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The nonlocal bistable equation: Stationary solutions on a bounded interval *

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Abstract

We discuss instability and existence issues for the nonlocal bistable equation. This model arises as the Euler-Lagrange equation of a nonlocal, van der Waals type functional. Taking the viewpoint of the calculus of variations, we prove that for a class of nonlocalities this functional does not admit nonconstant C^1 local minimizers. By taking variations along non-smooth paths, we give examples of nonlocalities for which the functional does not admit local minimizers having a finite number of discontinuities. We also construct monotone solutions and give a criterion for nonexistence of nonconstant solutions.

1 Introduction

We study the semilinear integral equation (the nonlocal bistable equation)

$$-J[u] + ju + f(u) = 0, \tag{1.1}$$

on the interval $(0,1)$, where

$$J[u](x) = \int_0^1 J(x,y)u(y)dy, \quad j(x) = \int_0^1 J(x,y)dy.$$

We assume $J \in W^{1,1}$ is symmetric, in the sense $J(x,y) = J(y,x)$ for $x, y \in (0,1)$, and $f \in C^1$ is a bistable function with three zeros: $-1, a \in (-1,1), 1$, with $f'(\pm 1) > 0$ and $f'(a) < 0$. Equation (1.1) (which has no boundary conditions) arises as the Euler-Lagrange equation of the functional (defined on $L^2(0,1)$)

$$I(u) = \frac{1}{4}\int_0^1\int_0^1 J(x,y)(u(x)-u(y))^2 dxdy + \int_0^1 W(u(x))\,dx, \tag{1.2}$$

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where W is a double-well function. Depending on the type of problem studied, one can impose the mass constraint $\int_0^1 u(x)dx = m$ on (1.2). This was done in the seminal work of van der Waals in 1892 [14], who simplified (1.2) by expanding the nonlocal part in power series and considering the first order approximation

$$I^{\rm loc}(u) = \frac{1}{2}\int_0^1 |u'(x)|^2 dx + \int_0^1 W(u(x))dx. \tag{1.3}$$

The gradient flow of (1.3) without the mass constraint:

$$u_t = u_{xx} - f(u), \quad u'(0) = u'(1) = 0, \tag{1.4}$$

is sometimes referred to as the Ginzburg-Landau or Allen-Cahn equation. (1.2) can be viewed as a model for materials whose constitutive relations are nonlocal (see [11, 12] for other examples). Namely, if u denotes the general phase field characterizing the state of a material, and if the energy density of the continuum is postulated to be $e = -J[u]u/2 + ju^2/2 + W(u)$, then the total free energy can be written as

$$I(u) = \int_0^1 e(x)dx = \int_0^1 \big(-\frac{1}{2}J[u]u + \frac{ju^2}{2} + W(u)\big)dx. \tag{1.5}$$

(1.2) can also be derived from elementary statistical mechanics. In [2] and [3] it was shown that (1.2) arises as the Helmholtz free energy of an Ising-like spin system with long range interactions. In particular, in this approach J can change sign and W does not have to be balanced. An infinite lattice system similar to (1.1) was studied in [3, 1].

Results on solutions of the whole line version of (1.1) (i.e., $-J*u+ju+f(u) = 0$) can be found in e.g., [4, 6, 7, 2, 8]. An interesting discovery was made in those papers: the existence of discontinuous solutions when $ju + f(u)$ is not monotone. In [8] and [9] we studied (1.1) for $ju + f(u)$ monotone and established, using singular perturbation techniques, another interesting phenomenon: the *existence* of nonconstant local minimizers of I for a class of sign-changing interaction kernels J. By comparison, the local functional (1.3) does not admit nonconstant local minimizers, the stationary solutions of (1.4) are metastable, and the evolution of (1.4) is through slow motion [10, 5]. In this paper we show that for $ju + f(u)$ monotone and a different wide class of J's which are nonnegative and translationally invariant (i.e., $J(x, y) = J(x-y)$), nonconstant local minimizers do *not* exist. In the case $ju + f(u)$ non-monotone (where solutions are in general discontinuous), we show in some examples that variations along non-smooth paths lead to a similar nonexistence result. Using an iteration method, we construct monotone solutions of (1.1) under certain assumptions. To our knowledge, it is the first nonperturbative existence result for a class of nonlocal equations such as (1.1). Finally, we give a criterion for nonexistence of stationary solutions of (1.1).

