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Sufficient conditions for functions to form Riesz bases in L_2 and applications to nonlinear boundary-value problems *

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Abstract

We find sufficient conditions for systems of functions to be Riesz bases in $L_2(0,1)$. Then we improve a theorem presented in [13] by showing that a "standard" system of solutions of a nonlinear boundary-value problem, normalized to 1, is a Riesz basis in $L_2(0,1)$. The proofs in this article use Bari's theorem.

1 Introduction

Early results in the study of basis properties of eigenfunctions of nonlinear ordinary differential operators can be found in the monograph by Makhmudov [5]. Because of its difficulty and the small number of publications on this question, basis properties has been established only for very simple nonlinear ordinary differential equations. Among the results in this direction, we have the following.

In [7, 8], Zhidkov presents an analysis of the equation

$$\begin{gathered}-u'' + f(u^2)u = \lambda u, \quad u = u(x), \quad x \in (0,1), \\ u(0) = u(1) = 0, \quad \int_0^1 u^2(x)\,dx = 1\,,\end{gathered}$$

where λ is a spectral parameter, $f(s)$ is a smooth nondecreasing function for $s \geq 0$, and all quantities are real. In these two publications, it is proved that the eigenfunctions $\{u_n\}$ $(n = 0, 1, 2, \dots)$ of this problem have precisely n zeros in $(0, 1)$. Furthermore, each eigenfunction is unique up to the coefficient ± 1. The main result states that the sequence of eigenfunctions $\{u_n\}$ $(n = 0, 1, 2, \dots)$ is a Bari basis in $L_2 = L_2(0,1)$, i.e., it is a basis and there exists an orthonormal basis $\{e_n\}$ $(n = 0, 1, 2, \dots)$ in L_2 for which $\sum_{n=0}^{\infty} \|u_n - e_n\|_{L_2}^2 < \infty$. Note that in [7] there are some errors which have been corrected in [9].

In [10, 11], a modified version of the above nonlinear eigenvalue problem is studied and similar basis properties for their eigenfunctions are obtained. In

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[12], an analog to the Fourier transform associated with an eigenvalue problem for a nonlinear ordinary differential operator on a half-line is considered.

The aim in the present publication is to improve the result in [13], where the following nonlinear problem is considered:

$$u'' = f(u^2)u\,, \quad u = u(x)\,, \quad x \in (0,1)\,, \tag{1.1}$$
$$u(0) = u(1) = 0\,. \tag{1.2}$$

Here there is no spectral parameter, and all variables are real. For the rest of this article, we will assume that

(F) The function $f(u^2)u$ is a continuously differentiable for $u \in \mathbb{R}$, $f(0) \geq 0$, and $f(+\infty) = -\infty$.

It is well known now (and partially proved in [13]) that under assumption (F): For each integer $n \geq 0$ problem (1.1)–(1.2) has a solution u_n which possesses precisely n zeros in $(0,1)$ and that generally speaking this solution is not unique.

Definition A sequence $\{u_n\}$ $(n = 0,1,2,\dots)$ of solutions to (1.1)–(1.2) is called standard if the solution u_n has precisely n zeros in $(0,1)$.

The main result in [13] states that there exists $s_0 < 0$ such that for $s < s_0$ any standard sequence of solutions $\{u_n\}$ is a basis in $H^s(0,1)$. In addition, the sequence $\{u_n/\|u_n\|_{H^s(0,1)}\}$ is a Riesz basis in $H^s(0,1)$. Here $H^s(0,1)$ is the usual Sobolev space with negative index s. In the present paper, we improve this result by showing the above properties of a standard system $\{u_n\}$ in L_2 (see Theorem 1.3 below), by first obtaining a general result on bases in L_2 (see Theorem 1.1 below). We believe that this result is of a separate interest.

