SOLUTIONS OF BOUNDARY-VALUE PROBLEMS IN DISCRETIZED VOLUMES

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ABSTRACT. The solution of a boundary-value problem in a volume discretized by finitely many copies of a tile is obtained via a Green's function. The algorithm for constructing the solution exploits results from graph and group theory. This technique produces integral equations on the internal and external boundaries of the volume and demonstrates that two permutation matrices characterize the symmetries of the volume. We determine the number of linearly independent solutions required over the tile and the conditions needed for two boundary-value problems to be isospectral. Our method applies group theoretical considerations to asymmetric volumes.

1. INTRODUCTION

The application of symmetry arguments to the solution of physical and mathematical problems has a long and fruitful tradition. The development of mathematical machinery in the form of group and graph theory gave particular impetus to their application, in particular, in modern physics. The advent of computers and the parallel flowering of numerical methods have not yet exploited group theoretical principles to the same extent. The development of nuclear power in the latter half of the last century required the solution of large multidimensional boundary-value problems of neutron transport which quickly taxed each advance in computational power. Early on the possibility of systematically exploiting the symmetries inherent in a nuclear reactor core using group theory were pointed out by Goldsmith (1963) and Theophilou and Tsilimigras (1969); application to the finite element formulation of boundary-value problems by Fässler (1976); applications to reactor control by Nieva and Christensen (1977); the numerical formulation, viability and the superior performance of a computer code for routine industrial application was demonstrated for the solution of the neutron diffusion equation Makai,(1980), Makai and Arkuszewski (1981); the application to the response matrix method Makai (1984); the explanation for the convergence of routinely applied iterative schemes Makai and Orechwa (1999) the domain decomposition for parallel applications Makai and Orechwa (1997); the optimization of sensor response in symmetric

volumes Makai and Orechwa (2000). Independently and somewhat later there appeared in the literature similar approaches in the context of model problems, see Allgower, Böhmer, Georg and Miranda (1992) and subsequent publications.

The success demonstrated by these implementations has given impetus to a search for possible further applications of group theoretic results to the numerical solution of boundary-value problems. In particular, we seek the solution of an elliptic equation in a volume $\mathfrak{V}$ which is obtained by gluing together copies of a given tile $\mathfrak{t}$. (Although only planar shapes are considered here, we shall use the term volume, with an eye on those problems in physics which are modeled as homogeneous and infinite in extent in the additional dimension.) The objective is to combine the language of numerical analysis with that of modern algebra through group and graph theory to boundary value problems on finite lattices.

To motivate the approach we can keep in mind two typical problems– one physical, the other mathematical– which have a common formulation.

(1) Determine the energy-band structure in a microscopic sample which contains a few Wigner-Seitz cells. This requires the solution of the Schrödinger equation in a finite region which consists of a finite number of cells. A similar problem arises in the determination of the neutron distribution in a finite periodic fuel lattice in a nuclear reactor.

(2) For the solution to a boundary-value problem with an elliptic operator in a finite volume, one possible algorithm divides the volume into congruent subvolumes (discretization). The solution is obtained iteratively from an initial guess by taking one subvolume with given boundary conditions and solving the equation in the subvolume; the solution inside the volume is expressed with the help of the solution on the boundaries then we pass on to the next subvolume. This type of numerical problem is encountered quite frequently.

In the solution of a linear elliptic equation over a structure $\mathfrak{V}$, which consists of a repetition of a tile $\mathfrak{t}$, we can exploit translational symmetry. In an infinite lattice, for example, the solution is usually expanded in terms of the eigenfunctions of the translation operator. In certain finite volumes, with Dirichlet boundary conditions, the image pile concept of Weinberg and Schweinler (1948) applies.

Assume that $\mathfrak{V} = G\backslash\mathfrak{t}$: that is, the orbit of $\mathfrak{t}$ under group G is just $\mathfrak{V}$. Then, if G commutes with operator $\mathcal{A}$ of the boundary value problem (to be defined later), we can reduce the solution procedure from $\mathfrak{V}$ to $\mathfrak{t}$. Sunada (1985) and Brooks (1988) have given a method for deriving G by means of the fundamental groups $\pi_1(\mathfrak{t})$ and $\pi_1(\mathfrak{V})$. When $\mathfrak{t}$ is not simply connected, G is the quotient of the fundamental group of $\mathfrak{t}$ and $\mathfrak{V}$. When $\mathfrak{t}$ is simply connected, Gordon, Webb, Wolpert (1992) made the case tractable.

An issue of interest and of practical consequence is the existence of equispectral volumes. When such volumes exist, the solids have the same spectrum, and the solutions over two discretized volumes can be transformed into each other. Thus, for a numerical solution we can choose the equispectral volume which is easiest to solve. Sunada (1985), Brooks (1988) and Buser (1988) have given a systematic method, based on group theory, for finding equispectral volumes.

Our first task is to find a formalism which accommodates some of the basic algebraic terms (e.g. covering group, connectivity matrix) in the numerical algorithms for the solution of boundary-value problems.

The Green's function of the tile is used to build a formal solution to the boundary value problem. We then exploit graph properties to show that an automorphism of a discretized volume can be described by two permutations. The first permutes nodes, while the second permutes node boundaries. The formal solution is expressed in terms of the solution along node boundaries. These functions can be determined iteratively from (3.11). In problems with Dirichlet boundary conditions the solution vanishes on external node boundaries; and thus, the number of unknowns is smaller. The formalism gives the number of linearly independent solutions over a node. In Section 5, we apply this approach to specific problems: In the first, we show that "collective modes" may exist; (i.e. there are situations where functions over two or more nodes together form a set of node functions in such a way that they are transformed into each other when they are reflected through a common internal boundary) the second example illustrates the usage of the covering group, wherein the solution procedure is reduced to a tile; the third example is taken from Gordon et al. (1992) and demonstrates the covering group for a non-trivial shape. Section 6 addresses isospectral volumes. Assume that connectivity matrices (see Section 3) $\mathbf{C}_1$ of volume $\mathfrak{V}_1$ and $\mathbf{C}_2$ of volume $\mathfrak{V}_2$ are equivalent so that $\mathbf{C}_2 = \mathbf{T}\mathbf{C}_1\mathbf{T}^{-1}$ and the edge connectivity patterns described by the block matrices of the geometry matrix in Section 3 are the same for $\mathfrak{V}_1$ and $\mathfrak{V}_2$. Then, the solution of a suitable boundary-value problem over the constituents of $\mathfrak{V}_1$ is a linear expression of the solutions over the constituents of $\mathfrak{V}_2$. Thus, by solving an algebraic problem (viz. that of finding equivalent connectivity matrices) we can solve boundary-value problems (viz. if the solution in $\mathfrak{V}_1$ is known we can also get the solution in $\mathfrak{V}_2$).

Our main results can be summarized as follows:

(1) Knowledge of the covering group may result in a reduced range where we have to find the solution. To demonstrate this, we present Example 3.
(2) The amount of reduction depends on the structure of the covering group. Existence of two-dimensional or higher representation tends to lessen the reduction.
(3) If the geometry matrices of volumes $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are similar, and the conditions stipulated in Lemma 2 are met, $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are isospectral.
(4) We show the solution of a homogeneous problem contains as many linearly independent node functions (defined later) as there are internal boundaries in the volume.

The matrices introduced in connection with the Green's function, can be used to formulate the necessary conditions for two discretized volumes to be isospectral. The discretized volumes can be ordered into classes with the help of the conjugate classes of the associated node permutations. Construction of a computational scheme based on these results may consist of these steps:

- find the symmetries of a discretized volume,
- find the covering group, and
- reduce the solution to a few tiles.