2 Nonexistence of local minimizers

(1.1) has a rich structure of solutions, whose properties in general depend both on the nonlocality J and on the nonlinearity f. In [8] and [9] we showed using the Γ-convergence method that by taking J and W having particular forms:

$$J(x,y) = \frac{1}{\epsilon} J^s\Big(\frac{x-y}{\epsilon}\Big) - \epsilon J^l(x,y), \quad W = W_0 + \epsilon W_1, \tag{2.1}$$

with $J^s \geq 0$, $\frac{js^2}{2} + W(s)$ convex in s, $W_0(-1) = W_0(1)$ and $|W_1(1) - W_1(-1)| < 2\int_0^1 J^l(0,y)dy$, there exist nonconstant local minimizers of I for $\epsilon > 0$ small enough. Observe that J in (2.1) changes sign for $\epsilon > 0$ small enough. The properties of these minimizers are as follows. Briefly speaking, if I_ϵ is the energy (1.2) corresponding to (2.1), then $\frac{1}{\epsilon} I_\epsilon$ Γ-converges to I_0, defined by

$$I_0(u) = \begin{cases} c_0 \frac{||Du||(0,1)}{2} + I^l(u) & \text{if u} \in \mathrm{BV}((0,1),\{-1,1\}), \\ \infty & \text{otherwise} \end{cases} \tag{2.2}$$

where $I^l(u) = -\frac{1}{4}\int_0^1\int_0^1 J^l(x,y)(u(x)-u(y))^2 dxdy + \int_0^1 W_1(u(x))dx$ and $c_0\frac{1}{2}(||Du||(0,1))$ is equal to a constant multiplied by the number of jumps u has. If an isolated local minimizer of I_0 exists in the space of step functions having a fixed number of jumps, then $\frac{1}{\epsilon} I_\epsilon$ also has a local minimizer, which is C^1 and L^2-close to the BV one. The layers $\xi_1, \dots, \xi_n$ of the minimizers of I_0 are determined from the system

$$J^l[u](\xi_i) = -\frac{1}{2} f_1(r)dr, \quad i = 1, \dots, n, \tag{2.3}$$

where $f_1 = W_1'$. (2.3) is in general difficult to solve. In [8] we considered J^l to be the Green's function of the linear differential equation $-\gamma^2 v'' + v = u$, $v'(0) = v'(1) = 0$, i.e.,

$$J^l(x,y) = \frac{1}{\gamma(e^{\frac{1}{\gamma}} - e^{-\frac{1}{\gamma}})}\Big[\cosh\Big(\frac{x+y-1}{\gamma}\Big) + \cosh\Big(\frac{|x-y|-1}{\gamma}\Big)\Big], \ \gamma > 0. \tag{2.4}$$

In this case, (2.3) can be written as a system of ODE's, and we showed that for every n, there exists a unique solution of (2.3) such that $u(\xi_1+) = -1$, and it is an isolated local minimum of (2.2). In [9], we considered more general $J^l(x,y) = J^l(x-y)$. This is a more complicated case, as (2.3) becomes progressively more difficult to solve with an increasing number of layers. On an interesting note, we found that if J^l changes sign, then in the class of critical points u of (2.2) having one jump at ξ such that $u(\xi+) = -1$, there are in general two local minima ξ^1, ξ^3, and a local maximum ξ^2 between them, i.e., $\xi^1 < \xi^2 < \xi^3$. We do not know if ξ^2 can be perturbed for small $\epsilon > 0$ to, say, a mountain pass solution of $\frac{1}{\epsilon} I_\epsilon$.

We now show that for a class of nonnegative J's there are no local minimizers of I.

