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Completeness of elementary solutions of second order elliptic equations in a semi-infinite tube domain *

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Abstract

Boundary-value problems for second order abstract differential equations on a semi-axis are considered in this article. We find isomorphisms for the corresponding operators and prove completeness of elementary solutions corresponding to subsets of eigenvalues. As an application of the abstract results, we study second order elliptic equations in semi-infinite tube domains. Our results can be applied to pure differential, integro-differential, functional-differential and equations with a shift.

Introduction

The question of completeness for systems of eigenvectors corresponding to the whole spectrum arises when solving non-stationary equations. However, when solving stationary equations the question changes to the completeness of systems corresponding to subsets of the spectrum. For general equations this question can be very difficult. For thermal conduction and elasticity systems [9, 10, 11], it is an open question. In this article, we consider only equations without mixed derivatives.

There are many articles and monographs devoted to the solvability of regular elliptic boundary-value problems in non-smooth bounded and unbounded domains [2, 7, 4, 3]. In this article, we obtain algebraic conditions for the solvability of boundary-value problems for second order elliptic equations on semi-infinite cylindrical domains. We also obtain conditions for the completeness of elementary solutions corresponding to subsets of eigenvalues. The presence of an abstract operator in our equation allows us to consider integro-differential equations, functional-differential equations, equations with a shift, in addition to pure differential equations. To the best of our knowledge, our results are new.

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Similar questions were considered by Yakubov and Yakubov [14] for fourth order elliptic equations and by Shkalikov [8] for second order elliptic equations in semi-infinite tube domains. In contrast to Shkalikov [8] who assumes that two supplementary Kondratiev problems do not have eigenvalues on the line $\operatorname{Re}\lambda = 1$, we find sufficient conditions for the completeness of root functions and elementary solutions corresponding to eigenvalues with $\operatorname{Re}\lambda_i < 0$.

We start by giving the notation and definitions to be used in this paper.

Let E be a Banach space and n a non-negative integer. Let $W_p^n((0,1);E)$ denote the Banach space of functions with values from E which have generalized derivatives up to order n on $(0,1)$. In this space, we consider the norm

$$\|u\|_{W_p^n((0,1);E)} := \sum_{k=0}^{n} \Big(\int_0^1 \|u^{(k)}(x)\|^p \, dx \Big)^{1/p} .$$

Let the standard Sobolev space be $W_p^n(0,1) := W_p^n((0,1);\mathbb{C})$.

Let E_0 and E_1 be two Banach spaces continuously embedded into the Banach space E: $E_0 \subset E$, $E_1 \subset E$. Such spaces are called an *interpolation couple* $\{E_0, E_1\}$. We also consider the Banach space

$$\begin{gathered} E_0 + E_1 := \big\{ u \mid u \in E : u = u_0 + u_1 \text{ with } u_j \in E_j,\ j = 0,1 \big\}, \\ \|u\|_{E_0+E_1} := \inf\{ \|u_0\|_{E_0} + \|u_1\|_{E_1} : u = u_0 + u_1,\ u_j \in E_j \}. \end{gathered}$$

Due to Triebel [12, 1.3.1], the functional

$$K(t,u) := \inf\{ \|u_0\|_{E_0} + t\|u_1\|_{E_1} : u = u_0 + u_1,\ u_j \in E_j \}$$

is continuous on $(0,\infty)$ in t, and

$$\min\{1,t\}\|u\|_{E_0+E_1} \le K(t,u) \le \max\{1,t\}\|u\|_{E_0+E_1} .$$

An *interpolation space* for $\{E_0, E_1\}$ by the K-method is defined as follows:

$$\begin{gathered} \|u\|_{(E_0,E_1)_{\theta,p}} := \Big(\int_0^\infty t^{-1-\theta p} K^p(t,u) \, dt \Big)^{1/p}, \quad 0 < \theta < 1,\ 1 \le p < \infty, \\ (E_0,E_1)_{\theta,p} := \{ u \mid u \in E_0 + E_1, \|u\|_{(E_0,E_1)_{\theta,p}} < \infty \}, \\ \|u\|_{(E_0,E_1)_{\theta,\infty}} := \sup_{t \in (0,\infty)} t^{-\theta} K(t,u), \quad 0 < \theta < 1, \\ (E_0,E_1)_{\theta,\infty} := \{ u \mid u \in E_0 + E_1, \|u\|_{(E_0,E_1)_{\theta,\infty}} < \infty \}. \end{gathered}$$

When ℓ is an non-negative integer, $W_p^\ell(G)$ is a standard Sobolev space. Let

$$B_{p,q}^s(G) := (W_p^{s_0}(G), W_p^{s_1}(G))_{\theta,q},$$

where s_0, s_1 are non-negative integers, $0 < \theta < 1$, $1 < p < \infty$, $1 \le q \le \infty$ and $s = (1-\theta)s_0 + \theta s_1$. Set $W_p^s(G) := B_{p,p}^s(G)$, where $0 < s$ is not an integer.

Let $\{E_0, E_1\}$ be an interpolation couple. Further, let $\ell = 1, 2, \dots,$ and $1 \le p \le \infty$. Then one sets

$$\begin{aligned} W_p^\ell((0,1); E_0, E_1) := \big\{u(t) \mid u(t) &\text{ is an } (E_0+E_1)\text{-valued function in } (0,1) \\ &\text{with } u(t) \in L_p((0,1); E_0),\ u^{(\ell)}(t) \in L_p((0,1); E_1)\big\} \\ \|u\|_{W_p^\ell((0,1);E_0,E_1)} := \|u(t)\|_{L_p((0,1);E_0)} &+ \|u^{(\ell)}(t)\|_{L_p((0,1);E_1)}, \end{aligned}$$

where $L_p((0,1); E) := W_p^0((0,1); E)$. It is known that $W_p^\ell((0,1); E_0, E_1)$ is a Banach space [12, Lemma 1.8.1]. One can also replace $(0,1)$ by $(0,\infty)$.

Let H be a Hilbert space. Consider a polynomial operator pencil equation in H

$$L(\lambda)u := \lambda^n u + \lambda^{n-1} A_1 u + \dots + A_n u = 0, \tag{0.1}$$

where n is a natural number and A_k are, generally speaking, unbounded operators in H. Let $H_n \subset H$ be a Hilbert space, such that operators A_k, $k = 1, \dots, n$, from H_n into H, are bounded.

A number λ_0 is called an *eigenvalue* of equation (0.1), or of the operator pencil $L(\lambda)$, if

$$L(\lambda_0)u = 0$$

has a nontrivial solution belonging to H_n. The nontrivial solution $u_0 \in H_n$ is called an *eigenvector* of equation (0.1), or of the operator pencil $L(\lambda)$ corresponding to the eigenvalue λ_0. A solution of the equation

$$L(\lambda_0)u_p + \frac{1}{1!}L'(\lambda_0)u_{p-1} + \dots + \frac{1}{p!}L^{(p)}(\lambda_0)u_0 = 0,$$

$u_p \in H_n$ is called a *p-associated vector* to the eigenvector u_0 of (0.1), or of the operator pencil $L(\lambda)$.

Eigenvectors and associated vectors are combined under the general name *root vectors* of equation (0.1), or of the operator pencil $L(\lambda)$. The dimension of the linear space of all root vectors corresponding to λ_0 is called *algebraic multiplicity* of λ_0.

A complex number λ is called a *regular point* of (0.1), or of the operator pencil $L(\lambda) : u \to L(\lambda)u$ which is bounded from H_n into H, if for any $f \in H$,

$$L(\lambda)u = f$$

has a unique solution $u \in H_n$ and $\|u\|_{H_n} \le C(\lambda)\|f\|$.

The complement of the set of regular points in the complex plane is called the *spectrum* of (0.1), or of the operator pencil $L(\lambda)$.