Notation

By $c, C, C_1, C_2, C', C'', \dots$ we denote positive constants. By $L_2(a,b)$ we denote the standard Lebesgue space of square integrable functions on the interval (a,b). In this space we introduce the standard inner product, and norm:

$$(g,h)_{L_2(a,b)} = \int_a^b g(x)h(x)dx\,, \quad \|g\|_{L_2(a,b)} = (g,g)^{1/2}_{L_2(a,b)}\,.$$

For short notation we will use $(\cdot,\cdot)$ and $\|\cdot\|$ respectively.

Let l_2 be the space of square summable sequences of real numbers. For a Banach space X with a norm $\|\cdot\|_X$, let $\mathcal{L}(X;X)$ be the linear space of linear bounded operators acting from X into X, equipped with the norm

$$\|A\|_{\mathcal{L}(X;X)} = \sup_{x\in X:\ \|x\|_X=1} \|Ax\|_X\,.$$

We also set $\|\cdot\| = \|\cdot\|_{\mathcal{L}(L_2;L_2)}$ for short notation.

Now, for convenience of readers, we define some well-known terms.

Definition A system $\{e_n\} \subset L_2(a,b)$ is called a basis in $L_2(a,b)$ if for any $g \in L_2(a,b)$ there exists a unique sequence $\{a_n\}$ of real numbers such that $g = \sum_{n=0}^{\infty} a_n e_n$ in $L_2(a,b)$.

There are several definitions of Riesz bases. In accordance with the classical paper by N. K. Bari [1], where this concept was introduced for the first time, we use the following definition.

Definition A basis $\{e_n\}$ in $L_2(a,b)$ is called a Riesz basis in this space when the series $\sum_{n=0}^{\infty} a_n e_n$, with real coefficients a_n, converges in $L_2(a,b)$ if and only if $\sum_{n=0}^{\infty} a_n^2 < \infty$.

Remark It is proved in [1] (see also [8]) that if $\{e_n\}$ is a Riesz basis in $L_2(a,b)$ in the sense of this definition, then there exist constants $0 < c < C$ such that

$$c\sum_{n=0}^{\infty} a_n^2 \le \Big\| \sum_{n=0}^{\infty} a_n e_n \Big\|^2_{L_2(a,b)} \le C \sum_{n=0}^{\infty} a_n^2$$

for all $\overline{a} = (a_0, a_1, a_2, \dots) \in l_2$. These estimates have been often used to define Riesz bases.

Definition A system of functions $\{g_n\}$ in $L_2(a,b)$ is called ω-linearly independent in $L_2(a,b)$ when $\sum_{n=0}^{\infty} a_n g_n = 0$, with a_n are real numbers, holds in $L_2(a,b)$ if and only if $0 = a_0 = a_1 = a_2 = \dots$.

Definition Two systems of functions $\{h_n\}$ and $\{e_n\}$ in $L_2(a,b)$ are called quadratically close in $L_2(a,b)$, if $\sum_{n=0}^{\infty} \|h_n - e_n\|^2_{L_2(a,b)} < \infty$.

Results

Theorem 1.1 *Let $\{h_n\}$ be a system of real-valued, three-times continuously differentiable functions. Assume that for each integer $n \ge 0$ the following holds:*

(a) $h_n\big(x + \frac{1}{n+1}\big) = -h_n(x)$ and $h_n\big(\frac{1}{2(n+1)} + x\big) = h_n\big(\frac{1}{2(n+1)} - x\big)$ for all $x \in \mathbb{R}$

(b) $h_n'(x) > 0$, $h_n''(x) \le 0$, and $h_n'''(x) \le 0$ for all $x \in \big(0, \frac{1}{2(n+1)}\big)$

(c) There exist $0 < c < C$ such that $c < h_n\big(\frac{1}{2(n+1)}\big) < C$ for all n.

Then, the system $\{h_n\}$ is a Riesz basis in L_2.

Remark Clearly, it follows from Theorem 1.1 that if a system of functions $\{h_n\}$ satisfies all the conditions of this theorem, except maybe (c), then it is a basis in L_2.

The next result follows from Theorem 1.1 by taking $h_n(x) = h((n+1)x)$.