Theorem 2.1 *Let $J(x,y) = J(x-y)$ and $j(\frac{1}{2}) + f'(s) > 0$ for $s \in [-1,1]$. Let $J'(x) \leq 0$ for $x \in (0,1)$ and $J(1) \geq 0$. Then any nonconstant solution of (1.1) is unstable, in the sense that it is not a local minimum of I.*

Proof. With the regularity of J and f we can compute the second variation of I. For every $w, \phi \in L^2(0,1)$

$$\frac{d^2 I(w+\epsilon\phi)}{d\epsilon^2}\Big|_{\epsilon=0} = \int_0^1 [-J[\phi]\phi + j\phi^2 + f'(w)\phi^2]dx. \tag{2.5}$$

It suffices to show that the right side of (2.5) is < 0 for $w = u$ and a particular choice of ϕ. Let us assume that u is a critical point of I, i.e., a solution of (1.1). Then (1.1) can be rewritten as

$$u = (j \cdot + f(\cdot))^{-1}(J[u]).$$

The regularity of J and f imply that u is in C^1. Differentiating (1.1) with respect to x, we deduce

$$-J[u'] + ju' + f'(u)u' = J(x-1)(u(x)-u(1)) - J(x)(u(x)-u(0)).$$

We now choose $\phi = u'$. Multiplying the last equation by u' and integrating over $(0,1)$, we obtain

$$\frac{d^2 I(u+\epsilon u')}{d\epsilon^2}\Big|_{\epsilon=0} = \int_0^1 [J(x-1)(u(x)-u(1)) - J(x)(u(x)-u(0))]u'(x)dx.$$

We break up this expression into four integrals and integrate by parts each one of them to get

$$\begin{aligned}
\int_0^1 J(x-1)u(x)u'(x)dx &= \frac{1}{2}[J(0)u(1)^2 - J(1)u(0)^2] - \frac{1}{2}\int_0^1 J'(x-1)u(x)^2 dx,\\
\int_0^1 J(x-1)u(1)u'(x)dx &= J(0)u(1)^2 - J(1)u(1)u(0) - \int_0^1 J'(x-1)u(1)u(x)dx,\\
\int_0^1 J(x)u(x)u'(x)dx &= \frac{1}{2}[J(1)u(1)^2 - J(0)u(0)^2] - \frac{1}{2}\int_0^1 J'(x)u(x)^2 dx,\\
\int_0^1 J(x)u(0)u'(x)dx &= J(1)u(0)u(1) - J(0)u(0)^2 - \int_0^1 J'(x)u(0)u(x)dx.
\end{aligned}$$

By completing the squares in the multiples of $J'(x)$ and $J'(x-1)$ we finally get

$$\begin{aligned}
\frac{d^2 I(u+\epsilon u')}{d\epsilon^2}\Big|_{\epsilon=0} \equiv R(u') \quad &= \quad -J(1)(u(1)-u(0))^2\\
&\quad -\frac{1}{2}\int_0^1 J'(x-1)[u(x)-u(1)]^2 dx\\
&\quad +\frac{1}{2}\int_0^1 J'(x)[u(x)-u(0)]^2 dx \le 0. \qquad (2.6)
\end{aligned}$$

We now show that actually $R(u') < 0$. Assume that $R(u') = 0$. Then also $R(|u'|) = 0$ ($\int_0^1\int_0^1 J(x-y)|u'(x)||u'(y)|dxdy \ge \int_0^1\int_0^1 J(x-y)u'(x)u'(y)dxdy$).

Consider the variational problem $\inf_{||\phi||_2=1} R(\phi)$. If this inf is less than 0, then there exists some ϕ_0 for which $R(\phi_0) < 0$, thus u is unstable. If it is equal to 0, then since it is achieved at $\phi_1 \equiv |u'|/||u'||_2$ we have

$$-J[|u'|] + j|u'| + f'(u)|u'| = \lambda|u'|.$$

If $|u'| = 0$ at some x_0, then $-J[|u'|](x_0) + j(x_0)|u'(x_0)| = 0$, which implies that $|u'| \equiv 0$ (an inductive argument is used if $|\mathrm{suppJ}| < 2$), a contradiction. So $|u'| > 0$. But then it is easily seen that $R(u') < 0$. □

The following example [9] shows that condition $J \geq 0$ might be relaxed somewhat.

