Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 11, pp. 1–26.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Blow-up of radially symmetric solutions of a non-local problem modelling Ohmic heating *

Dimitrios E. Tzanetis

Abstract

We consider a non-local initial boundary-value problem for the equation

$$u_t = \Delta u + \lambda f(u)/\Big(\int_\Omega f(u)\,dx\Big)^2, \quad x \in \Omega \subset \mathbb{R}^2,\ t > 0,$$

where u represents a temperature and f is a positive and decreasing function. It is shown that for the radially symmetric case, if $\int_0^\infty f(s)\,ds < \infty$ then there exists a critical value $\lambda^* > 0$ such that for $\lambda > \lambda^*$ there is no stationary solution and u blows up, whereas for $\lambda < \lambda^*$ there exists at least one stationary solution. Moreover, for the Dirichlet problem with $-s\,f'(s) < f(s)$ there exists a unique stationary solution which is asymptotically stable. For the Robin problem, if $\lambda < \lambda^*$ then there are at least two solutions, while if $\lambda = \lambda^*$ at least one solution. Stability and blow-up of these solutions are examined in this article.

1 Introduction

In this work we study the radially symmetric solutions to the non-local initial boundary-value problem

$$u_t = \Delta u + \frac{\lambda f(u)}{(\int_\Omega f(u)\,dx)^2}\,, \quad x \in \Omega,\ t > 0\,, \tag{1.1a}$$

$$\mathcal{B}(u) := \frac{\partial u}{\partial n} + \beta(x)u = 0\,, \quad t > 0\,,\ x \in \partial\Omega\,, \tag{1.1b}$$

$$u(x,0) = u_0(x)\,, \quad x \in \Omega\,,$$

where $u = u(x,t)$, Ω is a bounded domain of $\mathbb{R}^2$, λ is a positive parameter, $\partial\Omega$ and $\beta(x)$ are sufficiently smooth. The function f is continuous, positive and decreasing,

$$f(s) > 0\,, \quad f'(s) < 0\,, \quad s \geq 0\,. \tag{1.2}$$

* *Mathematics Subject Classifications:* 35B30, 35B40, 35K20, 35K55, 35K99.
Key words: Nonlocal parabolic equations, blow-up, global existence, steady states.

Submitted October 2, 2001. Published February 1, 2002.
Partially supported by the E.C. Human Capital and Mobility Scheme, under contract ERBCHRXCT 93-0409.

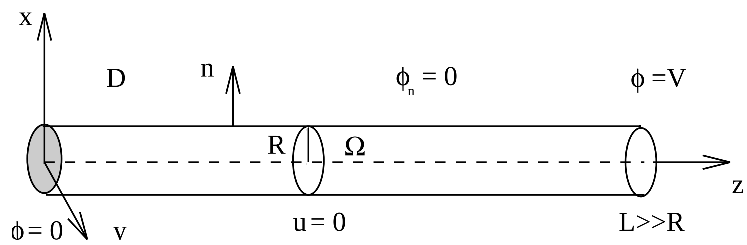

Figure 1: A long and thin cylindrical conductor

We also study, in Section 3, the case of f being the Heaviside function (which is neither continuous nor always positive). The equation (1.1a) arises by reducing the system of two equations

$$u_t = \nabla \cdot (k(u)\nabla u) + \sigma(u)\,|\nabla\phi|^2 \tag{1.3a}$$

$$\nabla \cdot (\sigma(u)\nabla\phi) = 0\,, \tag{1.3b}$$

to a simple, but still realistic equation. More precisely, (1.3a) is a parabolic equation while (1.3b) is an elliptic, u represents the temperature produced by an electric current flowing through a conductor, $\phi = \phi(x,t)$ is the electric potential, $k(u)$ is the thermal conductivity and $\sigma(u)$ is the electrical conductivity. Problem (1.3) models many physical situations especially in thermistors [1, 2, 3, 12], fuse wires, electric arcs and fluorescent lights. The conductivity σ may be either decreasing or increasing in u depending upon the nature of the conductor. Here we consider materials having constant thermal conductivity, e.g. $k(u) = 1$, and decreasing electrical conductivity, the latter allowing a thermal runaway to take place [18, 19].

Some questions concerning the steady problem to (1.3) were investigated by Cimatti [8, 9, 10], see also [3]. A similar problem to (1.3) with radial symmetry, Robin boundary conditions of the form $u_n + \beta u = 0$ and conductivity $\sigma(u) = \exp(-f(u)/\epsilon)$, $\epsilon \ll 1$ was discussed by Fowler et al. [12]. Some numerical results are also given for small β. See also Howison [14] for how the steady problem to (1.3) may be reduced to one nonlinear o.d.e. and Laplace's equation. Carrillo [6], has looked at the bifurcation diagram of the non-local elliptic problem with decreasing nonlinearity and Dirichlet boundary conditions, in $\Omega \subset \mathbb{R}^N$, See [5] for a similar study where Ω is a unit ball in $\mathbb{R}^N$. For an extended study of the structure of solutions of the non-local elliptic problem see [20].