The spectrum of (0.1), or of the operator pencil $L(\lambda)$, is called *discrete*, if:

a) All points which are not eigenvalues of (0.1) are regular points of (0.1)

b) The eigenvalues are isolated and have finite algebraic multiplicities

c) Infinity is the only limit point of the set of the eigenvalues of (0.1).

Consider the Cauchy problem for a differential-operator equation

$$L(D)u := u^{(n)}(t) + A_1 u^{(n-1)}(t) + \cdots + A_n u(t) = 0, \tag{0.2}$$

$$u^{(k)}(0) = v_{k+1}, \quad k = 0, \ldots, n-1, \tag{0.3}$$

where v_{k+1} are given elements of H, $D := \frac{d}{dt}$, and $t \geq 0$.

By [14, Lemma 1, p.56], a function

$$u(t) := \mathrm{e}^{\lambda_0 t}\Big(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + \cdots + u_k\Big) \tag{0.4}$$

is a solution of (0.2), if and only if the system of vectors $u_0, u_1, \cdots, u_k$ is a chain of root vectors of (0.1), corresponding to the eigenvalue λ_0. A solution of the form (0.4) is called an *elementary solution* of (0.2).

The possibility of approximating solutions of (0.2)–(0.3) by linear combinations of the elementary solutions, suggests that the vector $(v_1, v_2, \ldots, v_n)$ should be approximated by linear combinations of vectors of the form

$$(u(0), u'(0), \ldots, u^{(n-1)}(0)), \tag{0.5}$$

where $u(t)$ is an elementary solution of the form (0.4).

Let $\mathcal{H}$ be a Hilbert space, continuously embedded into the orthogonal sum of Hilbert spaces $\overset{n}{\oplus} H = H \oplus H \oplus \cdots \oplus H$.

A system of root vectors of (0.1) is called *n-fold complete in the space* $\mathcal{H}$, if the system of vectors (0.5) is complete in $\mathcal{H}$, i.e., the closure of a linear span of vectors (0.5) is equal to $\mathcal{H}$.

1 Abstract results for second order elliptic equations

In this section we prove completeness of a system of root vectors corresponding to a part of the spectrum of a quadratic operator pencil in a Hilbert space. Isomorphism and the completeness of elementary solutions corresponding to the eigenvalues λ_i with $\operatorname{Re}\lambda_i < 0$ for some special cases of abstract differential equations of the second order are established.

1.1 Completeness of a system of root vectors

Let us consider, in a Hilbert space H, the unbounded operator pencil:

$$L(\lambda) := \lambda^2 I + B. \tag{1.1}$$

Theorem 1 *Let the following conditions be satisfied:*

1. *B is a densely defined and closed operator in in a Hilbert space H;*

2. *There exists a Hilbert space H_1 for which the compact embeddings $H(B) \subset H_1 \subset H$ take place; $\overline{H_1}\big|_H = H$ and $\overline{H(B)}\big|_{H_1} = H_1$;*

3. *$s_j(J_1; H(B), H_1) \le Cj^{-p}$ and $s_j(J_2; H_1, H) \le Cj^{-p}$, $j = 1, \dots, \infty$, for some $p > 0$* [1]*;*

4. *There exist* [2] *rays ℓ_k with angles between neighboring rays less than $\frac{p\pi}{2}$ and a number η such that numbers λ from ℓ_k and with sufficiently large moduli are regular points for the operator pencil $L(\lambda)$ and*
$$\|L(\lambda)^{-1}\|_{B(H,H_1)} \le C|\lambda|^\eta, \quad \lambda \in \ell_k, \ |\lambda| \to \infty.$$

Then the spectrum of pencil (1.1) is discrete and a system of root vectors of pencil (1.1), corresponding to the eigenvalues λ_i with $\operatorname{Re}\lambda_i \le 0$, is complete in the spaces H_1 and $H(B)$.

Proof When applying a theorem from [14, p.65] or [13, Theorem 3.6, p.71]) to operator pencil (1.1), we have two-fold completeness of a system of root vectors of (1.1) in $H_1 \oplus H$ and $H(B) \oplus H_1$.

Let $v^0, v^1, v^2, \dots, v^s$ be a chain of root vectors of the operator pencil (1.1) corresponding to the eigenvalue λ_0, i.e.,
$$(\lambda_0^2 I + B)v^0 = 0, \tag{1.2}$$
$$(\lambda_0^2 I + B)v^1 + 2\lambda_0 v^0 = 0, \tag{1.3}$$
$$(\lambda_0^2 I + B)v^k + 2\lambda_0 v^{k-1} + v^{k-2} = 0, \quad k = 2, \dots, s. \tag{1.4}$$
Then $-v^0, v^1, -v^2, \dots, (-1)^{s-1}v^s$ is a chain of root vectors of the operator pencil $L(\lambda)$ corresponding to $-\lambda_0$, i.e.,
$$[(-\lambda_0)^2 I + B](-v^0) = 0,$$
which follows from (1.2),
$$[(-\lambda_0)^2 I + B]v^1 + 2(-\lambda_0)(-v^0) = 0,$$
which follows from (1.3), and
$$\begin{gathered}\big[(-\lambda_0)^2 I + B\big](-v^k) + 2(-\lambda_0)v^{k-1} + 2(-v^{k-2}) = 0, \quad \text{if } k \text{ is even},\\ \big[(-\lambda_0)^2 I + B\big]v^k + 2(-\lambda_0)(-v^{k-1}) + 2v^{k-2} = 0, \quad \text{if } k \text{ is odd},\end{gathered}$$
which follow from (1.4).

Let $v(t)$ be an elementary solution of the equation $u''(t) + Bu(t) = 0$, $t > 0$. Then
$$v(0) = \begin{cases} v^j & \text{if } v(t) \text{ corresponds to } \lambda_0,\\ (-1)^{j+1}v^j & \text{if } v(t) \text{ corresponds to } -\lambda_0. \end{cases}$$

[1]Singular numbers s_j of the compact operator A from a Hilbert space H into a Hilbert space H_1 are eigenvalues λ_j of the compact selfadjoint non-negative operator $(A^*A)^{\frac{1}{2}}$ in H.

[2]For $p > 4$ the existence of one such ray is enough.

By virtue of the above-mentioned two-fold completeness,

$$\Big\| \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} - \sum_{k=1}^{N} C_{kN} \begin{pmatrix} v_k(0) \\ v_k'(0) \end{pmatrix} \Big\|_{H_1 \oplus H} < \varepsilon \quad \text{for eigenvalues } \lambda_k$$

and

$$\Big\| \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} - \sum_{k=1}^{N} C_{kN} \begin{pmatrix} v_k(0) \\ v_k'(0) \end{pmatrix} \Big\|_{H(B) \oplus H_1} < \varepsilon \quad \text{for eigenvalues } \lambda_k,$$

then

$$\|F_1 - \sum_{k=1}^{N} \tilde{C}_{kN} v_k^j\|_{H_1} < \varepsilon \quad \text{for } \lambda_k \text{ with } \operatorname{Re}\lambda_k \leq 0$$

and

$$\|F_1 - \sum_{k=1}^{N} \tilde{C}_{kN} v_k^j\|_{H(B)} < \varepsilon \quad \text{for } \lambda_k \text{ with } \operatorname{Re}\lambda_k \leq 0.$$

□

1.2 Isomorphism of problems on the semi-axis

In a Hilbert space H, consider a boundary-value problem in $[0, \infty)$ for the second order elliptic equation

$$L(D)u := u''(x) + Bu(x) = f(x), \quad x > 0, \tag{1.5}$$

$$Lu := \alpha u(0) + \beta u'(0) = \varphi, \tag{1.6}$$

where α and β are complex numbers. Denote $L(\lambda) := \lambda^2 I + B$.