Theorem 2.2 *If in Theorem 2.1 we set $J(x) = b - m|x|$, $b, m > 0$, and $b \geq 3m/4$, then any nonconstant solution of (1.1) is unstable.*

Proof. Let u be a solution of (1.1). In (2.6) we break up $-(b-m)(u(1)-u(0))^2$ and use $\frac{m}{4}(u(1) - u(0))^2$ together with the other two terms in (2.6) to complete the square. We get

$$\begin{aligned} \frac{d^2 I(u + \epsilon u')}{d\epsilon^2}\Big|_{\epsilon=0} &= -\big(b - \frac{3m}{4}\big)[u(1) - u(0)]^2 \\ &\quad -m \int_0^1 \big[u(x) - \frac{1}{2}(u(1) + u(0))\big]^2 dx < 0. \end{aligned}$$

Note that J changes sign on $(-1, 1)$ if $\frac{3m}{4} \leq b < m$. □

In the case $j + f'$ changes sign the previous argument cannot be used. In fact, as was shown in [2], in some cases (1.1) has discontinuous solutions with discontinuities along arbitrarily prescribed interfaces, which are stable in L^∞ norm. However, we give two examples showing instability in L^2 sense. Recall that I is defined on $L^2(0, 1)$.

Theorem 2.3 *Let $J(x, y) = c$ and $c+f'(s) > 0$ on $[-1, 1]\setminus[s_1, s_2]$, $c+f'(s) < 0$ on (s_1, s_2). Then any nonconstant solution of (1.1) with a finite number of discontinuities is unstable, in the sense it is not a local minimum of I.*

Proof. Let u be a solution of (1.1). Note that $g(u) = \int_0^1 u(y)dy = const$, thus u is a step function. Assume that $u(x) = u_i$ on (ξ_i, ξ_{i+1}), $1 \leq i \leq k - 1$. For small $\epsilon > 0$, extend u to $(-\epsilon, 0)$ by setting $u(x) = u(0)$, so that $u(x - \epsilon)$ is defined on $(0, 1)$. It now suffices to note that the second directional derivative along the continuous but non-smooth path $u(\cdot - \epsilon)$ is negative:

$$\frac{d^2 I(u(\cdot - \epsilon))}{d\epsilon^2}\Big|_{\epsilon=0} = -\Big(\sum_{i=1}^{n} (-1)^{i+1} u_i\Big)^2 < 0.$$

3 Variations along non-smooth paths

In this section we further examine the role played by non-smooth paths of variations when studying the functional I. We show that when $ju + f(u)$ is not monotone, or in other words $\frac{ju^2}{2} + W(u)$ in (1.5) is not convex, variations along non-smooth paths select some special discontinuous solutions of (1.1).

We consider the following example. Let J be as in (2.4) with $\gamma = 1$. Let f be the piecewise linear function

$$f(u) = \begin{cases} u+1 & u < 0 \\ u-1 & u > 0. \end{cases} \tag{3.1}$$

Since the solutions we will consider jump across value 0, the discontinuity of f at 0 does not really violate the requirement $f \in C^1$. We may modify f to make it smooth after we have found jump solutions.

The equation (1.1) can be written as a system

$$\begin{gathered} -v'' + v = u \\ -v + u + u \pm 1 = 0 \\ v'(0) = v'(1) = 0 \end{gathered} \tag{3.2}$$

Denote the set of discontinuities of a solution by $\xi_1, \xi_2, \dots, \xi_k$. Assume $u < 0$ on $(0, \xi_1)$, $u > 0$ on (ξ_1, ξ_2), etc.

Denote the set of all such vectors ξ_i by A_k^-, where $-$ refers to the fact that $u < 0$ on $(0, \xi_1)$. Let

$$u = \frac{v - (\pm 1)}{2} \tag{3.3}$$

Substitute the second equation of (3.2) into the first to obtain

$$-v'' + \frac{v}{2} = -\frac{\pm 1}{2}, \ v'(0) = v'(1) = 0 \tag{3.4}$$

Let G be Green's function of this ODE. With the ξ_i's fixed we solve the last equation to find v and then u by (3.3). This way we have obtained a discontinuous solution of (1.1). The energy of this solution can be written as

$$\begin{aligned} I(u) &= \int_0^1 [-\frac{1}{2}J[u]u + \frac{j}{2}u^2 + W(u)]\,dx \\ &= \int_0^1 [-\frac{1}{2}u(v-u) + W(u)]\,dx \\ &= \int_0^2 [-\frac{1}{2}u(u\pm 1) + \frac{1}{2}(u \pm 1)^2]\,dx \\ &= \int_0^1 \frac{1}{2}(1 \pm u)\,dx \\ &= \frac{1}{4} + \frac{1}{4}[\int_0^{\xi_1} v\,dx - \int_{\xi_1}^{\xi_2} v\,dx + \int_{\xi_2}^{\xi_3} v\,dx - \dots] \end{aligned} \tag{3.5}$$

Now we treat the ξ_i's as variables and obtain a family of variations of I. Here the variations are taken along discontinuous solutions of (1.1). This family is continuous but not C^1 under the L^2-norm.