2. The Physical Problem

Let $\mathcal{A}$ be a linear, second order, elliptic operator with finite coefficients at every point x. Consider the problem

$$\begin{aligned}
\mathcal{A}\Phi(x) &= \lambda\Phi(x) \quad x \in \mathfrak{V} \\
\mathcal{B}\Phi(x) &= 0 \quad x \in \partial\mathfrak{V},
\end{aligned} \tag{2.1}$$

where $\mathcal{B}$ is a linear expression of Φ and of the normal gradient $\partial_n\Phi$, where n is the outward normal at $x \in \partial\mathfrak{V}$.

In vibration problems, for example, $\mathcal{A}$ is the Laplace-Beltrami operator, and $\mathcal{B} \equiv 1$. In solid-state physics, the Schrödinger equation is given with

$$\mathcal{A} = \frac{\hbar^2}{2m}\nabla^2 - V(x)$$

as the Hamiltonian operator and $\mathcal{B} \equiv 1$. Here $V(x)$ is a given potential. In neutron diffusion,

$$\mathcal{A} = -\langle D \rangle\nabla^2 + \Sigma,$$

where $\langle D \rangle$ is a diagonal matrix, its entries are the diffusion constants in the energy groups, Σ is a matrix describing the energy transfer processes. In the last example, the dependent variable is a vector, $\underline{\Phi}(x)$, called the neutron flux whose components give the distributions of neutrons in energy. On internal boundaries the flux and $\langle D \rangle\partial_n\underline{\Phi}(x)$ (i.e. the normal component of the current) must be continuous and $\mathcal{B} \equiv 1$.

Some of the above problems have a unique solution if the material properties (potential, cross-sections) have reasonable values, Habetler and Martino (1961). If the solution is unique, the solution may inherit the symmetries of $\mathfrak{V}$, see Kawohl (1998). The present work focuses on the conditions and numerical exploitation of the latter observation.

We address the following mathematical properties of the above physical problems. A symmetry of $\mathfrak{V}$ can be described as the direct product of a permutation of the nodes and a permutation of the sides of the node. By means of the Green's function of the tile (i.e. of the node), and knowledge of the solution along the edges the complete solution is determined. The Green's function of the tile determines the dimension of the solution. The number of edges where the solution is not zero limits the linearly independent function shapes over a tile. In Section 5, we apply the above considerations and construct a group G such that the orbit of $\mathfrak{t}$ under G just covers $\mathfrak{V}$. In Section 6, we give a condition for two volumes to be isospectral. The condition is formulated using the Green's function of the tile. We show that the isospectral volumes derived by Sunada (1988), Brooks (1988), Gordon et al. (1992) satisfy the condition we have specified.

3. Group theoretic, geometric and analysis background

Below we outline the group theoretic background applied throughout the present work; for group theory and geometry, we follow Stillwell (1993) and Babai (1995), for the response matrix method, as a numerical method for solving boundary-value problems, Weiss (1977).

Group theoretic background.

- *Generator.* In combinatorial topology, groups are described in terms of "generators" and "relations". A generator is a letter a_i, and has a formal inverse a_i^{-1}. A word w is any finite sequence

$$a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \ldots a_{i_k}^{\epsilon_k}$$

  of generators or their inverses, so that each $\epsilon_j = \pm 1$, where a_i^{+1} denotes a_i. The product $w_1 w_2$ of words w_1 and w_2 is the concatenation of the corresponding sequences. The empty word is denoted by 1. A relation is an equation $r = 1$ where r is a word. Words w and w' are called equivalent with respect to relations $r_j = 1$ if w is convertible to w' by a finite sequence of operations of the following types:
    (1) insertion or deletion of a subword r_j,
    (2) insertion or deletion of a subword $a_i a_i^{-1}$ or $a_i^{-1} a_i$.
- *Finitely presented group.* The structure $\langle a_1, a_2, \cdots : r_1, r_2, \ldots \rangle$ of equivalence classes of words in a_i with respect to the relations r_j, under the product operation, is a group G. The expression $\langle a_1, a_2, \cdots : r_1, r_2, \ldots \rangle$ is called a presentation of G. G is finitely presented if it has a presentation in which the sets $\{a_i\}$ and $\{r_j\}$ are finite.
- *Representation by matrices.* If G is a finite group, and $\mathbf{M}_g$ is a matrix, the isomorphism $g \leftrightarrow \mathbf{M}_g$ for all $g \in G$ is called a matrix representation of G.
- *Coset representation.* If A is a subgroup of G the sets $Ag = \{ag : a \in A\}$ for $g \in G$ are called right cosets of G modulo A. They constitute a partition of G, called the *right coset decomposition.*
- *Orbit.* Let d be an object on which the action of elements of G has been defined. A point e in the orbit of d under the group G is the point for which a group element g of G exists such that $\hat{g}d = e$, where $\hat{g}d$ is the object "g applied to d".
- *Group action on functions.* Let x denote the independent variables in the problem under consideration. With a matrix representation acting on x, the group action on a function $f(x)$ is usually defined as $gf(x) = f(g^{-1}x)$.

Geometric background. The geometric notions applied throughout the present work follow Stillwell (1993).

- *Graph.* A graph Γ is defined by its vertex set W and edge set E.
- *Symmetries of a graph.* In analogy to symmetries of volume $\mathfrak{V}$, we need the symmetries (automorphisms) of a graph. The automorphisms of a graph $\Gamma = (W, E)$ are permutation pairs P, p, where P permutes the elements of set W whereas p permutes elements of set E so that the incidence between vertices and edges is preserved, see Babai (1995).
- *Discretized volume.* A discretized volume is created by the following operations. We start from a tile $\mathfrak{t}$, which is a suitable polygon. We can reflect $\mathfrak{t}$ along a side if the vertices of the new copy lie on the boundary of the first tile, or entirely outside it. This gives a discretized volume consisting of two copies of $\mathfrak{t}$. The procedure can be repeated whenever the vertices of the new copy of the tile do not lie inside another already existing tile. We call the copies subvolumes or nodes. In the discretized volume, E^k ($k = 1, K$) denotes the set of edges, A subvolume $F_{m\alpha}$ ($\alpha = 1, n_F$) stands for the sides of node m. ϕ_m is the restriction of the solution $\Phi(x)$ to the m-th node. As

the nodes are congruent, we can map them into each other. Node $m = 1$ is considered as the reference and τ_m is an isometry from node $m = 1$ to the m-th node. Such an isometry is the reflection of a node through a joint boundary. With the consecutive application of that step, we can map any node in $\mathfrak{V}$ into a given node. We can define a linear transformation among the ϕ_m's as follows:

$$a_i \phi_i + a_j \phi_j = a_i \Phi \circ \tau_i + a_j \Phi \circ \tau_j.$$

We divide the edges as

$$E = E_i + E_e = \bigcup_{m=1}^{N} E_m = \bigcup_{m=1}^{N} \bigcup_{\alpha=1}^{n_F} F_{m\alpha} = \bigcup_{k=1}^{K} E^k,$$

where E_i, E_e are the set of internal and external edges; E_m is the set of edges belonging to node m, and $F_{m\alpha}$ stands for edge α of node m. An internal edge separates two nodes; an external edge lies on the external boundary $\partial\mathfrak{V}$; E^k is the k-th edge in $\mathfrak{V}$.