Theorem 2 *Let the following conditions be satisfied:*

1. *B is a densely defined and closed operator in a Hilbert space H;*
2. *$(1 + |\lambda|^2)\|L(\lambda)^{-1}\|_{B(H)} \leq C$, $\operatorname{Re}\lambda = 0$;*
3. *$|\alpha| + |\beta| \neq 0$; $\operatorname{Re}\alpha\beta^{-1} \leq 0$ if $\beta \neq 0$.*

Then the operator $\mathbb{L} : u \to \mathbb{L}u := (L(D)u, Lu)$ from $W_p^2((0,\infty); H(B), H)$ onto $L_p((0,\infty); H)\dot{+}(H(B), H)_{\frac{m}{2}+\frac{1}{2p},p}$, where $m = 0$ if $\beta = 0$ and $m = 1$ if $\beta \neq 0$, and $p > 1$ is an isomorphism.[3]

[3]Isomorphism means that the operator and its inverse are bounded.

Proof By Theorem 1.8.2 in [12], the operator $\mathbb{L}$ is continuous from the space $W_p^2((0,\infty);H(B),H)$ into $L_p((0,\infty);H)\dot{+}(H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}$. Let us prove that for any $f\in L_p((0,\infty);H)$ and any $\varphi\in(H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}$ problem (1.5)–(1.6) has a unique solution that belongs to $W_p^2((0,\infty);H(B),H)$. Let us show that a solution of problem (1.5)–(1.6) is represented in the form $u(x)=u_1(x)+u_2(x)$, where $u_1(x)$ is the restriction on $[0,\infty)$ of a solution $\tilde{u}_1(x)$ of the equation

$$\tilde{u}_1''(x)+B\tilde{u}_1(x)=\tilde{f}(x),\quad x\in\mathbb{R}, \tag{1.7}$$

where $\tilde{f}(x):=f(x)$ if $x\in[0,\infty)$ and $\tilde{f}(x):=0$ if $x\in(-\infty,0)$, and $u_2(x)$ is a solution of the problem

$$\begin{gathered}u_2''(x)+Bu_2(x)=0,\quad x>0,\\ \alpha u_2(0)+\beta u_2'(0)=-Lu_1+\varphi.\end{gathered} \tag{1.8}$$

Apply Theorem 1 of [14, p.250] to equation (1.7). Let $H_1:=(H(B),H)_{\frac{1}{2},2}$, $H_2:=H(B)$, $A_1:=0$, $A_2:=B$. Then, by virtue of [14, formula (1), p.39], we have

$$\|L(\lambda)^{-1}f\|_{H_1}=\|L(\lambda)^{-1}f\|_{(H(B),H)_{\frac{1}{2},2}}\le C\|L(\lambda)^{-1}f\|_{H(B)}^{1/2}\|L(\lambda)^{-1}f\|_H^{1/2}.$$

From condition (2) it follows that

$$|\lambda|^2\|L(\lambda)^{-1}f\|_H+\|L(\lambda)^{-1}f\|_{H(B)}\le C\|f\|_H,\quad f\in H,\ \operatorname{Re}\lambda=0. \tag{1.9}$$

Using the last inequality and the Young inequality [14, p.53], we have

$$\begin{aligned}|\lambda|\|L(\lambda)^{-1}f\|_{H_1}&\le C\|L(\lambda)^{-1}f\|_{H(B)}^{1/2}\big(|\lambda|^2\|L(\lambda)^{-1}f\|_H\big)^{1/2}\\ &\le C\big(\|L(\lambda)^{-1}f\|_{H(B)}+|\lambda|^2\|L(\lambda)^{-1}f\|_H\big)\le C\|f\|.\end{aligned}$$

Therefore, conditions (1)–(3) of [14, Theorem 1, p.250] are satisfied and, hence, (1.7) has a solution $\tilde{u}_1\in W_p^2(\mathbb{R};H(B),H_1,H)$. Then $u_1\in W_p^2((0,\infty);H(B),H)$.

Let us now prove that for any $\varphi\in(H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}$ problem (1.8) has a unique solution $u_2(x)$ that belongs to $W_p^2((0,\infty);\ H(B),H)$. By the above inequality (1.9), we have for $f\in H$, $\operatorname{Re}\lambda=0$

$$\|(\lambda^2I+B)^{-1}f\|_H\le C|\lambda|^{-2}\|f\|_H,\ \|(\lambda^2I+B)^{-1}f\|_H\le C\|f\|_H.$$

This implies that $\|(\lambda^2I+B)^{-1}f\|_H\le C(1+|\lambda|^2)^{-1}\|f\|_H$, $\operatorname{Re}\lambda=0$, i.e.,

$$\|R(\lambda,-B)\|\le C(1+|\lambda|)^{-1},\quad \arg\lambda=\pi\,.$$

Hence, as shown in Balakrishnan [1], there exists an operator $\mathrm{e}^{-x(-B)^{1/2}}$ and for some $\omega>0$,

$$\|\mathrm{e}^{-x(-B)^{1/2}}\|\le C\mathrm{e}^{-\omega x},\quad x\ge0.$$

Repeating the proof of [14, Lemma 1, p.263], one can show that an arbitrary solution of the equation in (1.8) that belongs to $W_p^2((0,\infty);H(B),H)$ has the form

$$u_2(x) = \mathrm{e}^{-x(-B)^{1/2}} g, \tag{1.10}$$

where $g \in (H(B),H)_{\frac{1}{2p},p}$ (and conversely). To this end one should use Theorem 3.2.11 in Krein [5]. Function (1.10) satisfies the boundary condition in (1.8) if

$$\alpha g - \beta(-B)^{1/2} g = \Phi, \tag{1.11}$$

where $\Phi = -Lu_1 + \varphi$. Since $u_1 \in W_p^2((0,\infty);H(B),H)$, by Theorem 1.8.2 in [12], $Lu_1 \in (H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}$. Then $\Phi \in (H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}$.

For $\beta = 0$, a solution of problem (1.8) has the form

$$u_2(x) = \alpha^{-1}\mathrm{e}^{-x(-B)^{1/2}}\Phi.$$

Since $\Phi \in (H(B),H)_{\frac{1}{2p},p}$, $u_2 \in W_p^2((0,\infty);H(B),H)$.

Let $\beta \neq 0$. From conditions (2) and (3), by T. Kato's theorem [14, p.31], it follows that (1.11) has a unique solution $g = (\alpha I - \beta(-B)^{\frac{1}{2}})^{-1}\Phi$. Then solutions of (1.8) have the form

$$u_2(x) = \mathrm{e}^{-x(-B)^{1/2}}(\alpha I - \beta(-B)^{1/2})^{-1}\Phi.$$

By Theorem 1.15.2 in [12], the operator $(-B)^{1/2}$ from $(H(B),H)_{\frac{1}{2p},p}$ onto $(H(B),H)_{\frac{p+1}{2p},p}$ is an isomorphism. Then $(\alpha I - \beta(-B)^{1/2})^{-1}\Phi \in (H(B),H)_{\frac{1}{2p},p}$, i.e., $u_2 \in W_p^2((0,\infty);H(B),H)$.

The uniqueness of a solution of problem (1.5)–(1.6) follows from the uniqueness of a solution of problem (1.8). Indeed, if problem (1.5)–(1.6) has two solutions $u(x)$, $\tilde{u}(x)$, then functions $u_2(x) := u(x) - u_1(x)$ and $\tilde{u}_2(x) := \tilde{u}(x) - u_1(x)$, where $u_1(x)$ is the restriction on $[0,\infty)$ of the solution $\tilde{u}_1(x)$ of (1.7), are two different solutions of problem (1.8), which is a contradiction. □

1.3 Completeness of elementary solutions of a problem on the semi-axis

In those cases when it is difficult to prove the applicability of the Fourier method, it is desirable at least to establish that a solution of an initial boundary-value problem may be approximated by linear combinations of elementary solutions. In a Hilbert space H, consider a boundary-value problem in $[0,\infty)$ for the second order elliptic equation

$$u''(x) + Bu(x) = 0, \quad x > 0, \tag{1.12}$$
$$\alpha u(0) + \beta u'(0) = \varphi. \tag{1.13}$$

Let us find conditions that allow building an approximation of a solution of (1.12)–(1.13) by linear combinations of elementary solutions of (1.12).