Theorem 1.2 *Let $h(x)$ be a real-valued three-times continuously differentiable function satisfying:*

(a) $h(1+x) = -h(x)$ *and* $h(1/2+x) = h(1/2-x)$ *for all* $x \in \mathbb{R}$

(b) $h'(x) > 0$, $h''(x) \le 0$ *and* $h'''(x) \le 0$ *for all* $x \in (0, 1/2)$

Then, the sequence of functions $h_n(x) = h((n+1)x)$, *where* $n = 0, 1, 2, \ldots$, *is a Riesz basis in* L_2.

The following statement also follows from Theorem 1.1, when applied to problem (1.1)–(1.2).

Theorem 1.3 *Let assumption (F) be valid and* $f(u^2) + 2u^2 f'(u^2) \le 0$ *for all sufficiently large* u. *Let* $\{u_n\}$ *be an arbitrary standard sequence of solutions of (1.1)–(1.2). Then, the sequence* $\{\|u_n\|^{-1}u_n\}$ *is a Riesz basis in* L_2.

To prove this theorem in Section 3, we exploit the following theorem.

Theorem 1.4 (Bari's Theorem) *Let* $\{e_n\}$ *be a Riesz basis in* $L_2(a,b)$ *and let a system* $\{h_n\} \subset L_2(a,b)$ *be* ω*-linearly independent and quadratically close to* $\{e_n\}$ *in* $L_2(a,b)$. *Then, the system* $\{h_n\}$ *is a Riesz basis in* $L_2(a,b)$.

This theorem, in a weaker form, was proved by N. K. Bari in [1]. In its current form it is proved, for example, in [4] and in [8].

We conclude the introduction by pointing out that the concept of a Riesz basis appeared for the first time in the middle of last century in the papers of N. K. Bari, as a result of developments in the general theory of orthogonal series and bases in infinite-dimensional spaces. Currently, this concept has important applications in areas such as wavelet analysis. Readers may consult [2, 6] for theoretical aspects of this field and [3] for applied aspects.

2 Proof of Theorem 1.1

Let $e_n(x) = \sqrt{2}\sin \pi(n+1)x$, $n = 0, 1, 2, \ldots$, so that $\{e_n\}$ is an orthonormal basis in L_2.

Lemma 2.1 *Let* g *satisfy condition (a) of Theorem 1.1 with* $n \ge 0$ *and let* g *be positive in* $(0, \frac{1}{n+1})$. *Then in the expansion*

$$g(\cdot) = \sum_{m=0}^{\infty} c_m e_m(\cdot),$$

understanding in the sense of L_2, *one has* $c_0 = \ldots = c_{n-1} = 0$ *and* $c_n > 0$.

Proof We follow the arguments in the proof of a similar statement in [13]. We have the above expansion in $L_2(0, \frac{1}{n+1})$ with $c_m = 0$ if $m \ne (n+1)(l+1) - 1$ for all integers $l \ge 0$ (this occurs because the functions $\{e_{(n+1)(m+1)-1}\}_m$ form an orthogonal basis in $L_2(0, \frac{1}{n+1})$). Therefore, $c_0 = \ldots = c_{n-1} = 0$. We observe that each $e_{(n+1)(m+1)-1}$ becomes zero at the points $\frac{1}{n+1}, \frac{2}{n+1}, \ldots, 1$.

Furthermore, due to condition (a) of Theorem 1.1 the function g is odd with respect to these points and each function $e_{(n+1)(m+1)-1}(x)$ is odd. Thus this expansion also holds in each space $L_2(\frac{1}{n+1},\frac{2}{n+1}), L_2(\frac{2}{n+1},\frac{3}{n+1}),\dots,L_2(\frac{n}{n+1},1)$. Finally, $c_n>0$ because $e_n(x)$ and $g(x)$ are of the same sign everywhere. $\square$

Due to Lemma 2.1, we have the following sequence of expansions:

$$h_n(\cdot)=\sum_{m=0}^{\infty} a_m^n e_m(\cdot) \quad \text{in } L_2, \tag{2.1}$$

with $a_0^n=\dots=a_{n-1}^n=0$ and $a_n^n>0$, for $n=0,1,2,\dots$.