- *Connectivity matrix.* The geometry of a discretized volume consisting of N copies of $\mathfrak{t}$ is characterized by a square matrix $\mathbf{C}$ called the connectivity matrix of volume $\mathfrak{V}$. $C_{ij} = 1$ if subvolumes i and j share an edge, 0 otherwise.
- *Fundamental group.* The product and inverse of closed curves is defined as follows. The natural product of two curves p, q which begin at P is their concatenation, the inverse p^{-1} of p will lie on top of p but with the opposite orientation. The fundamental group is a group of equivalence classes of closed paths in a discretized volume $\mathfrak{V}$.
- *Covering group.* If there is a group G, acting on a tile $\mathfrak{t}$ and its orbit is $\mathfrak{V}$, G is called covering group of $\mathfrak{V}$.
- *Chromatic number.* Like a geographic map, a discretized volume can also be colored. The minimal number of colors needed to color $\mathfrak{V}$ is called its chromatic number.
- *Quotient manifold.* Let us consider a manifold $\mathfrak{X}$, on which the action of elements of G is defined with x being equivalent to y if they lie in the same orbit of G. Let $\mathfrak{X}/G$ denote the set of equivalence classes, or, equivalently, the set of orbits of G. The equivalence class can be identified with the orbit of G passing through x. $\mathfrak{X}/G$ is called quotient manifold.

Analysis and numerical background. In this Section, we outline the basic notions of mathematical analysis applied throughout the present work.

- *Function space.* Throughout the present work, we are interested in functions given in a finite region $\mathfrak{V}$. The space of functions continuous along with its first derivatives is denoted by $C^1(\mathfrak{V})$.
- *Operator.* An operator $\mathcal{A}$ is a map $C^1(\mathfrak{V}) \to C^1(\mathfrak{V})$.
- *Boundary value problem (BVP).* The boundary-value problem under consideration is (2.1). We assume operators $\mathcal{A}, \mathcal{B}$ to be linear. The boundary value is prescribed on the boundary of $\mathfrak{V}$.
- *Green's function.* With the help of the Green's function $\mathcal{G}$, the boundary-value problem is transformed into an integral equation. In the sequel, we consider BVP where the solution is prescribed on the boundary (Dirichlet

type BVP). We assume the existence of the following Green's function in each node:

$$\mathcal{A}\mathcal{G}_{\text{node}}(x, x') = \lambda\mathcal{G}_{\text{node}}(x, x') \quad x, x' \in \mathfrak{V}_{\text{node}}$$
$$\mathcal{B}\mathcal{G}_{\text{node}}(x, x') = \delta(x - x') \quad x \in \mathfrak{V}_{\text{node}}, x' \in \partial\mathfrak{V}_{\text{node}}. \tag{3.1}$$

Here δ is Dirac's delta function. Since all the nodes are congruent, $\mathcal{G}_{\text{node}}$ is applicable to every node in $\mathfrak{V}$. This Green's function, when x' is restricted to side α, is written as $\mathcal{G}_{\text{node}\alpha}(x, x')$. Then, the solution in any node m is expressed in terms of this Green's function as a sum of integrals over the edges of the node as

$$\phi_m(x) = \sum_{F_{m\alpha} \in E} \int_{F_{m\alpha}} \mathcal{G}_{\text{node}\alpha}(x, x')q_{m\alpha}(x')dx'. \tag{3.2}$$

Here

$$q_{m\alpha}(x) = \Phi(x) \quad x \in F_{m\alpha}. \tag{3.3}$$

When x tends to the boundary of $\mathfrak{t}$, we have

$$\lim_{x \to \partial t} \mathcal{G}_{\text{node}\alpha}(x, x') = \begin{cases} \delta(x - x') & \text{if } x \in F_{m\alpha} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Thus,

$$\lim_{x \to \partial t} \int_{F_{m\beta}} \mathcal{G}_{\text{node}\alpha}(x, x')q_{m\beta}(x')dx' = \int_{F_{m\beta}} \delta_{\alpha\beta}\delta(x - x')q_{m\beta}(x')dx' = \delta_{\alpha\beta}q_{m\beta}(x), \tag{3.5}$$

were δ is the Kronecker symbol. The solution of the Dirichlet type BVP takes the form

$$\Phi(x) = \int_{x' \in \partial\mathfrak{V}} \mathcal{G}(x, x')f(x')dx' \tag{3.6}$$

where $\mathcal{G}(x, x')$ is the Green's function for volume $\mathfrak{V}$ and $f(x')$ is a given function on the boundary $\partial\mathfrak{V}$. The Green's function possesses the following properties:
  - If $\mathbf{O}\mathcal{A} = \mathcal{A}\mathbf{O}$ then $\mathcal{G}(\mathbf{O}x, \mathbf{O}x') = \mathcal{G}(x, x')$, i.e. the symmetries of $\mathcal{A}$ leave the Green's function invariant;
  - $\mathcal{G}(x, x')$ -along with its first-order derivatives- is continuous for all x, x' if $x \neq x'$ and neither x nor x' lies on the boundary;
  - $\mathcal{G}(x, x') = \mathcal{G}(x', x)$ and $\mathcal{G}(x, x') > 0$ if $x \neq x'$.
- *Geometry matrix.* The geometry matrix $\mathbf{H}$ expresses the connection between the nodes in discretized volume $\mathfrak{V}$. The solution belongs to C^1, hence, the normal gradients are continuous at internal faces, which can be expressed as

$$\begin{pmatrix} \mathbf{H}_{11} & \ldots & \mathbf{H}_{1N} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{N1} & \ldots & \mathbf{H}_{NN} \end{pmatrix} \begin{pmatrix} \underline{J}_1 \\ \vdots \\ \underline{J}_N \end{pmatrix} = 0. \tag{3.7}$$

The geometry matrix contains N blocks of order n_F in each row. In a row there are at most n_F non-zero blocks. The element (α, β) in block $\mathbf{H}_{mm'}$ is 1 if nodes m and m' are adjacent along side α of node m and side β of node m'. The blocks on the diagonal are diagonal matrices of order n_F. Element

$(\mathbf{H}_{mm})_\alpha = 1$ if side α of node m is internal and $(\mathbf{H}_{mm})_\alpha = 0$ if side α of node m is external. The vectors $J_{m\beta}(x)$ are defined in the following.

- *Reformulation of BVP.* Using the above Green's function, the boundary value problem is reformulated. In the context of the response matrix formalism, Weiss (1977), the normal gradient at face β of $\mathfrak{V}_m$ is

$$J_{m\beta}(x) = \partial_{n_\beta}\Phi(x); \quad x \in F_{m\beta}. \tag{3.8}$$

The gradient can then be expressed by the solution at the boundary as

$$J_{m\beta}(x) = \sum_{\alpha \in E_m} \int_{F_{m\alpha}} \partial_{n_\beta}\mathcal{G}_{\text{node}\alpha}(x, x')q_\alpha(x')dx' = \sum_{\alpha \in E_m} \mathcal{R}^{(m)}_{\beta\alpha}q^{(m)}_\alpha. \tag{3.9}$$

Let face α separate nodes k_α^+ and k_α^-. The continuity of the normal gradient is expressed by

$$J_{k_\alpha^+}(x) = \sum_{\beta \in E_{k_\alpha^+}} \mathcal{R}^{(k_\alpha^+)}_{\alpha\beta}q^{(k_\alpha^+)}_\beta(x') =$$

$$-J_{k_\alpha^-}(x) = \sum_{\beta \in E_{k_\alpha^-}} \mathcal{R}^{(k_\alpha^-)}_{\alpha\beta}q^{(k_\alpha^-)}_\beta(x').$$

This is a set of linear integral equations for the solutions on the boundaries. A nontrivial solution exists only if the null space of the operator acting on the "q"s is not empty. This condition is met at certain values of λ, called the eigenvalues of problem (2.1).