As it was mentioned in the introduction, the function

$$u_i(x) := \mathrm{e}^{\lambda_i x}\big(\frac{x^{k_i}}{k_i!}u_{i0} + \frac{x^{k_i-1}}{(k_i-1)!}u_{i1} + \dots + u_{ik_i}\big) \tag{1.14}$$

is a solution of (1.12) if and only if $u_{i0}, u_{i1}, \dots, u_{ik_i}$ is a chain of root vectors of the characteristic operator pencil (1.1) corresponding to the eigenvalue λ_i and (1.14) is called an elementary solution of (1.12).

Let $u_{10}, u_{11}, \dots, u_{1,r-1}$ be one of the maximal chains of root vectors of (1.1) corresponding to the eigenvalue μ. Then $\lambda_1 = \lambda_2 = \dots = \lambda_r = \mu$ and $k_1 = 0, k_2 = 1, \dots, k_r = r-1$. Note, it may happen that $\lambda_{r+1} = \mu$.

Lemma 3 *Let* $|\alpha|+|\beta| \neq 0$ *and* $\operatorname{Re}\alpha\beta^{-1} \leq 0$ *if* $\beta \neq 0$. *Then, if a system of root vectors* $\{u_{ip}\}$ *of (1.1) corresponding to eigenvalues* λ_i *with* $\operatorname{Re}\lambda_i < 0$ *is complete (a basis) in a Hilbert space* H *then a system of vectors* $\{(\alpha+\beta\lambda_i)u_{ip} + \beta u_{i,p-1}\}$, *where* $u_{i,-1} = 0$, *is also complete (a basis) in* H.

Proof Let $u_{10}, u_{11}, \dots, u_{1,r-1}$ be one of the maximal chains of root vectors of (1.1) corresponding to the eigenvalue μ with $\operatorname{Re}\mu < 0$. Show that we can uniquely define coefficients M_i with respect to coefficients C_i from the equation

$$\begin{aligned} C_1u_{10} + C_2u_{11} + \dots + C_ru_{1,r-1} =& M_1(\alpha+\beta\mu)u_{10} + M_2((\alpha+\beta\mu)u_{11} + \beta u_{10}) \\ &+ \dots + M_r((\alpha+\beta\mu)u_{1,r-1} + \beta u_{1,r-2}). \end{aligned}$$

Rewrite the last equation in the form

$$\begin{aligned} C_1u_{10} + C_2u_{11} + \dots + C_ru_{1,r-1} =& u_{10}(M_1(\alpha+\beta\mu) + \beta M_2) + u_{11}(M_2(\alpha+\beta\mu) \\ &+ \beta M_3) + \dots + u_{1,r-1}M_r(\alpha+\beta\mu). \end{aligned}$$

Therefore,

$$\begin{gathered} M_1(\alpha+\beta\mu) + \beta M_2 = C_1, \\ M_2(\alpha+\beta\mu) + \beta M_3 = C_2, \\ \vdots \\ M_{r-1}(\alpha+\beta\mu) + \beta M_r = C_{r-1}, \\ M_r(\alpha+\beta\mu) = C_r. \end{gathered}$$

If $\beta = 0$ then $\alpha \neq 0$ and $M_i = \frac{1}{\alpha}C_i,\ i = 1, \dots, r$. If $\beta \neq 0$ then $\alpha + \beta\mu \neq 0$ (since $\operatorname{Re}\alpha\beta^{-1} \leq 0$ and $\operatorname{Re}\mu < 0$). Therefore, starting from the last equation of the previous system we find that

$$M_r = \frac{C_r}{\alpha+\beta\mu}, \quad M_{r-1} = \frac{C_{r-1} - \beta M_r}{\alpha+\beta\mu}, \ \dots\ , \ M_1 = \frac{C_1 - \beta M_2}{\alpha+\beta\mu}.$$

□

Theorem 4 *Let the following conditions be satisfied:*

1. *B is a densely defined and closed operator in a Hilbert space H;*
2. *$s_j(J;H(B),H) \le Cj^{-q}$, $j = 1,\dots,\infty$, for some $q > 0$;*
3. *For $L(\lambda) := \lambda^2 I + B$, $(1+|\lambda|^2)\|L(\lambda)^{-1}\|_{B(H)} \le C$, $\operatorname{Re}\lambda = 0$;*
4. *$|\alpha| + |\beta| \neq 0$; $\operatorname{Re}\alpha\beta^{-1} \le 0$ if $\beta \neq 0$;*
5. *For $q \le 4$ there exist rays ℓ_k with angles between neighboring rays less than $\frac{q\pi}{4}$ and η such that*
$$\|L(\lambda)^{-1}\|_{B(H,H(B))} \le C|\lambda|^\eta, \qquad \lambda \in \ell_k,\ |\lambda| \to \infty;$$
6. *$\varphi \in (H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}$ for some $p > 1$, where $m = 0$ if $\beta = 0$ and $m = 1$ if $\beta \neq 0$.*

Then problem (1.12)–(1.13) has a unique solution $u \in W_p^2((0,\infty);H(B),H)$ and there exist numbers C_{in} such that

$$\lim_{n\to\infty} \sum_{k=0,2} \int_0^\infty \|u^{(k)}(x) - \sum_{i=1}^n C_{in} u_i^{(k)}(x)\|^p_{H_{2-k}}\, dx = 0, \tag{1.15}$$

where $H_0 = H, H_2 = H(B)$, $u_i(x)$ are elementary solutions (1.14) of equation (1.12) corresponding to the eigenvalue λ_i with $\operatorname{Re}\lambda_i < 0$.

Proof Consider in H an operator S such that $D(S) = H(B)$, $S = S^* \ge c^2 I$ (see, for example, Lions and Magenes [6, 1.2.1]). By Lemma 1 in [14, p.15] and condition (2), we have

$$s_j(J;H(B),H) = s_j(J;H(S),H) = s_j(JS^{-1};H,H) = \lambda_j(S^{-1}) \le Cj^{-q}.$$

Let $H_1 := (H(B),H)_{\frac12,2}$. Then $H_1 = (H(S),H)_{\frac12,2} = H(S^{1/2})$ and by Lemma 1 in [14, p.15],

$$\begin{gathered} s_j(J;H(B),H_1) = s_j(J;H(S),H(S^{1/2})) \\ = s_j(S^{1/2}JS^{-1};H,H) = \lambda_j(S^{-\frac12}) \le Cj^{-q/2}, \\ s_j(J;H_1,H) = s_j(J;H(S^{1/2}),H) = s_j(JS^{-\frac12};H,H) = \lambda_j(S^{-\frac12}) \le Cj^{-q/2}. \end{gathered}$$

Hence, by Theorem 1, a system of root vectors of pencil (1.1) corresponding to the eigenvalues λ_i, $\{u_i(0)\} = \{u_{ik_i}\}$, with $\operatorname{Re}\lambda_i < 0$ is complete in the spaces H_1 and $H(B)$. On the other hand, $\overline{H_1}|_H = H$. Then the same system of root vectors is complete in the space H and, therefore, in $(H(B),H)_{\theta,p}$, $0 < \theta < 1$ (see [12, Theorems 1.3.3 and 1.6.2]). Therefore, by virtue of Lemma 3, a system $\{\alpha u_i(0) + \beta u_i'(0)\}$ is also complete in $(H(B),H)_{\theta,p}$, $0 < \theta < 1$. Hence, there exist numbers C_{in} such that

$$\lim_{n\to\infty} \|\varphi - \sum_{i=1}^n C_{in}(\alpha u_i(0) + \beta u_i'(0))\|_{(H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}} = 0.$$

On the other hand, from Theorem 2 we have

$$\|u - \sum_{i=1}^{n} C_{in} u_i\|_{W_p^2((0,\infty);H(B),H)} \leq C\|\varphi - \sum_{i=1}^{n} C_{in}(\alpha u_i(0) + \beta u_i'(0))\|_{(H(B),H)_{\frac{m}{2}+\frac{1}{2p},p}}. \quad (1.16)$$

□

2 Boundary-value problems for second order elliptic equations

In this section we apply abstract results of section 1 to boundary-value problems for second order elliptic equations in semi-infinite tube domains. The corresponding isomorphism and completeness theorems are proved. Completeness theorems apply to eigenvalues λ_i with $\operatorname{Re}\lambda_i < 0$.