Lemma 2.2 *Under the assumptions of Theorem 1.1, the coefficients in (2.1) satisfy*

$$(a_n^n)^{-1}|a^n_{(n+1)(m+1)-1}|\le \frac{\pi}{2}(m+1)^{-2}$$

for all n and m. In addition, $a^n_{(n+1)(m+1)-1}=0$ if $m=2l+1$ for $l=0,1,2,\dots$.

Proof The second claim of this lemma is obvious because $e_{(n+1)(2l+2)-1}(x)$ is odd with respect to the middles of the intervals $(0,\frac{1}{n+1}),(\frac{1}{n+1},\frac{2}{n+1}),\dots,(\frac{n}{n+1},1)$ and the function $h_n(x)$ is even so that $a^n_{(n+1)(2l+2)-1}=(e_{(n+1)(2l+2)-1},h_n)=0$. Let us prove the first claim. Due to the properties of the functions h_n and $e_{(n+1)(m+1)-1}$, with $m=2l$, we have

$$\begin{aligned}
(a_n^n)^{-1}|a^n_{(n+1)(m+1)-1}| &= \frac{\left|\int_0^1 h_n(x)\sin\pi(n+1)(m+1)x dx\right|}{\int_0^1 h_n(x)\sin\pi(n+1)x dx}\\
&= \frac{\left|\int_0^{1/2(n+1)} h_n(x)\sin\pi(n+1)(m+1)x dx\right|}{\int_0^{1/2(n+1)} h_n(x)\sin\pi(n+1)x dx}\\
&= (m+1)^{-1}\frac{\left|\int_0^{1/2(n+1)} h_n'(x)\cos\pi(n+1)(m+1)x dx\right|}{\int_0^{1/2(n+1)} h_n'(x)\cos\pi(n+1)x dx}\\
&= (m+1)^{-1}\frac{\left|\int_0^1 h_n'(\frac{s}{2(n+1)})\cos\frac{\pi(m+1)s}{2}ds\right|}{\int_0^1 h_n'(\frac{s}{2(n+1)})\cos\frac{\pi s}{2}ds}.
\end{aligned}$$

Due to the conditions of Theorem 1.1, $h_n'(\frac{s}{2(n+1)})$ is a positive non-increasing concave function on $(0,1)$. Therefore,

$$\int_0^1 h_n'(\frac{s}{2(n+1)})\cos\frac{\pi s}{2}ds\ge h_n'(0)\int_0^1(1-s)\cos\frac{\pi s}{2}ds=\frac{4}{\pi^2}h_n'(0).$$

Using the same properties of h_n', one can easily see on its graph that

$$\begin{aligned}
\Big|\int_0^1 h_n'(\frac{s}{2(n+1)})\cos\frac{\pi(m+1)s}{2}ds\Big| &\le h_n'(0)\int_0^{1/(m+1)}\cos\frac{\pi(m+1)s}{2}ds\\
&= \frac{2}{\pi(m+1)}h_n'(0)\,.
\end{aligned}$$

Detailed arguments leading to a similar estimate are considered in [13]. We easily obtain now

$$(a_n^n)^{-1}|a_{(n+1)(m+1)-1}^n| \le \frac{\pi}{2}(m+1)^{-2},$$

which completes the proof. □

From the conditions on Theorem 1.1, we have $0 < c \le |a_n^n| \le C$. Then to prove this theorem, it suffices to prove that the system $\{\overline{h}_n\}$ with $\overline{h}_n = (a_n^n)^{-1}h_n$ is a Riesz basis in L_2.