We differentiate I with respect to ξ_i. For instance

$$\begin{aligned}
\frac{\partial I(u)}{\partial \xi_1} &= \frac{1}{2}v(\xi_1) + \frac{1}{4}\Big[\int_0^{\xi_1} \frac{\partial v}{\partial \xi_1}\,dx - \int_{\xi_1}^{\xi_2} \frac{\partial v}{\partial \xi_1}\,dx + \int_{\xi_2}^{\xi_3} \frac{\partial v}{\partial \xi_1}\,dx - \dots\Big] \\
&= \frac{1}{2}v(\xi_1) + \frac{1}{4}\Big[-\int_0^{\xi_1} G(x,\xi_1)\,dx + \int_{\xi_1}^{\xi_2} G(x,\xi_1)\,dx + \dots\Big] \\
&= \frac{1}{2}v(\xi_1) + \frac{1}{2}v(\xi_1) \\
&= v(\xi_1)
\end{aligned}$$

since

$$\begin{aligned}
\frac{\partial v}{\partial \xi_1} &= \frac{\partial}{\partial \xi_1}\Big[\int_0^{\xi_1} G(x,y)(-\frac{1}{2})\,dy + \int_{\xi_1}^{\xi_2} G(x,y)\frac{1}{2}\,dy + \dots\Big] \\
&= -G(x,\xi_1),
\end{aligned}$$

and in a similar way

$$\frac{\partial I(u)}{\partial \xi_i} = (-1)^{i+1}v(\xi_i). \tag{3.6}$$

Let us now look for solutions of I with k jump discontinuous points that are stationary with respect to these non-smooth paths of variations. Set $\frac{\partial I(u)}{\partial \xi_i} = 0$, we conclude that

$$v(\xi_i) = 0,\ \ i = 1, 2, \dots, k. \tag{3.7}$$

Two conclusions are drawn from (3.7). First, according to (3.3) at ξ_1 u must jump from $-1/2$ to $1/2$, and in general at ξ_i u jumps from $(-1)^{i+1}/2$ to $(-1)^i/2$. The two numbers $-1/2$ and $1/2$ are precisely the two global minima of the function $\frac{u^2}{2} + W(u)$ that appears in (1.5). Also see [13] for the role played by these two numbers.

The second conclusion is that the ξ_i's are equally distributed. This is because we may solve the equation (3.4) on each (ξ_i, ξ_{i+1}) with the boundary conditions (3.7). Then the continuity of v' across the ξ_i's requires that the sub-intervals (with the exception of $(0, \xi_1)$ and $(\xi_k, 1)$) all have the same length. The two end intervals have half the length. Such a solution u is unique in A_k^-.

Let us now compute the energy of this particular u. Because of (3.5) and $\xi_1 = 1/(2k)$ we deduce

$$I(u) = \frac{1}{4} - 2kv'(\frac{1}{\xi_1}).$$

Because v satisfies (3.4) and $v'(0) = v(\xi_1) = 0$, we solve the ODE to find

$$v(x) = -\frac{\sinh\frac{x}{\sqrt{2}}}{\cosh\frac{1}{2k\sqrt{2}}}.$$

So

$$I(u) = \frac{1}{4} - \frac{k}{2\sqrt{2}} \tanh \frac{1}{2k\sqrt{2}}, \tag{3.8}$$

which is increasing in k. A similar computation in A_k^+ shows that the critical point u in A_k^+ has the same energy.

We remark that u a local maximum of I in A_k^- with respect to ξ_i. Suppose this is not true. Since there is only one critical point of I in A_k^-, the maximum of I with respect to ξ_i must be achieved at some $\overline{u}$ on the boundary of the domain of ξ_i, which is identified by the union of all $A_m^{\pm}$ with $m < k$. Suppose $\overline{u}$ is in $A_m^{\pm}$. Consider in this $A_m^{\pm}$ the stationary solution u^* with respect to the similar non-smooth paths of variations. Ask whether u^* is local maximum of I in $A_m^{\pm}$. If it is, then we have a contradiction, since $I(u) \le I(\overline{u}) \le I(u^*)$ contradicting the fact that the expression (3.8) is increasing in k.