Let us collect the gradients at the faces of node m into a vector $\underline{J}_m = (J_{m\alpha}, \alpha = 1, n_F)$. The continuity of the gradients at the N_i internal faces takes the form (3.7), where N is the number of nodes in $\mathfrak{V}$. The matrix in (3.7) is called a geometry matrix, Weiss (1977), and contains N blocks of order n_F in each row. As described above, in a row there are at most n_F non-zero blocks. The element (α, β) in block $\mathbf{H}_{mm'}$ is 1 if nodes m and m' are adjacent along side α of node m and side β of node m'. The blocks on the diagonal are diagonal matrices of order n_F. Element $(\mathbf{H}_{mm})_\alpha = 1$ if side α of node m is internal and $(\mathbf{H}_{mm})_\alpha = 0$ if side α of node m is external. Thus, the geometry matrix and the connectivity matrix of $\mathfrak{V}$ are related as

$$\begin{pmatrix} \mathbf{H}_{11} & \ldots & \mathbf{H}_{1N} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{N1} & \ldots & \mathbf{H}_{NN} \end{pmatrix} = \mathbf{C} \times \mathbf{H}_{ij} + \begin{pmatrix} \mathbf{E} - \mathbf{P}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \mathbf{E} - \mathbf{P}^{(N)} \end{pmatrix}, \tag{3.11}$$

where $\mathbf{C}$ is the connectivity matrix and the direct product ($\times$) replaces 1 in position mm' by $\mathbf{H}_{mm'}$, $\mathbf{E}$ is the n_F order identity matrix. The n_F order matrix $\mathbf{P}^{(m)}$ projects out the external sides of node m. If edge α is on the external boundary $\partial\mathfrak{V}$ and it is in node m, then element (α, α) in block $\mathbf{H}_{mm}$ specifies a linear relationship between the solution and its normal gradient. Let constants $a^{(k)}$ and $b^{(k)}$ belong to the boundary condition at edge E^k. We drop superscript k and characterize the general boundary condition

$$a\Phi(x) + b\partial_n\Phi(x) = 0 \tag{3.12}$$

by a and b. Different edges may be characterized by different (a, b), which may also depend on the position.

We collect solutions along the n_F sides of node m into vector $\underline{q}_m$. Since in node m the gradient is expressible by $\underline{q}_m$, and assuming that constants a and b are the same in a given node, we get

$$
\begin{pmatrix} \mathbf{H}_{11} & \dots & \mathbf{H}_{1N} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{N1} & \dots & \mathbf{H}_{NN} \end{pmatrix} \begin{pmatrix} \mathcal{R}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \mathcal{R}^{(N)} \end{pmatrix} \begin{pmatrix} \underline{q}_1 \\ \vdots \\ \underline{q}_N \end{pmatrix} =
$$
$$
\begin{pmatrix} \mathbf{P}^{(1)}(b^{(1)} * \mathcal{R}^{(1)} + a^{(1)}) & & 0 \\ & \ddots & \\ 0 & & \mathbf{P}^{(N)}(b^{(N)} * \mathcal{R}^{(N)} + a^{(N)}) \end{pmatrix} \begin{pmatrix} \underline{q}_1 \\ \vdots \\ \underline{q}_N \end{pmatrix}. \quad (3.13)
$$

Here $\underline{q}_m$ is the solution vector of n_F components on the sides of node m. In this equation, the external boundary condition has been implemented as well. It should be noted that the left-hand side of (3.13) vanishes on external boundaries, whereas the right hand side on internal boundaries. When node m has at least one external boundary, the block $\mathbf{H}_{mm}$ on the diagonal is an n_F order diagonal matrix describing the boundary condition at the boundaries of node m lying also on $\partial\mathfrak{V}$.

Equation (3.13) is the key to the solution of problem (2.1-3). Equation (3.13) is an integral equation, the unknowns to be determined being $\underline{q}_1(x), \dots, \underline{q}_N(x)$. The kernel is implicitly in the elements of $\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(N)}$ and involves the gradients of the Green's function. Equation (3.13) is a homogeneous problem: a nontrivial solution exists only when the null space of the matrix is not empty. This is achieved by a suitable choice of the eigenvalue λ in (3.1). Computer programs are available to solve (3.13), see e.g. Dorning (1979). We introduce the following compact notation to shorten (3.13):

$$
\mathbf{H}\mathcal{R}\mathbf{q} = \mathcal{S}\mathbf{q} \quad (3.14)
$$

with obvious notation.

There are also numerical procedures to determine the largest eigenvalue and the associated values of the solution at the boundaries. This is the solution we are interested in; this solution is unique up to normalization. Once the solution values at the boundaries are known, we can use (3.2) to get the solution at arbitrary x in $\mathfrak{V}$. When the material composition is the same in all the nodes, we have $\mathcal{R}^{(m)} = \mathcal{R}$ for $m = 1, \dots, N$.

4. Structure of the Solution to a BVP

We start with an analysis of the discretized volume $\mathfrak{V}$. The main question is to determine when two apparently different discretized volumes are actually identical.

Definition 1. Let tile $\mathfrak{t}$ have n_F straight line sides. Let $\mathfrak{V}$ be a volume composed of N copies of a tile $\mathfrak{t}$, so that any two adjacent copies share a side of the tile. Graph Γ is created in the following manner. N copies of $\mathfrak{t}$ form the vertex set W of the graph, and vertex i and j are connected by an edge of color α if tile i and j share a side α of $\mathfrak{t}$. This graph is referred to as $\Gamma(\mathfrak{V}, t)$.

Definition 2. Two discretized volumes $\mathfrak{V}$ and $\mathfrak{V}'$ are called equivalent, if there exists a map $i \leftrightarrow i'$ such that if nodes i_1 and i_2 in $\mathfrak{V}$ share an edge of type α then nodes i'_1 and i'_2 in $\mathfrak{V}'$ also share an edge of type α.
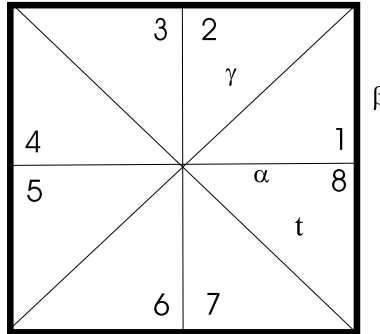

FIGURE 1. Square composed of 8 triangles

Figure 1 is an illustration for the introduced terms. There tile $\mathfrak{t}$ is an isosceles right triangle, the square volume $\mathfrak{V}$ consists of 8 copies of $\mathfrak{t}$. Tile $\mathfrak{t}$ has edges α, β, γ but edge β is always on the external boundaries. There are 8 vertices and 8 edges in $\Gamma(\mathfrak{V}, t)$. Nodes 1 and 2 are connected by a γ type edge, nodes 2 and 3 are connected by an α type edge. In this case the graph $\Gamma(\mathfrak{V}, t)$ is closed.

Since the ϕ_m's form a linear space, we can ask the "dimension" of the solution. In other words, we can ask the minimum number of functions over the tile with the help of which we can build the solution of problem (1). Finally, exploiting the fact that all the nodes can be mapped to $\mathfrak{t}$, we can define the dimension of a function given on $\mathfrak{V}$.

Definition 3. Let $\Phi(x)$ be given for any $x \in \mathfrak{V}$. Let $\phi_m(x)$ denote the restriction of $\Phi(x)$ to $x \in \mathfrak{V}_m \subset \mathfrak{V}$ which we call node function and consider it as a function of the position in tile $\mathfrak{t}$. $Dim(\Phi(x))$ is the maximum number of linearly independent node functions.

Given the above dimension of the solution, we can seek a relationship between the solutions belonging to two problems, which are formulated with the same operators $\mathcal{A}$ and $\mathcal{B}$ but for different volumes $\mathfrak{V}_1$ and $\mathfrak{V}_2$.