2.1 An isomorphism

In the semi-infinite strip $\Omega := [0,\infty) \times [0,1]$, consider a principally boundary-value problem for an elliptic equation of the second order,

$$Lu := D_x^2 u(x,y) + b(y) D_y^2 u(x,y) + Mu(x,\cdot)\big|_y = f(x,y), \quad (x,y) \in \Omega, \quad (2.1)$$

$$Pu := \gamma u(0,y) + \delta D_x u(0,y) = \varphi(y), \qquad y \in [0,1], \quad (2.2)$$

$$\begin{aligned} L_1 u &:= \alpha_1 D_y u(x,0) + \alpha_0 u(x,0) = 0, \quad x \in [0,\infty), \\ L_2 u &:= \beta_1 D_y u(x,1) + \beta_0 u(x,1) = 0, \quad x \in [0,\infty), \end{aligned} \quad (2.3)$$

where α_ν, β_ν, γ, δ are complex numbers, $D_x := \frac{\partial}{\partial x}$, $D_y := \frac{\partial}{\partial y}$. The corresponding spectral problem is

$$\lambda^2 u(y) + b(y) u''(y) + Mu\big|_y = 0, \quad y \in [0,1], \quad (2.4)$$

$$\begin{aligned} \tilde{L}_1 u &:= \alpha_1 u'(0) + \alpha_0 u(0) = 0, \\ \tilde{L}_2 u &:= \beta_1 u'(1) + \beta_0 u(1) = 0. \end{aligned} \quad (2.5)$$

Let $m_\nu := \operatorname{ord} L_\nu$, $W_{p,q}^{\ell,s}(\Omega) := W_p^\ell((0,\infty); W_q^s(0,1), L_q(0,1))$, $L_{p,q}(\Omega) := W_{p,q}^{0,0}(\Omega)$.

Theorem 5 *Let the following conditions be satisfied:*

1. $b \in C[0,1]$, $b(y) > 0$;
2. $|\alpha_1| + |\alpha_0| \neq 0$ *and* $|\beta_1| + |\beta_0| \neq 0$;

3. *The operator M from $W_2^2(0,1)$ into $L_2(0,1)$ is compact.* *This is equivalent to* $\forall\varepsilon>0$, $\|Mu\|_{L_2(0,1)} \le \varepsilon\|u\|_{W_2^2(0,1)} + C(\varepsilon)\|u\|_{L_2(0,1)}$, $u \in W_2^2(0,1)$ *(see Lemma 4 and Remark 5 [14, p.45]);*

4. *The spectral problem (2.4)–(2.5) does not have eigenvalues on the straight line* $\operatorname{Re}\lambda = 0$;

5. $|\gamma|+|\delta| \neq 0$; $\operatorname{Re}\gamma\delta^{-1} \le 0$ *when* $\delta \neq 0$.

Then the operator $\mathbb{L} : u \to \mathbb{L}u := (Lu, Pu)$ *from* $W_{p,2}^{2,2}(\Omega; L_\nu u = 0, \nu = 1,2)$ *onto* $L_{p,2}(\Omega)\dot{+}B_{2,p}^{2-m-\frac{1}{p}}((0,1); \tilde{L}_\nu u = 0, m_\nu < \frac{3}{2} - m - \frac{1}{p})$, *if* $p>1$ *and* $p \neq 2$, *or* $p=2$ *and* $m_\nu \neq 1-m$, *is an isomorphism, where* $m=0$ *if* $\delta=0$; $m=1$ *if* $\delta \neq 0$.

Remark In the case $p=2$ and $m_\nu = 1-m$, $(W_2^2((0,1); \tilde{L}_\nu u = 0, \nu = 1,2), L_2(0,1))_{\frac{m}{2}+\frac{1}{4},2} = B_{2,2}^{\frac{3}{2}-m}((0,1);\ \tilde{L}_\nu u = 0,\ m_\nu < 1-m;\ \tilde{L}_\nu u \in \widetilde{B}_{2,2}^{\frac{1}{2}}(0,1),$ $m_\nu = 1-m)$ (see Triebel [12, 4.3.3]) should be written instead of $B_{2,2}^{\frac{3}{2}-m}((0,1); \tilde{L}_\nu u = 0, m_\nu < 1-m)$. $\widetilde{B}_{p,q}^s(G) := \{u \mid u \in B_{p,q}^s(\mathbb{R}^r), \operatorname{supp}(u) \subset \overline{G}\}$. From the introduction, $B_{2,2}^s = W_2^s$. Moreover, by virtue of Theorem 6 of Grisvard and Seeley [14, p.45], $(W_2^2((0,1); \tilde{L}_\nu u = 0, \nu = 1,2), L_2(0,1))_{\frac{m}{2}+\frac{1}{4},2} \supset (W_2^2((0,1); \tilde{L}_\nu u = 0, \nu = 1,2), L_2(0,1))_{\frac{1}{5},2} = W_2^{\frac{8}{5}}((0,1); \tilde{L}_\nu u = 0, \nu = 1,2)$. Then, for a unique solvability (and not an isomorphism) it is enough to take $\varphi \in W_2^{\frac{8}{5}}((0,1); \tilde{L}_\nu u = 0, \nu = 1,2)$.

Proof Let us denote $H := L_2(0,1)$. Consider an operator B defined by

$$\begin{gathered} D(B) := W_2^2((0,1); \tilde{L}_\nu u = 0, \nu = 1,2), \\ Bu := b(y)u''(y) + Mu\big|_y. \end{gathered} \tag{2.6}$$

Then problem (2.1)–(2.3) can be rewritten in the form

$$\begin{gathered} u''(x) + Bu(x) = f(x), \\ \gamma u(0) + \delta u'(0) = \varphi, \end{gathered} \tag{2.7}$$

where $u(x) := u(x,\cdot), f(x) := f(x,\cdot)$ are functions with values in the Hilbert space $H := L_2(0,1)$ and $\varphi := \varphi(\cdot)$ is an element of H.

Let us apply Theorem 2 to problem (2.7). From Theorem 1 [14, p.111] (or Theorem 1.7 [13, p.100]) it follows that the operator $(\lambda^2 I + B)^{-1}$ is bounded in $L_2(0,1)$ (see below for the proof). A bounded operator is closed. The inverse operator to a closed operator is also closed. Therefore, $\lambda^2 I + B$ is a closed operator. This implies that the operator B is closed, i.e., condition (1) of Theorem 2 is fulfilled.