Lemma 2.3 *Let $\{g_n\}$ be a sequence of functions such that g_n satisfies condition (a) on Theorem 1.1 and is positive in $(0, \frac{1}{n+1})$. Then the system $\{g_n\}$ is ω-linearly independent in L_2.*

The proof of this lemma is rather simple. We refer the reader to the proof of similar statements in [8, 11, 13]. □

Let $b_m^n = (a_n^n)^{-1}a_m^n$, and let Id be the unit operator in L_2. For positive integers m, let B_m be the operator mapping e_n into $b_{(n+1)(m+1)-1}^n e_{(n+1)(m+1)-1}$, $B_m \in \mathcal{L}(L_2; L_2)$. Also let $B = \sum_{m=1}^\infty B_m$. Then for each m,

$$\|B_m\| \le \sup_n |b_{(n+1)(m+1)-1}^n| = b_m.$$

Furthermore, by Lemma 2,

$$\sum_{m=1}^\infty b_m \le \frac{\pi}{2}\sum_{l=1}^\infty (2l+1)^{-2} \le \frac{\pi}{2}\int_{1/2}^\infty (2x+1)^{-2}dx = \pi/8\,;$$

hence, $B \in \mathcal{L}(L_2; L_2)$ and $\|B\| \le \pi/8 < 1$. Therefore, the operator $A = \mathrm{Id} + B$ has a bounded inverse $A^{-1} = \mathrm{Id} + \sum_{n=1}^\infty (-1)^n B^n$. Note also that $Ae_n = \overline{h}_n$. Hence, as proved in [4], $\{\overline{h}_n\}$ is a Riesz basis in $L_2(0,1)$. For the convenience of the reader, we present a short proof of this statement.

Take an arbitrary $v \in L_2$ and let $u = A^{-1}v = \sum_{n=0}^\infty c_n e_n \in L_2$ where c_n are real coefficients. Then, $\sum_{n=0}^\infty c_n^2 < \infty$ because $\{e_n\}$ is an orthonormal basis in L_2. Since the series $\sum_{n=0}^\infty c_n e_n$ converges in L_2, we have $v = Au = \sum_{n=0}^\infty c_n A e_n = \sum_{n=0}^\infty c_n \overline{h}_n$ where all infinite sums also converge in L_2. Therefore, in view of Lemma 2.3, the system $\{\overline{h}_n\}$ is a basis in L_2 and, if $\sum_{n=0}^\infty c_n^2 < \infty$, then the series $\sum_{n=0}^\infty c_n h_n$ converges in L_2. Conversely, let a series $u = \sum_{n=0}^\infty c_n \overline{h}_n$ converge in L_2. Then $A^{-1}u = \sum_{n=0}^\infty c_n e_n$ in L_2; hence $\sum_{n=0}^\infty c_n^2 < \infty$. Thus, $\{\overline{h}_n\}$ is a Riesz basis in L_2, and the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.3

As was proved in [13], any solution u_n of problem (1.1)–(1.2), that possesses precisely n zeros in $(0,1)$, satisfies condition (a) of Theorem 1.1. In addition,

u_n is strictly monotone ($u_n'(x) \neq 0$) in the interval $(0, \frac{1}{2(n+1)})$. Let $\overline{u} > 0$ be an arbitrary number such that $f(u^2) < 0$ and $f(u^2) + 2u^2 f'(u^2) \leq 0$ for all $u \geq \overline{u}$. Let $\{u_n\}$ be an arbitrary standard system of solutions of problem (1.1)–(1.2). We assume that $u_n(x) > 0$ for $x \in (0, \frac{1}{n+1})$ for each n which is possible without loss of generality due to the invariance of (1.1) when $u(x)$ is replaced by $-u(x)$. Due to the standard comparison theorem $\max_{u \in [0, u_n(1/2(n+1))]} |f(u^2)| \to +\infty$ as $n \to \infty$; hence $u_n(\frac{1}{2(n+1)}) \to +\infty$ as $n \to \infty$. For n sufficiently large, we denote by $x_n \in (0, \frac{1}{2(n+1)})$ the point for which $u_n(x_n) = \overline{u}$. Then

$$u_n(\frac{1}{2(n+1)}) - \overline{u} = \int_{x_n}^{1/2(n+1)} u_n'(x)dx = u_n'(\tilde{x}_n)(\frac{1}{2(n+1)} - x_n)$$