The first problem is to reduce the number of data to describe $\mathfrak{V}$. To this end the following automorphisms are exploited in the reduction.

Definition 4. Operator $\mathbf{O}$ is called a symmetry of the boundary-value problem (2.1-3) if (i), $\mathcal{A}\mathbf{O} = \mathbf{O}\mathcal{A}$ (ii), $\mathcal{B}\mathbf{O} = \mathbf{O}\mathcal{B}$ and (iii), $\mathbf{O}$ is invertible and maps $\mathfrak{V}$ into itself.

Lemma 4.1. *Let $\mathbf{O}$ be a symmetry of problem (2.1). Then we can associate a permutation $\mathbf{P}$ of the nodes and a permutation p of the n_F node sides with $\mathbf{O}$.*

Proof. A symmetry maps $\mathfrak{V}$ into itself, so it is an automorphism of $\Gamma(\mathfrak{V}, t)$. This ensures preservation of incidence relation between vertices and edges. Thus, a general symmetry may have two effects: mapping nodes into nodes and permutation of node faces as stated. $\square$

Proposition 1. *The permutation matrices, which preserve the adjacency relation among node sides, form a group G. The elements of that group leave the matrix $\mathcal{R}$ in (3.13) invariant.*

Proof: Each element $\mathcal{R}^{(k)}$ of block matrix $\mathcal{R}$ is a cyclic matrix Diaconis (1990), its rows differ only in the element order. The permutations preserving adjacency form a group because there is a unit transformation; they are closed for matrix multiplication; matrix multiplication is associative, each matrix is invertible thus each has an inverse. Hence, the permutations under consideration form group G. $\mathcal{R}^{(k)}$ is cyclic, thus its element $\mathcal{R}_{ij}$ depends only on the adjacency of edges i and j, which adjacency is untouched by elements of group G. $\square$

Proposition 2. *A symmetry of (3.13) orders the nodes into cycles. The nodes in one cycle are permuted among each other. For the node-side permutation p, when the maximal orbit length is L, then $p^L = 1$.*

Proof: A symmetry maps a node into another node, this map is a permutation of nodes. According to Lemma 4.1, a symmetry can be given by a permutation $\mathbf{P}$ of nodes, actually $\mathbf{P}$ is a matrix of order N, and a node-side permutation p. The permutation $\mathbf{P}$ can be ordered into cycles, the nodes in a cycle are permuted among themselves. Applying a given symmetry as many times as the maximum cycle length, we get the original node- and node-side numbering, i.e. we get the identity transformation, thus, the associated side permutation matrix satisfies $p^L = 1$ as stated. $\square$

A corollary of Proposition 2 is that the length of an orbit in a node permutation associated with a symmetry can not be longer than the maximum order of an element in group G.

Unless otherwise stated, we consider Dirichlet problems: $\mathcal{B} \equiv 1$, where the solution vanishes on the external boundaries and we deal only with internal boundaries. The K non-zero boundary functions q_k and the solutions ϕ_m in the N nodes are collected into their respective vectors[1]:

$$\begin{aligned} \underline{q} &= (q_1, \ldots, q_K) \\ \vec{\phi} &= (\phi_1, \ldots \phi_N). \end{aligned} \tag{4.1}$$

Clearly,

$$\vec{\phi}(x) = \mathcal{M}(x, x')\underline{q}(x'), \tag{4.2}$$

where operator $\mathcal{M}$ has K columns and N rows. In a row, $\mathcal{M}$ may have at most n_F non-zero elements. $\mathcal{M}$ is also a matrix, its element $\mathcal{M}_{m\alpha}$ is defined by

$$\mathcal{M}_{m\alpha}(x, x') * q_\alpha(x') = \int_{E_\alpha} \mathcal{G}_{\text{node}\alpha(m)}(x, x') q_\alpha(x') dx'. \tag{4.3}$$

Here the notation $\alpha(m)$ refers to boundary type α in node m. The following definition aims at separating a linear integral operator from the matrix. The latter carries information on the structure of $\mathfrak{V}$, the former depends on the boundary value to be solved.

---

[1]Note the formal difference between $\underline{q}$ and $\mathbf{q}$. The former involves the solution as function along the K internal faces, the latter consists of N vectors of length n_F. The two vectors contain the same information.

Definition 5. We write the linear operator $\mathcal{M}(x, x')$ as a product of a matrix in which the elements may take the value 0 or 1, and a diagonal matrix, whose entries are integral operators:

$$\mathcal{M}(x, x') = \mathbf{Q} * \mathcal{G}_K(x, x'), \tag{4.4}$$

where $\mathbf{Q}$ is a $K \times M$ matrix and $\mathcal{G}_K$ is a diagonal matrix of order K, in the kth row its entry is $\mathcal{G}_\alpha(x, x')$ provided that boundary k is of α type.

Proposition 3. $\dim(\vec{\phi}) = \operatorname{rank}(\mathbf{Q})$.

Proof: Because $\vec{\phi}(x)$ is a linear expression (cf. Eqs. (4.2) and (4.4)) of the rows in matrix $\mathbf{Q}$, the statement follows.   □

When the solution to (2.1) is unique and $\mathbf{O}$ is a symmetry of (2.1) then $\mathbf{O} : \underline{q} \to \underline{q}$ and $\mathbf{O} : \vec{\phi} \to \vec{\phi}$. So a symmetry of problem (2.1)-(3) leaves the set of equations determining the solution along boundaries invariant.

Let us return to the continuity of the normal gradients. Now the matrix in (3.7) is a square matrix. To determine $Dim(\underline{q})$, we note that if $\mathbf{O}$ is a symmetry of $\mathfrak{V}$, then it transforms internal edges into internal edges and external edges into external edges. The symmetries of $\mathfrak{V}$ form a group G. Furthermore, the symmetries of $\mathfrak{V}$ permute edges among themselves. Symmetry transforms a vertex emanating j edges into another vertex emanating j edges. If the symmetries of $\mathfrak{V}$ commute with operator $\mathcal{A}$, a representation of the symmetry group is also an eigenspace of $\mathcal{A}$. If $\mathfrak{V}_0 = \mathfrak{V}/G$ then the solution in $\mathfrak{V}_0$ fully describes a given representation. Consequently, E/G fully describes the edges and $\underline{q}/G$ fully describes the internal faces. Now, let

$$\underline{\phi}_0(x) = \underline{\phi}(x)/G$$
$$\underline{q}_0(x) = \underline{q}(x)/G,$$

so that the zero subscripted functions refer to nodes in $\mathfrak{V}/G$. We write (4.2) on $\mathfrak{V}/G$ as

$$\underline{\phi}_0(x) = \mathcal{M}_0(x, x')\underline{q}(x'), \quad x, x' \in \mathfrak{V}/G. \tag{4.5}$$

We have arrived at the following statement.

Proposition 4. *Let G denote the symmetry group of $\mathfrak{V}$. Then, $\dim(\vec{\phi}/G) = \operatorname{rank}(\mathcal{M}_0)$.*

Since a symmetry of $\mathfrak{V}$ permutes the internal edges, we can easily formulate how to obtain a symmetry of $\mathfrak{V}$.