Consider the problem

$$
\begin{gathered}
\lambda^2 u(y) + b(y)u''(y) + Mu\big|_y = f(y), \quad y \in (0,1), \\
\alpha_1 u'(0) + \alpha_0 u(0) = 0, \\
\beta_1 u'(1) + \beta_0 u(1) = 0.
\end{gathered} \tag{2.8}
$$

By condition (1), the equation $1 + b(y)\omega^2 = 0$ has roots $\omega_1(y) = i\frac{1}{\sqrt{b(y)}}$ and $\omega_2(y) = -i\frac{1}{\sqrt{b(y)}}$. Then

$$
\begin{aligned}
\underline{\omega} &:= \inf_{y\in[0,1]} \min\{\arg\omega_1(y), \arg\omega_2(y) + \pi\} = \frac{\pi}{2}, \\
\overline{\omega} &:= \sup_{y\in[0,1]} \max\{\arg\omega_1(y), \arg\omega_2(y) + \pi\} = \frac{\pi}{2}.
\end{aligned}
$$

When choosing $\omega_1(y) = -i\frac{1}{\sqrt{b(y)}}$, $\omega_2(y) = i\frac{1}{\sqrt{b(y)}}$ we get that $\underline{\omega} = \overline{\omega} = -\frac{\pi}{2}$. Therefore, from Theorem 1 [14, p.111] it follows that condition (2) of Theorem 2 is satisfied. Indeed, for a solution of problem (2.8), from formula (5) [14, p.112] for $\ell = 2$, $q = 2$, $\gamma = 0$ and $\underline{\omega} = \overline{\omega} = \frac{\pi}{2}$ we have

$$
\begin{gathered}
|\lambda|^2 \|u\|_{L_2(0,1)} + \|u\|_{W_2^2(0,1)} \le C\|f\|_{L_2(0,1)}, \\
f \in L_2(0,1), \quad \varepsilon < \arg\lambda < \pi - \varepsilon, \quad |\lambda| \to \infty.
\end{gathered}
$$

and for $\underline{\omega} = \overline{\omega} = -\frac{\pi}{2}$,

$$
\begin{gathered}
|\lambda|^2 \|u\|_{L_2(0,1)} + \|u\|_{W_2^2(0,1)} \le C\|f\|_{L_2(0,1)}, \\
f \in L_2(0,1), \quad \pi + \varepsilon < \arg\lambda < 2\pi - \varepsilon, \quad |\lambda| \to \infty.
\end{gathered}
$$

These two inequalities and condition (4) give us condition (2) of Theorem 2.

By a theorem of Grisvard and Seeley (see, e.g., [14, Theorem 6, p.45]), we have

$$
\begin{aligned}
(H(B), H)_{\theta,p} &= (W_2^2((0,1); \tilde{L}_\nu u = 0, \nu = 1,2), L_2(0,1))_{\theta,p} \\
&= B_{2,p}^{2(1-\theta)}((0,1); \tilde{L}_\nu u = 0, m_\nu < 2(1-\theta) - \frac{1}{2}),
\end{aligned}
$$

if there does not exist a number m_ν such that $m_\nu = 2(1-\theta) - \frac{1}{2}$. Consequently,

$$
(H(B), H)_{\frac{m}{2}+\frac{1}{2p},p} = B_{2,p}^{2-m-\frac{1}{p}}((0,1); \tilde{L}_\nu u = 0, m_\nu < \frac{3}{2} - m - \frac{1}{p}).
$$

If there exists $m_\nu = 2(1-\theta) - \frac{1}{2}$ then see the corresponding remark to Theorem 5. So, for problem (2.7) all conditions of Theorem 2 are fulfilled, from which the statement of Theorem 5 follows. □

In the semi-infinite domain $\Omega := [0,\infty) \times G$, where $G \subset \mathbb{R}^r$, $r \ge 2$, is a bounded domain with an $(r-1)$-dimensional smooth boundary ∂G, consider a principally boundary-value problem for an elliptic equation of the second order

$$Lu := D_x^2 u(x,y) + \sum_{j,k=1}^{r} b_{jk}(y) D_j D_k u(x,y) + Mu(x,\cdot)\big|_y = f(x,y), \tag{2.9}$$

$$Pu := \gamma u(0,y) + \delta D_x u(0,y) = \varphi(y), \quad y \in G, \tag{2.10}$$

$$L_1 u := \sum_{|\alpha|\le m_1} b_{1\alpha}(y') D_y^\alpha u(x,y') = 0, \;\; (x,y') \in [0,\infty) \times \partial G, \tag{2.11}$$

where γ, δ are complex numbers, $m_1 \le 1$, $y := (y_1, \dots, y_r)$, $D_x := \frac{\partial}{\partial x}$, $D_y^\alpha := D_1^{\alpha_1} \cdots D_r^{\alpha_r}$, $D_j = \frac{\partial}{\partial y_j}$. Let $W_{p,q}^{\ell,s}(\Omega) := W_p^\ell((0,\infty); W_q^s(G), L_q(G))$, $L_{p,q}(\Omega) := W_{p,q}^{0,0}(\Omega)$.

The corresponding spectral problem is

$$\lambda^2 u(y) + \sum_{j,k=1}^{r} b_{jk}(y) D_j D_k u(y) + Mu\big|_y = 0, \quad y \in G, \tag{2.12}$$

$$\tilde{L}_1 u := \sum_{|\alpha|\le m_1} b_{1\alpha}(y') D_y^\alpha u(y') = 0, \quad y' \in \partial G. \tag{2.13}$$

Let us denote $H := L_2(G)$ and consider the operator B which is defined by

$$\begin{gathered} D(B) := W_2^2(G; \tilde{L}_1 u = 0), \\ Bu := \sum_{j,k=1}^{r} b_{jk}(y) D_j D_k u(y) + Mu\big|_y. \end{gathered} \tag{2.14}$$

Theorem 6 *Let the following conditions be satisfied:*

1. $b_{jk} \in C(\overline{G})$, $b_{1\alpha} \in C^{2-m_1}(\overline{G})$, $\partial G \in C^2$;
2. *If* $y \in \overline{G}$, $\sigma := (\sigma_1, \dots, \sigma_r) \in \mathbb{R}^r$, $|\sigma| + |\lambda| \ne 0$ *then*
$$\lambda^2 - \sum_{j,k=1}^{r} b_{jk}(y)\sigma_j\sigma_k \ne 0, \quad \operatorname{Re}\lambda = 0;$$
3. $\sum_{|\alpha|=m_1} b_{1\alpha}(y')\sigma^\alpha \ne 0$ *for any vector* σ *normal to* ∂G *at the point* $y' \in \partial G$;
4. *Let* y' *be any point on* ∂G, *the vector* σ' *tangent and the vector* σ *normal to* ∂G *at the point* $y' \in \partial G$. *Consider the following ordinary differential problem*
$$\big[\lambda^2 - \sum_{j,k=1}^{r} b_{jk}(y')\big(\sigma_j' + \sigma_j \frac{d}{dt}\big)\big(\sigma_k' + \sigma_k \frac{d}{dt}\big)\big] u(t) = 0, \; t > 0, \; \operatorname{Re}\lambda = 0, \tag{2.15}$$
$$\sum_{|\alpha|=m_1} b_{1\alpha}(y')\big(\sigma' + \sigma \frac{d}{dt}\big)^\alpha u(t)\big|_{t=0} = h_1; \tag{2.16}$$

problem (2.15)–(2.16) should have only one solution that with all its derivatives tend to zero as $t \to \infty$ for any number $h_1 \in \mathbb{C}$;

5. $|\gamma| + |\delta| \neq 0$; $\operatorname{Re} \gamma\delta^{-1} \leq 0$ *when* $\delta \neq 0$;
6. *The spectral problem (2.12)–(2.13) does not have eigenvalues on the line* $\operatorname{Re} \lambda = 0$;
7. *The operator M from $W_2^2(G)$ into $L_2(G)$ is compact.*

Then the operator $\mathbb{L} : u \to \mathbb{L}u := (Lu, Pu)$ from $W_{p,2}^{2,2}(\Omega; L_1 u = 0)$ onto $L_{p,2}(\Omega)\dot{+}\ B_{2,p}^{2-m-\frac{1}{p}}(G; \tilde{L}_1 u = 0, m_1 < \frac{3}{2} - m - \frac{1}{p})$, if $p > 1$ and $p \neq 2$, or $p = 2$ and $m_1 \neq 1 - m$, [4] *is an isomorphism, where $m = 0$ if $\delta = 0$; $m = 1$ if $\delta \neq 0$.*

Proof Problem (2.9)–(2.11) can be rewritten in the form

$$u''(x) + Bu(x) = f(x), \quad x > 0, \tag{2.17}$$
$$\gamma u(0) + \delta u'(0) = \varphi, \tag{2.18}$$

where $u(x) := u(x, \cdot), f(x) := f(x, \cdot)$ are functions with values in the Hilbert space $H := L_2(G)$, $\varphi := \varphi(\cdot)$ is an element of H, the operator B is defined by the equalities (2.14).