If it is not, we repeat this process until it stops at $k = 0$. But there the lone element ± 1 is trivially a local maximum. So in conclusion I achieves a local maximum at u.

4 Existence of monotone solutions, nonexistence of solutions

We now turn our attention to the existence of nonconstant solutions of (1.1) for nonlocalities other than (2.4). We construct monotone solutions for a wide class of J's.

Assume f is odd, $J(x, y) = J(x - y)$ and J is decreasing on $[0, 1]$. Note that here J is allowed to change sign. In addition, in the case $j(\frac{1}{2}) + f'(s) \ge 0$ for $s \in [-1, 1]$, we assume

$$\int_x^{x-1} tJ(t)dt > (x - \frac{1}{2})f'(0), \quad \text{for} x \in (\frac{1}{2}, 1] \tag{4.1}$$

and

$$J(\frac{1}{2}) < -f'(0). \tag{4.2}$$

Note that the slightly weaker $J(\frac{1}{2}) \le -f'(0)$ follows from (4.1). Also, note that (4.1) guarantees that J is not constant. Otherwise, if $J = c$, then (4.1) is equivalent to $f'(0) < -c$, which violates $j(\frac{1}{2}) + f'(s) = c + f'(s) \ge 0$. For $J = c$ and $j(\frac{1}{2}) + f'(s) \ge 0$ on $[-1, 1]$, it can be easily determined that (1.1) has only constant solutions, whose values are the zeros of f.

Conditions (4.1) and (4.2) relate the nonlocal effect with $f'(0)$. As an example, let us choose $J(x) = b - m|x|$, $b, m > 0$. $j(\frac{1}{2}) + f'(s) \ge 0$ is equivalent to $m \le 4(b + f'(s))$. (4.1) and (4.2) are then satisfied if and only if $m > 2(b + f'(0))$.

With these assumptions we have the following.

Theorem 4.1 *There exists an increasing solution U of (1.1), such that $U(x) = -U(1 - x)$ (i.e., $U(\cdot + \frac{1}{2})$) is odd. Moreover, with U' denoting the pointwise derivative of U and assuming $J(\frac{1}{2}) < J(0)$,*

1. *In the case* $j(\frac{1}{2}) + f'(s) > 0$ *for* $s \in [-1, 1]$, $U' > 0$.
2. *In the case* $j(\frac{1}{2})+f'(s) > 0$ *for* $s \in [-1, -u_0) \cup (u_0, 1]$ *and* $j(\frac{1}{2})+f'(s) < 0$ *for* $s \in (-u_0, u_0)$, $U(\frac{1}{2}\pm) = \pm u_0$ *and* $U'(x) > 0$ *for* $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.
3. *In the case* $j(\frac{1}{2}) + f'(s) > 0$ *for* $s \in [-1, 0) \cup (0, 1]$ *and* $j(\frac{1}{2}) + f'(0) = 0$, $U'(0\pm) = +\infty$ *and* $U'(x) > 0$ *for* $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

Proof. U is constructed from an iteration scheme. Let

$$u_0(x) = \begin{cases} -1, & x \in (0, \frac{1}{2}) \\ 1, & x \in (\frac{1}{2}, 1) \end{cases} .$$

Define the sequence $\{u_n\}$, $n = 0, 1, 2, \dots$, by

$$J[u_n](x) + (j(\frac{1}{2}) - j(x))u_n(x) = g(u_{n+1}(x)), \tag{4.3}$$

where $g(s) \equiv j(\frac{1}{2})s+f(s)$. Note that since J is decreasing on $(0, 1)$, $j(\frac{1}{2})-j(x) \geq 0$ for $x \in (0, 1)$.

First, $u_n(x) = -u_n(1-x)$ and J even imply

$$J[u_n](x) = -\int_0^1 J(x-y)u_n(1-y)dy = -\int_0^1 J(1-x-y)u_n(y) = -J[u_n](1-x).$$

Since g is odd, it then easily follows that $u_{n+1}(x) = -u_{n+1}(1-x)$ as well.