Proposition 5. *Let $\mathbf{O}$ be a symmetry of problem (2.1). Assume that the solution to problem (2.1) is unique. Then $\mathbf{O}\mathcal{R}\mathbf{O}^{-1} = \mathcal{R}$, $\mathbf{O}\mathbf{H}\mathbf{O}^{-1} = \mathbf{H}$ and $\mathbf{O}\mathbf{q} = \mathbf{q}$.*

Proof: According to Definition 4, item (i), $\mathbf{O}$ commutes with $\mathcal{A}$, thus according to the first property of the Green's function $\mathbf{O}\mathcal{R} = \mathcal{R}\mathbf{O}$. By definition, $\mathbf{O}$ is an automorphism of $\Gamma(\mathfrak{V}, t)$ so adjacency and non-adjacency are preserved. Applying $\mathbf{O}$ to (3.14) we get from Definition 4, item (ii): $\mathbf{S}\mathbf{O} = \mathbf{O}\mathbf{S}$ and $(\mathbf{O}(\mathbf{H}\mathcal{R})\mathbf{O}^{-1}\mathbf{O})\mathbf{q} = \mathbf{O}\mathbf{S}\mathbf{q} = \mathbf{S}\mathbf{O}\mathbf{q}$. Since $\mathbf{O}$ is a symmetry, $\mathbf{O}\vec{\phi}$ is also a solution but by the uniqueness $\mathbf{O}\vec{\phi} = \vec{\phi}$ and consequently $\mathbf{O}\mathbf{q} = \mathbf{q}$. Comparing the above equation to (3.14), and using $(\mathbf{O}(\mathbf{H}\mathcal{R})\mathbf{O}^{-1}) = (\mathbf{O}\mathbf{H}\mathbf{O}^{-1})\mathcal{R}$ we conclude $\mathbf{O}\mathbf{H}\mathbf{O}^{-1} = \mathbf{H}$ as stated.   □

5. Application of the Covering Group

Given $\mathfrak{V}$ and $\mathfrak{t}$, we are seeking a group G such that the orbit of the group elements applied to $\mathfrak{t}$ is just $\mathfrak{V}$, i.e. $\mathfrak{V} = G\backslash\mathfrak{t}$. We follow Buser's recipe as given in Buser et al. (1994). The permutation representation of G is obtained through the following steps. If $\mathfrak{V}$ consists of N copies of $\mathfrak{t}$, group G will be represented by permutations on N letters. G is constructed by means of n_F generators. We enumerate the copies of $\mathfrak{t}$ from 1 to N. We construct the first generator from those copies that are adjacent along side α, say, $(i_{\alpha1}, j_{\alpha1}), (i_{\alpha2}, j_{\alpha2}), \ldots, (i_{\alpha m}, j_{\alpha m})$ make $a = (i_{\alpha1}, j_{\alpha1}), (i_{\alpha2}, j_{\alpha2}), \ldots, (i_{\alpha m}, j_{\alpha m})$ The procedure is repeated for each side, thus we obtain n_F generators. G is the group generated by the n_F generators. That group permutates the N nodes among each other. Since most finite groups can be presented on 2 generators, this representation is quite general. When n_F is even, we can reduce the number of generators by numbering the faces of $\mathfrak{t}$ as $(\alpha, \beta, \ldots)$ along the first $n_F/2$ sides and using the second $n_F/2$ sides as "conjugated" sides, gluing side α to side α conjugated.

Example 1. If the covering group has two- or three-dimensional conjugate classes, the corresponding irreducible representations contain collective modes to be determined over several nodes. In this example, $\mathfrak{t}$ is an isosceles right triangle and $\mathfrak{V}$ consists of 8 copies of $\mathfrak{t}$. $\mathfrak{V}$ is obtained with consecutive copies of $\mathfrak{t}$ glued sequentially along the hypotenuse (γ) and along another side (α), see Fig. 1. In this case words can be written directly without resorting to a permutation representation. The 8 copies of $\mathfrak{t}$ are obtained by applying $\gamma, \alpha\gamma, \gamma\alpha\gamma, \alpha\gamma\alpha\gamma, \gamma\alpha\gamma\alpha\gamma, \alpha\gamma\alpha\gamma\alpha\gamma,$ $\gamma\alpha\gamma\alpha\gamma\alpha\gamma$ and the identity element. These 8 elements form a group isomorphic to D_4. There are 8 internal boundaries, so in the solution of a general boundary condition problem we have 8 independent functions over $\mathfrak{t}$. The 8 functions are derived from 4 functions over $\mathfrak{t}$, corresponding to even and odd functions along sides γ and α. By means of the character table of group D_4, any function $f(x)$ given for $x \in \mathfrak{V}$ can be decomposed into irreducible components as

$$f_i(x) = \sum_{g \in D_4} \chi_i^*(g) * f(g^{-1}x). \tag{5.1}$$

Thus, the irreducible components of the solution in $\mathfrak{V}$ along any side of type β represent a linear combination of the boundary values along the 8 external faces of the 8 copies of $\mathfrak{t}$ that make up $\mathfrak{V}$.

In the case of a Dirichlet problem, we have to solve equation (2.1) over $\mathfrak{t}$ with the boundary condition solution being zero along sides β. In this case $t = \mathfrak{V}/G$ and $\mathfrak{t}$ has 3 sides of which one is always an external boundary.

According to Proposition 3, $Dim(\vec{\phi}) = 8$. Thus, the most general boundary condition involves 8 linearly independent solutions in a tile $\mathfrak{t}$. There are four one dimensional representations of the group D_4, the corresponding solutions obey the boundary conditions in Table 1.

We use the term *reflective* for normal gradient equals zero, and *black* for function equals zero on the boundary. There are two, two-dimensional irreps, which are determined for two attached copies of $\mathfrak{t}$; hence, these values are fixed at two α type faces and at a γ type face, see Table 2.

Example 2. The covering group usually leads to a reduction in problem size. To illustrate this, let us consider a square shaped tile and let $\mathfrak{V}$ be composed of two squares, see Fig. 2. The length of the side of the tile is a.

Irrep	Boundary condition at	
	side α	side γ
A_1	reflective	reflective
A_2	black	black
B_1	black	reflective
B_2	reflective	black

TABLE 1. Boundary conditions for 1D Irreps of symmetries of Fig. 1

Irrep	Boundary condition at		
	side α	side γ	side α
E_1	black	None	reflective
E_2	black	None	reflective
E_3	reflective	reflective	black
E_4	reflective	black	black

TABLE 2. Boundary conditions for 2D Irreps of symmetries of Fig. 1
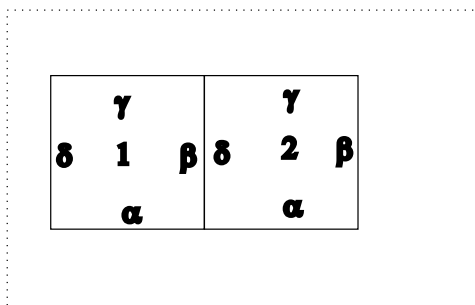
FIGURE 2. $\mathfrak{V}$ composed of two squares

Here node No. 2 can be obtained by a translation along the horizontal axis from node No. 1, and $\mathfrak{V}$ is obtained by $mod\,2$ addition. The covering group has 2 elements and is isomorphic to the inversion group. In contrast to the previous example, here the group elements map an internal side into an external one (sides β and δ) and vice versa. There is one internal side so in a Dirichlet problem the solution is the same in the left and right squares. In the general case, we decompose the prescribed value of the solution along the boundary into irreps of the covering group and solve one boundary-value problem in one node for each irrep. Let us consider the problem

$$\Delta\Phi(x) + B^2\Phi(x) = 0, \quad x \in \mathfrak{V}.$$

Its solution is $\Phi(x,y) = cos(\mu x)cos(\nu y)$ with $\mu^2 + \nu^2 = B^2$.