Apply Theorem 2 to problem (2.17)–(2.18). From Theorem 1 in [14, p.207] it follows that the operator $(\lambda^2 I + B)^{-1}$ is bounded in $L_2(0, 1)$. A bounded operator is closed. The inverse operator to a closed operator is also closed. Therefore, $\lambda^2 I + B$ is a closed operator. This implies that the operator B is closed, i.e., condition (1) of Theorem 2 is fulfilled. On the other hand, from Theorem 1 in [14, p.207] and condition (6), condition (2) of Theorem 2 follows. The last part of the proof is similar to that in the proof of Theorem 5. □

2.2 Completeness of elementary solutions

Let us consider, in the semi-infinite strip $\Omega := [0, \infty) \times [0, 1]$, a principally boundary-value problem for an elliptic equation of the second order,

$$D_x^2 u(x, y) + b(y) D_y^2 u(x, y) + Mu(x, \cdot)\big|_y = 0, \tag{2.19}$$
$$\gamma u(0, y) + \delta D_x u(0, y) = \varphi(y), \quad y \in [0, 1], \tag{2.20}$$
$$\begin{aligned} L_1 u &:= \alpha_1 D_y u(x, 0) + \alpha_0 u(x, 0) = 0,\ x \in [0, \infty), \\ L_2 u &:= \beta_1 D_y u(x, 1) + \beta_0 u(x, 1) = 0,\ x \in [0, \infty), \end{aligned} \tag{2.21}$$

and the corresponding spectral problem (2.4)–(2.5), where α_ν, β_ν are complex numbers, $D_x := \frac{\partial}{\partial x}$, $D_y := \frac{\partial}{\partial y}$; $m_\nu := \operatorname{ord} L_\nu$ and $W_{p,q}^{\ell,s}(\Omega) := W_p^\ell((0, \infty); W_q^s(0, 1), L_q(0, 1))$.

[4] See the corresponding remark of Theorem 5.

As it was mentioned in the introduction, a function of the form

$$u_i(x,y) := \mathrm{e}^{\lambda_i x}\big(\frac{x^{k_i}}{k_i!}u_{i0}(y) + \frac{x^{k_i-1}}{(k_i-1)!}u_{i1}(y) + \cdots + u_{ik_i}(y)\big) \tag{2.22}$$

becomes an elementary solution of problem (2.19), (2.21) if and only if a system of functions $u_{i0}(y), u_{i1}(y), \dots, u_{ik_i}(y)$ is a chain of root functions of problem (2.4)–(2.5) corresponding to the eigenvalue λ_i. See the corresponding remark in subsection 1.3.

Theorem 7 *Let the following conditions be satisfied:*

1. *The conditions of Theorem 5 are fulfilled;*
2. $\varphi \in B^{2-m-\frac{1}{p}}_{2,p}((0,1); \tilde{L}_\nu u = 0, m_\nu < \frac{3}{2} - m - \frac{1}{p})$ *if* $p > 1$ *and* $p \neq 2$, *or* $p = 2$ *and* $m_\nu \neq 1 - m$[5], *where* $m = 0$ *if* $\delta = 0$; $m = 1$ *if* $\delta \neq 0$.

Then problem (2.19)–(2.21) has a unique solution $u \in W^{2,2}_{p,2}(\Omega)$, *and there exist numbers* C_{in} *such that*

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty \Big(&\|D_x^2 u(x,\cdot) - \sum_{i=1}^n C_{in} D_x^2 u_i(x,\cdot)\|^p_{L_2(0,1)}\\ &+ \|u(x,\cdot) - \sum_{i=1}^n C_{in} u_i(x,\cdot)\|^p_{W_2^2(0,1)}\Big)\,dx = 0,\end{aligned}$$

where $u(x,y)$ *is a solution of problem (2.19)–(2.21) and* $u_i(x,y)$ *is the elementary solution (2.22) of problem (2.19), (2.21) corresponding to the eigenvalue* λ_i *with* $\operatorname{Re}\lambda_i < 0$.

Proof Apply Theorem 4 to problem (2.19)–(2.21). In $H := L_2(0,1)$, consider an operator B which is defined by equality (2.6). Then, problem (2.19)–(2.21) can be rewritten in the form

$$u''(x) + Bu(x) = 0, \quad x > 0, \tag{2.23}$$

$$\gamma u(0) + \delta u'(0) = \varphi, \tag{2.24}$$

where $u(x) := u(x,\cdot)$ is a function with values in the Hilbert space $H := L_2(0,1)$ and $\varphi := \varphi(\cdot)$ is an element of H.

Conditions (1) and (3) of Theorem 4 have been checked in the proof of Theorem 5. By virtue of Triebel [12, formula 4.10.2/14],

$$s_j(J; W_2^2(0,1), L_2(0,1)) \sim j^{-2}. \tag{2.25}$$

Since $W_2^2((0,1); \tilde{L}_\nu u = 0, \nu = 1,2)$ is a subspace of $W_2^2(0,1)$ then, by Lemma 3 in [14, p.17], from (2.15) it follows that

$$s_j(J; H(B), H) \le C s_j(J; W_2^2(0,1), L_2(0,1)) \le C j^{-2},$$

[5]See the corresponding remark of Theorem 5.

i.e., condition (2) of Theorem 4 is fulfilled for $q = 2$. We have shown in the proof of Theorem 5 that for $\varepsilon < \arg\lambda < \pi - \varepsilon$ or $\pi + \varepsilon < \arg\lambda < 2\pi - \varepsilon$ and $|\lambda| \to \infty$,

$$|\lambda|^2 \|u\|_{L_2(0,1)} + \|u\|_{W_2^2(0,1)} \le C\|f\|_{L_2(0,1)}, \quad f \in L_2(0,1).$$

This gives us condition (5) of Theorem 4 for $q = 2$ and $\eta = 0$. Condition (6) of Theorem 4 one can see in the proof of Theorem 5.