We show by induction that $u_n'(x) \geq 0$ for $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $n \geq 1$, where u_n' denotes the pointwise derivative of u_n. First assume that $g'(s) > 0$ for $s \in [-1, 0) \cup (0, 1]$. Inverting (4.3), we see that u_{n+1} is C^1 on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Thus we can differentiate (4.3) on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ to get:

$$\begin{aligned} & -J(x-1)u_n(1) + J(x)u_n(0) + J(x-\frac{1}{2})(u_n(\frac{1}{2}+) - u_n(\frac{1}{2}-)) \\ & + \int_0^1 J(x-y)u_n'(y)dy + (J(x-1) - J(x))u_n(x) + (j(\frac{1}{2}) - j(x))u_n'(x) \\ & = \; g'(u_{n+1}(x))u_{n+1}'(x). \end{aligned} \tag{4.4}$$

Since J is decreasing, the integral appearing on the left side of (4.4) can be estimated as follows:

$$\int_0^1 J(x-y)u_n'(y)dy \geq \int_0^x J(x)u_n'(y)dy + \int_x^1 J(1-x)u_n'(y)dy. \tag{4.5}$$

With this estimate, (4.4) becomes

$$\begin{aligned} & [J(x-\frac{1}{2}) - J(\frac{1}{2} + |x-\frac{1}{2}|)][u_n(\frac{1}{2}+) - u_n(\frac{1}{2}-)] + (j(\frac{1}{2}) - j(x))u_n'(x) \\ & \leq \; g'(u_{n+1}(x))u_{n+1}'(x). \end{aligned}$$

Since $u_n'(x) \geq 0$ and J is decreasing on $(0,1)$, also $u_{n+1}'(x) \geq 0$ for $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

We now show that (4.3) has the following monotonicity property. If v and w are two functions such that $v(x) = -v(1-x)$ and $w(x) = -w(1-x)$, with $v(x) \leq w(x)$ for $\frac{1}{2} \leq x \leq 1$, and $\tilde{v}$ and $\tilde{w}$ are defined by (4.3), namely $J[v](x) + (j(\frac{1}{2}) - j(x))v(x) = g(\tilde{v}(x))$ and similarly for $\tilde{w}$, then $\tilde{v} \leq \tilde{w}$ for $\frac{1}{2} \leq x \leq 1$.

First note that if u is such that $u(x) = -u(1-x)$ and $u(x) \geq 0$ for $x \in (\frac{1}{2}, 1)$, then J being even and decreasing on $(0,1)$ implies that

$$J[u](x) = \int_{\frac{1}{2}}^{1} [J(x-y) - J(1-x-y)]u(y)dy \geq 0. \tag{4.6}$$

For $x \in (\frac{1}{2}, 1)$, (4.3), (4.6) and $v(x) \leq w(x)$ imply that $0 \geq g(\tilde{v}) - g(\tilde{w}) = g'(c(x))(\tilde{v} - \tilde{w})$ for some function $c(x)$, thus since $g' > 0$ we get $\tilde{v} \leq \tilde{w}$.

To show by induction that $\{u_n(x)\}$ is nonincreasing in n for $x \in (\frac{1}{2}, 1)$, it now suffices to note by direct computation that $u_0(x)$ is a supersolution of (4.3) on $(\frac{1}{2}, 1)$:

$$J[u_0](x) + (j(\frac{1}{2}) - j(x))u_0(x) \leq g(u_0(x)).$$

We can now set $U(x) \equiv \lim_{n\to\infty} u_n(x)$. Clearly, U solves (1.1). To guarantee that U is not the constant solution 0, we show that $\underline{u}(x) \equiv m(x - \frac{1}{2})$ is a subsolution of (4.3) on $(\frac{1}{2}, 1)$:

$$J[\underline{u}](x) - j(x)\underline{u} = m\int_{x}^{x-1} tJ(t)dt \geq m(x - \frac{1}{2})(f'(0) + \epsilon) \geq f(m(x - \frac{1}{2})) = f(\underline{u}),$$

for m and $\epsilon > 0$ small enough, where we used (4.1) and (4.2).