Since the covering group is isomorphic to the inversion group, we decompose the solution into irreducible representations of the covering group, i.e. into an even and
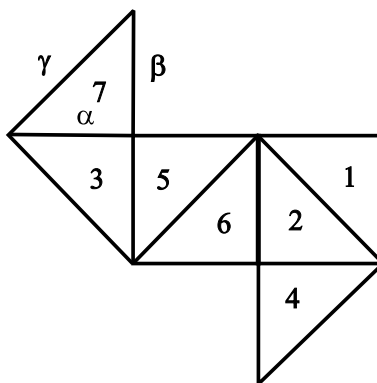
FIGURE 3. $\mathfrak{V}$ composed of 7 triangles

an odd component in x:

$$\Phi_{\text{even}}(x,y) = u_0 \frac{\cos(\mu x)}{\cos(\mu * a/2)} \cos(\nu y)$$

$$\Phi_{\text{odd}}(x,y) = u_1 \frac{\sin(\mu x)}{\sin(\mu a/2)} \cos(\nu y).$$

If we choose

$$u_0 = \cos^2(\mu * a/2)$$
$$u_1 = \sin^2(\mu * a/2),$$

the even and odd solutions are exact. Thus, we have reduced the solution of the problem to a fraction of $\mathfrak{V}$ using the covering group.

Example 3. The covering group may introduce simplifications even in asymmetric volumes. Notwithstanding, it is not trivial how to find the covering group even for simple volumes. To demonstrate this, we present an example after Gordon et al. (1992) and Buser et al. (1994). In Fig. 3, $\mathfrak{V}$ is composed of 7 isosceles right triangles. The edges of $\mathfrak{t}$ are labeled as α, β, γ. Adjacent triangles are attached along the appropriate edges, e.g. 7 and 3 share an α edge. As γ is always the hypotenuse, and at an edge only two types of side are incident, the types of edges are unique in Fig. 3. To find the covering group, we follow Buser (1988). We make the map ($\alpha \to a, \beta \to b$ and $\gamma \to c$). Since along side α we map nodes 7 and 3 into each other, as well as 6 and 2, the permutation representations of a, b and c are readily available:

$$a = (7,3)(6,2)$$
$$b = (2,4)(3,5)$$
$$c = (5,6)(1,2).$$

Group G, generated by a, b, and c has 168 elements (in group theoretic calculations the GAP program was used, see GAP (1995)), and G is isomorphic to $SL(3,2)$ and $PSL(2,7)$. Let the subgroup $A = Stabilizer(G,1)$ be the stabilizer of subvolume 1, that subgroup has 24 elements. A coset representation of G is

$$G = \cup_{i=1}^{7} G_i = \cup_{i=1}^{7} A * u_i$$

where

$$u_1 = () \quad u_2 = (1,2)(5,6)$$
$$u_3 = (1,3,2)(5,6,7) \quad u_4 = (1,4,2)(3,5,6)$$
$$u_5 = (1,5,7,6,3,4,2) \quad u_6 = (1,6,3,7,5,4,2)$$
$$u_7 = (1,7,4,2)(3,6).$$

The u_i elements are transformed under the generators a, b and c into some L_j if $u_i * s \in G_j$ for some j; and s is a generator of G. In the following table, indices j of $u_i * s \in G_j$ are given for $s = a, b$ and c and $i = 1, \ldots, 7$: which corresponds to Fig. 3. Gordon, et al. (1992) have shown that Fig. 3 can also be considered as a Cayley graph of G.

i	$u_i * a$	$u_i * b$	$u_i * c$
1	1	1	2
2	6	4	1
3	7	5	3
4	4	2	4
5	5	3	6
6	2	6	5
7	3	7	7

TABLE 3. Coset indices for products $u_i * s$, for $s = a, b, c$

6. Isospectral Plane Manifolds

Further computational applications of isospectral, simply connected manifolds, depend on the ability to generate such manifolds. Based on the relationship between the connectivity matrix and the geometry matrix, we show that if $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are volumes with respective equivalent geometry matrices $\mathbf{C}_1$ and $\mathbf{C}_2$, then the solutions of (2.1) on the boundaries are also equivalent. The statement is formulated in Lemma 6.1.

Lemma 6.1. *Let volumes $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be such that (a) Each consists of N copies of tile $\mathfrak{t}$. There are the same number of copies of $\mathfrak{t}$ in $\mathfrak{V}_1$ and $\mathfrak{V}_2$ which are filled with the same material. (b) The associated geometry matrices take the form $\mathbf{H}_1 = \mathbf{C}_1 \times \mathbf{H}_{ij} + \mathbf{S}_1$ and $\mathbf{H}_2 = \mathbf{C}_2 \times \mathbf{H}_{ij} + \mathbf{S}_2$, where $\mathbf{C}_1$ and $\mathbf{C}_2$ are the connectivity matrices of order N, and $\mathbf{H}_{ij}$ are the edge connection matrices of order n_F introduced in connection with (3.13). $\mathbf{S}_i, i = 1, 2$ refer to the external boundary, they are diagonal block matrices of order N, the order of each block being n_F. (c) Let $\mathbf{C}_2 = \mathbf{T}\mathbf{C}_1\mathbf{T}^{-1}$, and $\mathcal{R}_2 = \mathbf{T}\mathcal{R}_1\mathbf{T}^{-1}$, where matrix $\mathbf{T}$ is of order N. (d) As to the external boundaries, we assume $\mathbf{T}\mathbf{S}_1 = \mathbf{S}_2\mathbf{T}$. Then, from*

$$\mathbf{H}_1 \mathcal{R}_1 \mathbf{q}_1 = \mathbf{S}_1 \mathbf{q}_1 \tag{6.1}$$

and

$$\mathbf{H}_2 \mathcal{R}_2 (\mathbf{q}_2) = \mathbf{S}_2 (\mathbf{q}_2) \tag{6.2}$$

it follows that $\mathbf{q}_2 = \mathbf{T}\mathbf{q}_1$.

Proof: From conditions (a), (b), (c) and (d), the geometry matrices are equivalent: $\mathbf{H}_2 = \mathbf{T}\mathbf{H}_1\mathbf{T}^{-1}$ and because of condition (a), the following relationship holds for the response matrices: $\mathcal{R}_2 = \mathbf{T}\mathcal{R}_1\mathbf{T}^{-1}$. Now, multiplying (6.1) by $\mathbf{T}$ we get

$$\mathbf{T}\mathbf{H}_1\mathbf{T}^{-1}\mathbf{T}\mathcal{R}_1(\mathbf{T}^{-1}\mathbf{T})\mathbf{q}_1 = \mathbf{T}\mathbf{S}_1\mathbf{q}_1 = \mathbf{S}_2\mathbf{T}\mathbf{q}_1, \tag{6.3}$$

where, in the last equation we have used condition (d). By means of the relationships between the geometry matrices and the response matrices of $\mathfrak{V}_1$ and $\mathfrak{V}_2$, the above equation becomes $\mathbf{H}_2\mathcal{R}_2\mathbf{T}\mathbf{q}_1 = \mathbf{S}_2\mathbf{T}\mathbf{q}_1$, from which we get $\mathbf{q}_2 = \mathbf{T}\mathbf{q}_1$ as stated. $\square$

According to Lemma 6.1, two volumes are equivalent if their connectivity matrices are similar and the geometry matrices contain the same blocks. This means that the nodes in $\mathfrak{V}_1$ should match the nodes in $\mathfrak{V}_2$ with regard to the external and internal faces. Such volume pairs are generated by group theoretic means Buser (1988), Brooks (1988), Gordon et al. (1992). In such volume pairs, the solutions on every boundary of $\mathfrak{V}_1$ will be a linear combination of the boundary fluxes of $\mathfrak{V}_2$. Because of (4.2), this implies that the node functions in $\mathfrak{V}_1$ are linear expressions of the node functions in $\mathfrak{V}_2$. In a number of applications, it suffices to show that the dominant eigenvalues are the same for some volumes $\mathfrak{V}_1$ and $\mathfrak{V}_2$ and the associated eigenfunctions can be transformed into each other. If we know the solutions in $\mathfrak{V}_1$ and $\mathfrak{V}_2$ and there is a matrix transforming one solution into the other, the geometry- and response matrices of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are also related.