So, for problem (2.23)–(2.24) all conditions of Theorem 4 have been checked and the statement of Theorem 7 follows. □

In the semi-infinite domain $\Omega := [0,\infty) \times G$, where $G \subset \mathbb{R}^r$, $r \ge 2$, is a bounded domain with an $(r-1)$-dimensional smooth boundary ∂G, consider a principally boundary-value problem for an elliptic equation of the second order

$$D_x^2 u(x,y) + \sum_{j,k=1}^{r} b_{jk}(y) D_j D_k u(x,y) + Mu(x,\cdot)\big|_y = 0, \tag{2.26}$$

$$\gamma u(0,y) + \delta D_x u(0,y) = \varphi(y), \quad y \in G, \tag{2.27}$$

$$L_1 u := \sum_{|\alpha| \le m_1} b_{1\alpha}(y') D_y^\alpha u(x,y') = 0, \ (x,y') \in [0,\infty) \times \partial G, \tag{2.28}$$

and the corresponding spectral problem

$$\lambda^2 u(y) + \sum_{j,k=1}^{r} b_{jk}(y) D_j D_k u(y) + Mu\big|_y = 0, \quad y \in G, \tag{2.29}$$

$$\tilde{L}_1 u := \sum_{|\alpha| \le m_1} b_{1\alpha}(y') D_y^\alpha u(y') = 0, \quad y' \in \partial G, \tag{2.30}$$

where $m_1 \le 1$, $y := (y_1,\dots,y_r)$, $D_x := \frac{\partial}{\partial x}$, $D_y^\alpha := D_1^{\alpha_1} \cdots D_r^{\alpha_r}$, $D_j = \frac{\partial}{\partial y_j}$. As above, $W_{p,q}^{\ell,s}(\Omega) := W_p^\ell((0,\infty); W_q^s(G), L_q(G))$.

A function of the form

$$u_i(x,y) := \mathrm{e}^{\lambda_i x}\big(\frac{x^{k_i}}{k_i!} u_{i0}(y) + \frac{x^{k_i-1}}{(k_i-1)!} u_{i1}(y) + \dots + u_{ik_i}(y)\big) \tag{2.31}$$

becomes an elementary solution of problem (2.26), (2.28) (see the introduction) if and only if a system of functions $u_{i0}(y), u_{i1}(y), \dots, u_{ik_i}(y)$ is a chain of root functions of the spectral problem (2.29)–(2.30) corresponding to the eigenvalue λ_i. See the corresponding remark in subsection 1.3.

Consider in $H := L_2(G)$ the operator B which is defined by the equalities in (2.14).

Theorem 8 *Let the following conditions be satisfied:*

1. *The conditions of Theorem 6 are fulfilled;*

2. *There exist rays ℓ_k with angles between neighbouring rays less than $\frac{\pi}{2r}$ such that for $y \in \overline{G}$, $\sigma \in \mathbb{R}^r$, $|\sigma| + |\lambda| \neq 0$, $\lambda \in \ell_k$, the following is true:*

$$\lambda^2 + \sum_{j,k=1}^{r} b_{jk}(y)\sigma_j\sigma_k \neq 0;$$

3. *Let y' be any point on ∂G, the vector σ' tangent and the vector σ normal to ∂G at the point $y' \in \partial G$. Consider the following ordinary differential problem*

$$\Big[\lambda^2 + \sum_{j,k=1}^{r} b_{jk}(y')\big(\sigma'_j + \sigma_j \frac{d}{dt}\big)\big(\sigma'_k + \sigma_k \frac{d}{dt}\big)\Big]u(t) = 0, \quad t \geq 0,\ \lambda \in \ell_k, \tag{2.32}$$

$$\sum_{|\alpha|=m_1} b_{1\alpha}(y')\big(\sigma' + \sigma\frac{d}{dt}\big)^{\alpha}u(t)\big|_{t=0} = h_1; \tag{2.33}$$

problem (2.32)–(2.33) should have only one solution that with all its derivatives tend to zero as $t \to \infty$ for any number $h_1 \in \mathbb{C}$;

4. *$\varphi \in B_{2,p}^{2-m-\frac{1}{p}}(G; \tilde{L}_1 u = 0, m_1 < \frac{3}{2} - m - \frac{1}{p})$ if $p > 1$ and $p \neq 2$, or $p = 2$ and $m_1 \neq 1 - m$[6], where $m = 0$ if $\delta = 0$; $m = 1$ if $\delta \neq 0$.*

Then problem (2.26)–(2.28) has a unique solution $u \in W_{p,2}^{2,2}(\Omega)$, and there exist numbers C_{in} such that

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\infty}\Big(&\|D_x^2 u(x,\cdot) - \sum_{i=1}^{n} C_{in} D_x^2 u_i(x,\cdot)\|_{L_2(G)}^p \\ &+ \|u(x,\cdot) - \sum_{i=1}^{n} C_{in} u_i(x,\cdot)\|_{W_2^2(G)}^p\Big)\,dx = 0,\end{aligned}$$

where $u(x,y)$ is a solution of problem (2.26)–(2.28) and $u_i(x,y)$ is the elementary solution (2.31) of problem (2.26), (2.28) corresponding to the eigenvalue λ_i with $\operatorname{Re}\lambda_i < 0$.

Proof Apply Theorem 4 to problem (2.26)–(2.28). Problem (2.26)–(2.28) can be rewritten in the form

$$u''(x) + Bu(x) = 0, \quad x > 0, \tag{2.34}$$

$$\gamma u(0) + \delta u'(0) = \varphi, \tag{2.35}$$

where $u(x) := u(x,\cdot)$ is a function with values in the Hilbert space $H := L_2(G)$ and $\varphi := \varphi(\cdot)$ is an element of H.

[6]See the corresponding remark of Theorem 5.

Conditions (1) and (3) of Theorem 4 have been checked in Theorem 6. By formula 4.10.2/14 in [12],

$$s_j(J; W_2^2(G), L_2(G)) \sim j^{-\frac{2}{r}}. \tag{2.36}$$

Since $W_2^2(G; \tilde{L}_1 u = 0)$ is a subspace of $W_2^2(G)$ then, by Lemma 3 in [14, p.17], from (2.36) it follows that

$$s_j(J; H(B), H) \le C s_j(J; W_2^2(G), L_2(G)) \le C j^{-\frac{2}{r}},$$

i.e., condition (2) of Theorem 4 is fulfilled for $q = \frac{2}{r}$. By Theorem 1 in [14, p.207], from conditions (2) and (3), condition (5) of Theorem 4, for $q = \frac{2}{r}$, follows.

Condition (6) of Theorem 4 one can see in the proof of Theorem 5. So, for problem (2.34)–(2.35) all conditions of Theorem 4 have been checked and the statement of the theorem follows. □

The results of this paper can be applied to the thermal conduction problem from [9] in the case when there are not mixed derivatives in the equation. We get completeness of a system of root functions of the corresponding spectral problem and completeness of elementary solutions of the original problem for eigenvalues λ_i with $\operatorname{Re}\lambda_i < 0$. Moreover, since the corresponding operator B of the thermal conduction problem is selfadjoint then one can get a basis property theorem instead of completeness Theorem 7 (for $p = 2$). But the latter needs some additional considerations.

A few examples of the operator M which satisfies conditions of Theorems 5 and 6 and, therefore, Theorems 7 and 8 are the following. Let G denote the interval $(0, 1)$ or a bounded domain in $\mathbb{R}^r$, $r \ge 2$, with an $(r-1)$-dimensional smooth boundary.

1. If $b_j \in L_2(G)$, then the operator

$$Mu := \sum_{j=0}^{1} b_j(x) u^{(j)}(x)$$

from $W_2^2(G)$ into $L_2(G)$ is compact.

2. If $b_{ji} \in L_2(G)$ and $\varphi_{ji}(x)$ are functions mapping $\overline{G}$ into itself and belong to $C(\overline{G})$, then the operator

$$Mu := \sum_{j=0}^{1} \sum_{i=1}^{N_j} b_{ji}(x) u^{(j)}(\varphi_{ji}(x)),$$

from $W_2^2(G)$ into $L_2(G)$ is compact.

3. If $B_j(x, y)$ are kernels such that for some $\sigma > 1$

$$\int_G |B_j(x, y)|^\sigma dy + \int_G |B_j(x, y)|^\sigma dx \le C$$

then the operator

$$Mu := \sum_{j=0}^{2} \int_G B_j(x,y)u^{(j)}(y)\, dy$$

from $W_2^2(G)$ into $L_2(G)$ is compact.

The proofs can be found in [14, p.201].

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