for some $\tilde{x}_n \in (x_n, \frac{1}{2(n+1)})$. Since $u_n'(x_n) \geq u_n'(\tilde{x}_n)$ (because $f(u^2) < 0$ for $u > \overline{u}$ and, therefore, $u_n''(x) < 0$ for $x \in (x_n, \frac{1}{2(n+1)})$), we derive

$$u_n'(x_n) \geq \frac{3}{2} u_n(\frac{1}{2(n+1)})(n+1)$$

for all sufficiently large n. Since in view of (1.1), $\sup_n \max_{x \in [0, x_n]} |u_n''(x)| \leq C'$, we have $\min_{x \in [0, x_n]} |u_n'(x)| \geq u_n(\frac{1}{2(n+1)})(n+1)$ for all sufficiently large n. Therefore,

$$0 < x_n \leq (n+1)^{-1} \big[u_n(\frac{1}{2(n+1)})\big]^{-1} \tag{3.1}$$

for all sufficiently large n.

Using u_n and n large, we now want to construct a function h_n that satisfies the conditions of Theorem 1.1. Introduce the linear function $l_n(x) = \frac{\overline{u}}{x_n} x$ which is equal to 0 at $x = 0$ and to $\overline{u} = u_n(x_n)$ at $x = x_n$. Multiply (1.1), with $u = u_n$, by $2u_n'(x)$ and integrate the result from 0 to x. Then

$$\{[u_n'(x)]^2 + F(u_n^2(x))\}' = 0, \quad x \in \mathbb{R}, \tag{3.2}$$

where $F(s) = -\int_0^s f(t)dt$. Due to condition (F), $F(u^2) \to +\infty$ as $u \to \infty$, therefore, without loss of generality, we can assume that $\overline{u} > 0$ and is large enough so that $|\overline{u} f(\overline{u}^2)| > |u f(u^2)|$ and $F(\overline{u}^2) > F(u^2)$ for all $u \in [0, \overline{u})$. Then from (3.2), it follows that

$$u_n'(x_n) < u_n'(x), \quad x \in [0, x_n), \tag{3.3}$$

for all sufficiently large n. By (3.3), we have

$$\overline{u} = \int_0^{x_n} u_n'(x)dx > x_n u_n'(x_n);$$

therefore,

$$u_n'(x_n) < \frac{\overline{u}}{x_n} = l_n'(x) \tag{3.4}$$

for all sufficiently large n.

Take a sufficiently small $\Delta \in (0, \frac{x_n}{2})$ and define a continuous function $\omega_1(x)$ equal to $u_n'''(x)$ for $x \in [x_n, \frac{1}{2(n+1)}]$ such that $u_n'''(x) \le \omega_1(x) \le 0$ for $x \in [x_n - \Delta, x_n]$ and $\omega_1(x) = 0$ for $x \in [0, x_n - \Delta)$. We define $g_1(x)$ to be equal to $u_n(x)$ for $x \in [x_n, \frac{1}{2(n+1)}]$, and for $x \in [0, x_n)$ to be given by the rules:

$$\begin{aligned} g_1''(x) &= u_n''(x_n) - \int_x^{x_n} \omega_1(t)dt, \\ g_1'(x) &= u_n'(x_n) - \int_x^{x_n} g_1''(t)dt, \\ g_1(x) &= u_n(x_n) - \int_x^{x_n} g_1'(t)dt. \end{aligned} \tag{3.5}$$

Then $g_1(x)$ is three times continuously differentiable in $[0, \frac{1}{2(n+1)}]$ and satisfies condition (b) of Theorem 1.1. It is easy to see that if $\Delta > 0$ is sufficiently small, then $g_1(x_n - \Delta)$ and $g_1'(x_n - \Delta)$ are arbitrary close to $u_n(x_n)$ and $u_n'(x_n)$, respectively, and $g_1''(x)$ is arbitrary close to $u_n''(x_n)$ for all $x \in [0, x_n - \Delta]$. Now, due to our choice of $\overline{u} > 0$, for $\Delta > 0$ and sufficiently small, $g_1(0)$ is arbitrary close to

$$u_n(x_n) - x_n u_n'(x_n) + \frac{x_n^2}{2} u_n''(x_n)\,. \tag{3.6}$$

This expression is negative because

$$0 = u_n(0) = u_n(x_n) - x_n u_n'(x_n) + \int_0^{x_n} dx \int_x^{x_n} u_n''(t)dt$$

where the last term in the right-hand side of this equality is larger than the last term in (3.6), due to our choice of $\overline{u}$ and (1.1). We have defined a function $g_1(x)$ satisfying $g_1(0) < 0$.