Lemma 6.2. *Let us consider two volumes, $\mathfrak{V}_1$ and $\mathfrak{V}_2$, each consisting of N copies of a tile $\mathfrak{t}$. Let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ have the same number of internal and external faces. If the eigenvalue problem (3.13) leads to*

$$\mathbf{H}_i\mathcal{R}_i\mathbf{q}_i = \mathbf{S}_i\mathbf{q}_i, \quad i = 1, 2 \tag{6.4}$$

let the following relations hold between the solutions $\underline{\phi}_1$ and $\underline{\phi}_2$:

$$\underline{\phi}_1(x) = \mathbf{U}\underline{\phi}_2(x), \tag{6.5}$$

where $\mathbf{U}$ is invertible, furthermore,

$$\mathbf{S}_1\mathbf{U}\mathbf{q}_2 = \mathbf{V}\mathbf{S}_2\mathbf{q}_2 \tag{6.6}$$

where $\mathbf{V}$ is invertible, then the eigenvalues of problem (3.13) are the same on $\mathfrak{V}_1$ and $\mathfrak{V}_2$ and the geometry- and response matrices of the two problems are similar: $\mathbf{V}\mathbf{H}_2 = \mathbf{H}_1\mathbf{U}$ and $\mathcal{R}_1\mathbf{U} = \mathbf{U}\mathcal{R}_2$.

Proof: First, from (6.5) letting x tend to the boundary of $\mathfrak{t}$ we get $\mathbf{q}_1 = \mathbf{U}\mathbf{q}_2$. Substituting this into (6.4) with $i = 1$ we get $\mathbf{H}_1\mathcal{R}_1\mathbf{U}\mathbf{q}_2 = \mathbf{S}_1\mathbf{U}\mathbf{q}_2$. Now using (6.6), we obtain

$$\mathbf{V}^{-1}\mathbf{H}_1\mathcal{R}_1\mathbf{U}\mathbf{q}_2 = \mathbf{S}_2\mathbf{U}\mathbf{q}_2. \tag{6.7}$$

From this, we see

$$\mathbf{V}^{-1}\mathbf{H}_1\mathbf{U}\mathbf{U}^{-1}\mathcal{R}_1\mathbf{U}\mathbf{q}_2 = (\mathbf{V}^{-1}\mathbf{H}_1\mathbf{U})(\mathbf{U}^{-1}\mathcal{R}_1\mathbf{U})\mathbf{q}_2 = \mathbf{H}_2\mathcal{R}_2\mathbf{q}_2, \tag{6.8}$$

with the correspondence $\mathbf{H}_2 = \mathbf{V}^{-1}\mathbf{H}_1\mathbf{U}$ and $\mathcal{R}_2 = \mathbf{U}\mathcal{R}_1\mathbf{U}^{-1}$, as stated. $\square$

Sunada (1985), Brooks (1988) and Gordon et al. (1992) have introduced a systematic way of finding isospectral shapes.

Definition 6. Let G be a finite group, and let G_1 and G_2 be subgroups of G. Then, (G, G_1, G_2) is a Sunada triple if the left $R[G]$-modules $R[G/G_1]$ and $R[G/G_2]$ determined by the G-sets G/G_1 and G/G_2 are orthogonally isomorphic, where the

inner products are those for which G/G_1 and G/G_2 are orthogonal bases. Gordon et al. (1992).

The Sunada condition is equivalent to the assertion that G_1 and G_2 are almost conjugate subgroups of G, i.e., there is a bijection of G_1 with G_2 carrying every element $\xi \in G_1$ to a conjugate element $g\xi g^{-1} \in G_2$, where the conjugating element g may depend upon ξ.

Theorem 6.3. *(Sunada) Let $\mathfrak{M}$ be a compact Riemannian manifold, G a finite group acting on $\mathfrak{M}$ by isometries. Suppose that (G, G_1, G_2) is a Sunada triple, and that G_1 and G_2 act freely on $\mathfrak{M}$. Then the quotient manifolds $\mathfrak{M}_1 = G_1\backslash\mathfrak{M}$ and $\mathfrak{M}_2 = G_2\backslash\mathfrak{M}$ are isospectral.*

Based on a proof Berard (1992) of the Sunada theorem, the requirement that G_1 and G_2 act freely, can be removed.

Theorem 6.4 (Berard)**.** *Let $\mathfrak{M}$ be a compact Riemannian manifold upon which a finite group G acts by isometries. Suppose that (G, G_1, G_2) is a Sunada triple. Then the quotient orbifolds $\mathfrak{O}_1 = G_1\backslash\mathfrak{M}$ and $\mathfrak{O}_2 = G_2\backslash\mathfrak{M}$ are isospectral. If $\mathfrak{M}$ has a boundary, then $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are isospectral both with Dirichlet and Neumann boundary conditions.*

We have to show that $\mathfrak{O}_1$ and $\mathfrak{O}_2$ satisfy the conditions postulated in Lemma 6.2. $\mathfrak{O}_1$ and $\mathfrak{O}_2$ consist of the same number of copies of tile t, they have the same number of external faces because there is a one-to-one relationship between the elements of G_1 and G_2. The $R[G]$ modules allow the construction of matrix **U**.

7. Concluding Remarks

We have dealt with the solution of a linear elliptic equation in a finite volume consisting of congruent nodes. In Section 4, we described the utilized basic concepts and notations. The condition that the solution must be continuous along with its first derivative gives an integral equation for the solution along the node boundaries. The solution is given by means of the tile's Green's function. The $\mathcal{M}$ matrix is factored into two components: the first component **Q** is a matrix, it contains all information on the geometry of the investigated volume $\mathfrak{V}$ and allows one to establish the number of independent node functions in the solution. The second component associates an integral operator to every boundary and depends only on the equation to be solved. In Section 5, we dealt with examples.

Example 1 illustrates the existence of collective modes when the covering group has a multidimensional irrep. A collective mode spreads over several nodes and its components are transformed into each other under an element of the covering group. A detailed investigation of the covering group in the context of Quantum Mechanics may result in a systematic way of finding finite volumes in which the energy levels of electrons are the same. In connection with *Example 2*, we demonstrated how the covering group reduces the size of the problem. The asymmetric volume of *Example 3* also has a covering group, the numerical solution is reducible to a part of the seven triangles.

In Section 6, we have formulated conditions for eigenvalue problems in volumes $\mathfrak{V}_1$ and $\mathfrak{V}_2$ to be isospectral. If the response and geometry matrices meet the stipulated conditions, the solutions of the two problems can be transformed into each other. The postulated conditions are consistent with those of a few other known isospectral problems.

We presented a formulation of the boundary-value problem, which is applicable in practice and capable of accommodating the connectivity matrix and the covering group. This facilitates combining group- and graph theory with numerical methods. Graph- and group theory offers a means to make the investigation of boundary value problems easier. The structure of the response matrix is similar to the structure of the connectivity matrix, allowing us to find equivalent problems, see Section 6. The covering group has made it possible to reduce the solution in an asymmetric discretized volume. The structure of the covering group limits the achievable reduction: The higher the dimension of the irreducible subspaces induced by the covering group, the less reduction achieved. Thus, some of the group theoretical techniques becomes applicable to asymmetric volumes, as in Section 6.

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