u(1,t)|^2 - k'(t)\Box u(1,t)).$$

Lemma 3.1 *For a strong solution of the system (1.1)-(1.3),*

$$\begin{aligned} \frac{d}{dt}F(t) &\leq -\frac{\tau}{2}|u_t(1,t)|^2 + \frac{\tau}{2}k^2(t)|u_0(1)|^2 + \frac{\tau}{2}k'(t)|u(1,t)|^2 \\ &\quad -\frac{\tau}{2}k''(t)\Box u(1,t) + \frac{1}{2}\int_0^1 \mu'|u_x|^2 dx. \end{aligned}$$

Proof. Multiplying (1.1) by u_t and integrating over $[0,1]$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |u_t|^2 + \mu(t)|u_x|^2 dx = \mu(t)u_x(1,t)u_t(1,t) + \frac{1}{2}\int_0^1 \mu'|u_x|^2 dx. \tag{3.1}$$

Using (2.2) and Lemma 2.1 the conclusion follows. Q.E.D.

The following Lemma plays an important role in the construction of the Lyapunov functional. Let us define the functional

$$\psi(t) = \int_0^1 x u_x u_t dx\,.$$

Lemma 3.2 *The strong solution of (1.1)-(1.3) satisfies*

$$\begin{aligned} \frac{d}{dt}\psi(t) &\leq -\frac{1}{2}\int_0^1 (|u_t|^2 + \mu(t)|u_x|^2)dx + c|u_t(1,t)|^2 + ck^2(t)|u_0(1)|^2 \\ &\quad + ck(0)k(t)|u(1,t)|^2 + ck(0)|k'|\Box u(1,t). \end{aligned}$$

Proof. From (1.1) it follows that

$$\begin{aligned} \frac{d}{dt}\psi(t) &= \int_0^1 x u_{xt} u_t dx + \int_0^1 x u_x u_{tt} dx \\ &= \frac{1}{2}\int_0^1 x\frac{d}{dx}|u_t|^2 dx + \frac{1}{2}\int_0^1 x\mu(t)\frac{d}{dx}|u_x|^2 dx \\ &= -\frac{1}{2}\int_0^1 (|u_t|^2 + \mu(t)|u_x|^2)dx + \frac{1}{2}|u_t(1,t)|^2 + \frac{1}{2}\mu(t)|u_x(1,t)|^2. \end{aligned} \tag{3.2}$$

Note that

$$-k(0)u(1,t) - k' * u(1,t)$$

$$
\begin{aligned}
&= -\int_0^t k'(t-s)[u(1,s)-u(1,t)]ds - k(t)u(1,t)\\
&\le \Big(\int_0^t |k'(s)|ds\Big)^{1/2}[|k'|\Box u(1,t)]^{1/2} + k(t)|u(1,t)| \qquad (3.3)\\
&\le |k(t)-k(0)|^{1/2}[|k'|\Box u(1,t)]^{1/2} + k(t)|u(1,t)|.
\end{aligned}
$$

Using (2.2) and (3.3), it follows that

$$
\begin{aligned}
&\mu(t)|u_x(1,t)|^2 \qquad (3.4)\\
&\le c\{|u_t(1,t)|^2 + k^2(t)|u_0(1)|^2 + k(0)|k'|\Box u(1,t) + k(0)k(t)|u(1,t)|^2\}.
\end{aligned}
$$

Substituting (3.4) into (3.2), the conclusion of the lemma follows. Q.E.D.

Let us introduce the functional

$$\mathcal{L}(t) = NF(t) + \psi(t), \qquad (3.5)$$

with $N > 0$. It is not difficult to see that $\mathcal{L}(t)$ satisfies

$$q_0F(t) \le \mathcal{L}(t) \le q_1F(t), \qquad (3.6)$$

for some positive constants q_0 and q_1. Finally, we shall show the main result of this Section.