Take now a sufficiently small $\Delta \in (0, \frac{x_n}{2})$ and a continuous function $\omega_2(x) \le 0$ which is equal to $u_n'''(x)$ for $x \in [x_n, \frac{1}{2(n+1)}]$ and to 0 for $x \in [0, x_n - \Delta)$, such that

$$\int_{x_n-\Delta}^{x_n} \omega_2(x)dx = u_n''(x_n).$$

Then, defining the function $g_2(x)$ just as $g_1(x)$ in (3.5) with the substitution of ω_2 in place of ω_1 and of g_2 in place of g_1, we get that if $\Delta > 0$ is sufficiently small, then $g_2(x_n - \Delta)$ and $g_2'(x_n - \Delta)$ are arbitrary close, respectively, to $u_n(x_n)$ and $u_n'(x_n)$, and $g_2''(x) = 0$ for $0 \le x \le x_n - \Delta$. Therefore, due to (3.4), $g_2(0) > 0$ if $\Delta > 0$ is sufficiently small, for all sufficiently large n. We have defined a function $g_2(x)$ satisfying $g_2(0) > 0$.

Now, consider the family of functions $g_\lambda(x) = \lambda g_1(x) + (1 - \lambda) g_2(x)$ where $\lambda \in [0, 1]$. Clearly, there exists a unique $\lambda_0 \in (0, 1)$ such that $g_{\lambda_0}(0) = 0$. Extend $g_{\lambda_0}(x)$ continuously on the entire real line by the rules:

$$g_{\lambda_0}(\frac{1}{n+1} + x) = -g_{\lambda_0}(x), \quad g_{\lambda_0}(\frac{1}{2(n+1)} + x) = g_{\lambda_0}(\frac{1}{2(n+1)} - x)$$

and denote the obtained function by $h_n(x)$. This function satisfies conditions (a) and (b) of Theorem 1.1. In addition, by Theorem 1.1(b), $h_n''(x_n) \le h_n''(x) \le 0$ for all $x \in [0, x_n)$.

So far, we have constructed h_n for n sufficient large. For small values of n, we use arbitrary functions h_n satisfying the conditions of Theorem 1.1. Therefore the sequence $\{h_n\}$ $(n = 0, 1, 2, \dots)$ satisfies the conditions (a) and (b) of Theorem 1.1.

Let $\alpha_n = [h_n(\frac{1}{2(n+1)})]^{-1}$. Then, by Theorem 1.1, the system $\{\alpha_n h_n\}$ is a Riesz basis in L_2. Furthermore, by Lemma 2.3, the system $\{\alpha_n u_n\}$ is ω-linearly independent in L_2. Also, due to (1.1) and by construction, there exists $C_1 > 0$ such that

$$|u_n''(x)| = \max_{u \in [0, \overline{u}]} |uf(u^2)| \le C_1$$

and

$$\max_{x \in [0, x_n]} |h_n''(x)| = |h_n''(x_n)| = |u_n''(x_n)| = |\overline{u} f(\overline{u}^2)| \le C_1$$

for all n sufficiently large. Hence,

$$|u_n'(x) - h_n'(x)| \le C_2 x_n$$

for all n sufficiently large and all $x \in [0, x_n]$. Hence, due to (3.1),

$$\|\alpha_n u_n - \alpha_n h_n\|^2 \le C_3 x_n^4 \le C_4 (n+1)^{-4}$$

for all n sufficiently large. Therefore, the systems $\{\alpha_n u_n\}$ and $\{\alpha_n h_n\}$ are quadratically close in L_2. In view of Bari's Theorem, the proof of Theorem 1.3 is complete.

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