In the case $j(\frac{1}{2}) + f'(s) > 0$ for $s \in [-1, -u_0) \cup (u_0, 1]$ and $j(\frac{1}{2}) + f'(s) < 0$ for $s \in (-u_0, u_0)$, the construction is similar, except that $\underline{u}$ is now taken to be $\underline{u}(x) = H(x - \frac{1}{2})u_0$, where H is the Heaviside function. That $\underline{u}$ is indeed a subsolution on $(\frac{1}{2}, 1)$, follows from (4.6) and $g(u_0) = 0$.

We now show that $U' > 0$. First, in the case $j(\frac{1}{2}) + f'(s) > 0$ for $s \in [-1, 1]$, as we discussed before, u is in C^1. Using (4.4) and (4.5), we see that $J(\frac{1}{2}) < J(0)$ implies

$$[j(\frac{1}{2}) - j(x)]U'(x) < g'(U(x))U'(x),$$

thus $U' > 0$. The other two cases are discussed in a similar way. In particular, the regularity of $J[U]$ implies that $U(\frac{1}{2}\pm) = \pm u_0$. $\square$

Note that for the whole line version of (1.1), the increasing solution $\hat{U}$ of $-J * u + ju + f(u) = 0$ has the property $\hat{U}' > 0$ under the milder assumption $J \geq 0$ [4]. One cannot expect the same property to hold in Theorem 4.1 (recall that the discontinuous increasing solution for $J = c$ is piecewise constant, as was discussed before).

For $J \geq 0$, the C^1 solutions constructed in Theorem 4.1 are unstable, by Theorem 2.1. We do not know if they are unique in the class of increasing functions. Recall that for the scaled local (1.4) equation $-\epsilon^2 u'' + f(u) = 0$, $u'(0) = u'(1) = 0$, where $f(u) = u^3 - u$, for $\epsilon_{n+1} \le \epsilon \le \epsilon_n$ there exist n solutions (such that $u(0) < 0$), where $\epsilon_i = \sqrt{f'(0)}/2\pi i$. The existence of similar nonmonotone solutions of (1.1) is left as an open problem.

As was noted before, for $J = c$ and $c + f'(s) \ge 0$, (1.1) has only constant solutions. This nonexistence result can be improved in the following way.

Theorem 4.2 *Let $J(x,y) \ge 0$ and $j(x) + f'(s) > 0$ for $s \in [-1,1]$, where $j(x) = \int_0^1 J(x,y)dy$. Then, assuming*

$$\max_{x\in[0,1]} \int_0^1 |x-y||J_x(x,y)|dy < \min_{s\in[-1,1]} \min_{x\in[0,1]} [j(x) + f'(s)], \tag{4.7}$$

there are no nonconstant solutions of (1.1).

Proof. First, note that from the comparison principle $J(x,y) \ge 0$ implies that any solution u of (1.1) is such that $|u| \le 1$. Also, $j(x) + f'(s) > 0$ implies that u is C^1. We write (1.1) as $J[u] = ju + f(u)$, then differentiate it and estimate both sides in the following way:

$$\begin{aligned} &\big[\max_{x\in[0,1]} u'(x)\big] \int_0^1 |x-y||J_x(x,y)|dy \\ \ge\ & \int_0^1 J_x(x,y)[\int_0^1 (y-x)u'(x-t(x-y))dt]dy \\ =\ & \int_0^1 J_x(x,y)(u(y)-u(x))dy \\ =\ & (j(x)+f'(u))u'(x). \end{aligned}$$

We take the maximum of both sides of this inequality to get

$$\big[\max_{x\in[0,1]} u'(x)\big] \int_0^1 |x-y||J_x(x,y)|dy \ge \min_{s\in[-1,1]} \min_{x\in[0,1]} [j(x)+f'(s)] \max_{x\in[0,1]} u'(x).$$

If u is nonconstant, we divide both sides by $\max_{x\in[0,1]} u'(x)$ to get a contradiction. $\square$

To illustrate this theorem, we again choose $J(x) = b - m|x|$, $b, m > 0$ (as was already noted, $j(\frac{1}{2}) + f'(s) > 0$ is equivalent to $m < 4(b + f'(s))$. Then condition (4.7) is equivalent to $m < b + \min_{s\in[-1,1]} f'(s)$.

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