Theorem 3.3 *Assume that the initial data $(u_0, u_1) \in V \times L^2(0,1)$ and that the resolvent k satisfies (2.1). Then there exist positive constants α_1 and γ_2 such that*

$$F(t) \le \alpha_1 e^{-\gamma_2 t}F(0), \quad \forall t \ge 0\,.$$

Proof. We will suppose that $(u_0, u_1) \in H^2(0,1) \cap V \times V$ and satisfies (2.3); our conclusion will follow by standard density arguments. Using Lemmas 3.1 and 3.2 we get

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \le\; & -\frac{\tau}{2}N|u_t(1,t)|^2 + \frac{\tau}{2}Nk^2(t)|u_0(1)|^2 + \frac{\tau}{2}Nk'(t)|u(1,t)|^2\\
& -\frac{\tau}{2}Nk''\Box u(1,t) - \frac{1}{2}\int_0^1(|u_t|^2 + \mu(t)|u_x|^2)dx + c|u_t(1,t)|^2\\
& + ck^2(t)|u_0(1)|^2 + ck(0)k(t)|u(1,t)|^2 + ck(0)|k'|\Box u(1,t). \qquad (3.7)
\end{aligned}
$$

Then, choosing N large enough we obtain

$$\frac{d}{dt}\mathcal{L}(t) \le -q_2F(t) + ck^2(t)F(0),$$

where $q_2 > 0$ is a small constant. Here we use (2.1) to conclude the following estimates for the corresponding two terms appearing in Lemma 3.1.

$$
\begin{aligned}
&-\tfrac{\tau}{2}k''\Box u(1,t) \le c_1 k'\Box u(1,t),\\
&\tfrac{\tau}{2}k'|u(1,t)|^2 \le -c_1 k|u(1,t)|^2\,.
\end{aligned}
$$

Finally, in view of (3.6) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -\gamma_1 \mathcal{L}(t) + ck^2(t)F(0).$$

Using the exponential decay of the resolvent kernel k we conclude

$$\mathcal{L}(t) \le \{\mathcal{L}(0) + c\}e^{-\gamma_2 t}$$

for all $t \ge 0$, where $\gamma_2 = min(\gamma_0, \gamma_1)$. From (3.6) the conclusion follows. Q.E.D.

4 Polynomial rate of decay

The proof of the existence of global solutions for (1.1)-(1.3) with resolvent kernel k decaying polynomially is essentially the same as in Section 2. Here our attention will be focused on the uniform rate of decay when the resolvent k decays polynomially such as $(1+t)^{-p}$. In this case, we will show that the solution also decays polynomially with the same rate. We shal use the following hypotheses:

$$\begin{gathered} 0 < k(t) \le b_0(1+t)^{-p}, \\ -b_1 k^{\frac{p+1}{p}} \le k'(t) \le -b_2 k^{\frac{p+1}{p}}, \\ b_3(-k'(t))^{\frac{p+2}{p+1}} \le k''(t) \le b_4(-k'(t))^{\frac{p+2}{p+1}} \end{gathered} \tag{4.1}$$

where $p > 1$ and b_i, $i = 0, 1 \dots, 4$, are positive constants. Also we assume that

$$\int_0^\infty |k'(t)|^r dt < \infty \quad \text{if} \quad r > \frac{1}{p+1}. \tag{4.2}$$

The following lemmas will play an important role in the sequel.

Lemma 4.1 *Let m and h be integrable functions, $0 \le r < 1$ and $q > 0$. Then, for $t \ge 0$,*

$$\begin{aligned} &\int_0^t |m(t-s)h(s)|ds \\ &\quad \le \Big(\int_0^t |m(t-s)|^{1+\frac{1-r}{q}}|h(s)|ds\Big)^{\frac{q}{q+1}} \Big(\int_0^t |m(t-s)|^r |h(s)|ds\Big)^{\frac{1}{q+1}}. \end{aligned}$$

Proof. Let

$$v(s) := |m(t-s)|^{1-\frac{r}{q+1}}|h(s)|^{\frac{q}{q+1}}, \quad w(s) := |m(t-s)|^{\frac{r}{q+1}}|h(s)|^{\frac{1}{q+1}}.$$

Then using Hölder's inequality with $\delta = \frac{q}{q+1}$ for v and $\delta^* = q+1$ for w, we arrive to the conclusion. Q.E.D.

Lemma 4.2 *Let $p > 1$, $0 \leq r < 1$ and $t \geq 0$. Then for $r > 0$,*

$$\begin{aligned}
&(|k'|\Box u(1,t))^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}} \\
&\leq 2\Big(\int_0^t |k'(s)|^r ds ||u||^2_{L^\infty((0,T),H^1(0,1))}\Big)^{\frac{1}{(1-r)(p+1)}} (|k'|^{1+\frac{1}{p+1}}\Box u(1,t)),
\end{aligned}$$

and for $r = 0$,

$$\begin{aligned}
&(|k'|\Box u(1,t))^{\frac{p+2}{p+1}} \\
&\leq 2\Big(\int_0^t ||u(.,s)||^2_{H^1(0,1)} dst ||u(.,s)||^2_{H^1(0,1)}\Big)^{\frac{1}{p+1}} (|k'|^{1+\frac{1}{p+1}}\Box u(1,t))
\end{aligned}$$

Proof. The above inequality is a immediate consequence of Lemma 4.1 with

$$m(s) := |k'(s)|, \quad h(s) := |u(x,t) - u(x,s)|^2, \quad q := (1-r)(p+1),$$

and t fixed. Q.E.D.

Lemma 4.3 *Let $\alpha > 0$, $\beta \geq \alpha + 1$, and $f \geq 0$ be differentiable function satisfying*

$$f'(t) \leq \frac{-\bar{c}_1}{f(0)^{\frac{1}{\alpha}}} f(t)^{1+\frac{1}{\alpha}} + \frac{\bar{c}_2}{(1+t)^\beta} f(0)$$

for $t \geq 0$ and some positive constants $\bar{c}_1$, $\bar{c}_2$. Then there exists a constant $\bar{c}_3 > 0$ such that for $t \geq 0$,

$$f(t) \leq \frac{\bar{c}_3}{(1+t)^\alpha} f(0).$$

Proof. Let $t \geq 0$ and

$$F(t) = f(t) + \frac{2\bar{c}_2}{\alpha}(1+t)^{-\alpha} f(0).$$

Then

$$\begin{aligned}
F' &= f' - 2\bar{c}_2(1+t)^{-(\alpha+1)} f(0) \\
&\leq \frac{-\bar{c}_1}{f(0)^{\frac{1}{\alpha}}} f^{1+\frac{1}{\alpha}} - \bar{c}_2(1+t)^{-(\alpha+1)} f(0),
\end{aligned}$$

where we used $\beta \geq \alpha + 1$. Hence

$$F' \leq \frac{-c}{f(0)^{\frac{1}{\alpha}}}(f^{1+\frac{1}{\alpha}} + (1+t)^{-(\alpha+1)} f(0)^{1+\frac{1}{\alpha}}) \leq \frac{-c}{F(0)^{\frac{1}{\alpha}}} F^{1+\frac{1}{\alpha}}.$$

Integration yields

$$F(t) \leq \frac{F(0)}{(1+ct)^\alpha} \leq \frac{c}{(1+t)^\alpha} f(0) \quad \text{hence} \quad f(t) \leq \frac{\bar{c}_3}{(1+t)^\alpha} f(0)$$

for some $\bar{c}_3$, which proves Lemma 4.3. Q.E.D.

Theorem 4.4 *Assume that* $(u_0, u_1) \in V \times L^2(0,1)$ *and that the resolvent* k *satisfies (4.1). Then there exists a positive constant* c *for which*

$$F(t) \le \frac{c}{(1+t)^{p+1}} F(0).$$

Proof. We will suppose that $(u_0, u_1) \in H^2(0,1) \cap V \times V$ and satisfies (2.3); our conclusion will follow by standard density arguments. We define the functional $\mathcal{L}$ as (3.5) and we have the equivalence to the energy term F as given in (3.6) again. The negative term

$$-ck(t)|u(1,t)|^2$$

can be obtained from Lemma 3.2 and estimate

$$k(t)|u(1,t)|^2 \le c \int_0^1 \mu(t)|u_x|^2 dx.$$

From Lemmas 3.1 and 3.2, using the properties of k'' from the assumption (4.1) for the term

$$-\frac{\tau}{2} k'' \Box u(1,t),$$

we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \quad &\le \quad -c_1\Big(\int_0^1 |u_t|^2 + \mu(t)|u_x|^2 dx + k(t)|u(1,t)|^2 \\ &\quad + N(-k')^{1+\frac{1}{p+1}} \Box u(1,t)\Big) + c_2 k^2(t) F(0). \end{aligned} \tag{4.3}$$

Applying Lemma 4.2 with $r > 0$ we obtain

$$\begin{aligned} &(-k')^{1+\frac{1}{p+1}} \Box u(1,t) \\ &\ge \quad \frac{c}{(\int_0^t |k'|^r ds)^{\frac{1}{(1-r)(p+1)}} F(0)^{\frac{1}{(1-r)(p+1)}}} ((-k')\Box u(1,t))^{1+\frac{1}{(1-r)(p+1)}} \end{aligned}$$

(with $c = c(r)$). On the other hand, we have

$$\begin{aligned} &(k(t)|u(1,t)|^2 + \int_0^1 |u_t|^2 + \mu(t)|u_x|^2 dx)^{1+\frac{1}{(1-r)(p+1)}} \\ &\quad \le \quad cF(0)^{\frac{1}{(1-r)(p+1)}} \Big(k(t)|u(1,t)|^2 + \int_0^1 |u_t|^2 + \mu(t)|u_x|^2 dx\Big). \end{aligned}$$

From (4.2) and (4.3) and using the last two inequalities, we conclude that

$$\begin{aligned} &\frac{d}{dt}\mathcal{L}(t) \\ &\le \quad -\frac{c}{F(0)^{\frac{1}{(1-r)(p+1)}}} [(k(t)|u(1,t)|^2 + \int_0^1 |u_t|^2 + \mu(t)|u_x|^2 dx)^{1+\frac{1}{(1-r)(p+1)}} \\ &\quad + ((-k')\Box u(1,t))^{1+\frac{1}{(1-r)(p+1)}}] + ck^2(t)F(0). \end{aligned}$$

Using (3.6) and the above inequality can be written as

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + ck^2(t)F(0). \tag{4.4}$$

Applying the Lemma 4.3 with $f = \mathcal{L}$, $\alpha = (1-r)(p+1)$ and $\beta = 2p$ we have

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{\alpha}}\mathcal{L}(0). \tag{4.5}$$

Choosing r such that

$$\frac{1}{p+1} < r < \frac{p}{p+1},$$

and taking into account the inequality (4.5), we obtain

$$\int_0^\infty F(s)ds \leq c\int_0^\infty \mathcal{L}(s)ds \leq c\mathcal{L}(0), \tag{4.6}$$

$$t||u(.,t)||^2_{H^1(0,1)} \leq ct\mathcal{L}(t) \leq c\mathcal{L}(0), \tag{4.7}$$

$$\int_0^t ||u(.,s)||^2_{H^1(0,1)} \leq c\int_0^\infty \mathcal{L}(t)dt \leq c\mathcal{L}(0). \tag{4.8}$$

Estimates (4.6)-(4.8) together with Lemma 4.2 (case $r = 0$) imply that

$$(-k')^{1+\frac{1}{p+1}}\Box u(1,t) \geq \frac{c}{F(0)^{\frac{1}{p+1}}}((-k')\Box u(1,t))^{1+\frac{1}{p+1}}.$$

Using the same arguments as in the derivation of (4.4), we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + ck^2(t)F(0).$$

Applying Lemma 4.3, we obtain

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}}\mathcal{L}(0).$$

Then from (3.6), we conclude

$$F(t) \leq \frac{c}{(1+t)^{p+1}}F(0),$$

which completes the present proof. Q.E.D.

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