The two-dimensional mathematical problem for the single equation can be derived by considering a long and thin cylindrical conductor D, $(x,y,z) \in D \subset \mathbb{R}^3$, of length L, $R \ll L$ where R is the radius of the cross-section Ω of D, see Figure 1.

The curved surface of D is electrically insulated, i.e. $\phi_n = 0$, and $u = 0$ or more generally $u_n + \beta u = 0$ for $\beta \in [0,\infty]$, ($\beta = 0$ gives $u_n = 0$ while $\beta = \infty$ gives $u = 0$). Also $\phi(x,y,0,t) = 0$, $\phi(x,y,L,t) = V$ at the ends of the conductor, thus V is the potential difference. Here the temperature u is taken to be initially independent of z ($u = 0$ is likely to be of practical interest). Also the z-derivatives of u are neglected, so the model gives z-independence of u for $t > 0$. Moreover we stress that the model is most definitely only supposed to apply in the bulk of the device. Thus taking the thermal conductivity to be constant, and neglecting the end effects, problem (1.3) can then be reduced, as in [18], to a single non-local equation

$$u_t = \Delta u + \lambda\sigma(u)/(\int_\Omega \sigma(u)\,dx)^2, \tag{1.4}$$

where $\lambda = I^2/|\Omega|^2 \geq 0$, I is the electric current which we suppose to be constant and $|\Omega|$ is the measure of Ω. On the other hand, by assuming the voltage V to be constant, $\phi_x = V/L$, problem (1.3) takes the more standard semi-linear parabolic form:

$$u_t = \Delta u + \lambda\sigma(u)\,, \quad x \in \Omega, \quad \text{where} \quad \lambda = V^2/L^2 \geq 0. \tag{1.5}$$

Finally, taking the more general case of a conductor connected in series with a resistance R_0 under a constant voltage E, then (1.3) gives, on using

$$E = I\,R_0 + V = \Big[I + R_0\,|\Omega| \int_\Omega \sigma(u)\,dx\Big]\,V\,,$$

the non-local equation

$$u_t = \Delta u + \lambda\sigma(u)/\Big[a + b\int_\Omega \sigma(u)\,dx\Big]^2, \;\; x \in \Omega\,, \;\; t > 0, \;\; a, b > 0. \tag{1.6}$$

For the derivation of equations (1.3)-(1.6), as well as a complete study of the one-dimensional model for a decreasing $\rho(u)$, (the electrical resistivity $\sigma(u) = 1/\rho(u)$ is increasing), see [18, 19]. In [18, 19] it was shown that for $\int_0^\infty f(s)\,ds < \infty$ there is some critical value λ^* such that for $\lambda > \lambda^*$ there is no steady state and u blows up globally, for $\lambda = \lambda^*$, and $f(s) = \exp(-s)$, again there is no steady state and u exists globally in time but is unbounded. Moreover, for $\lambda < \lambda^*$, as well as for any $\lambda > 0$ provided now that $\int_0^\infty f(s)\,ds = \infty$, where a unique steady state exists, this steady state is globally asymptotically stable. A global existence and divergence result for the solution of (1.7) (see below), when $f(s) = e^{-s}$, is also proved in [16].

Chafee [7] considered a related model $u_t = u_{xx} - g(u) + \lambda f(u)/(\int_{-1}^1 f(u)\,dx)^2$. It was found that there is a λ^* such that for $\lambda < \lambda^*$ there is a homogeneous steady state which is globally asymptotically stable. There are conditions under which the homogeneous steady state is unstable and there are then two stable inhomogeneous steady states. Other works concerning the blow-up in non-local parabolic problems are [4, 22, 23].

Finally we wish to study problem (1.1), which comes from (1.4) by setting $\sigma(u) = f(u)$, in the radially symmetric case. Therefore we take Ω to be the unit ball and the initial data are taken to be radially symmetric and decreasing,

($u(x,0)=u_0(r)$ and $u_0'(r)<0\,,\ \ r=|x|$). Thus our problem, in case of Dirichlet boundary conditions ($\beta(x)=\infty$), takes the form:

$$u_t = u_{rr} + \frac{u_r}{r} + \frac{\lambda f(u)}{4\pi^2(\int_0^1 f(u)\,r\,dr)^2}\,, \quad 0<r<1\,, \quad t>0\,, \tag{1.7a}$$

$$u(1,t)=u_r(0,t)=0\,, \quad t>0\,, \tag{1.7b}$$

$$u(r,0)=u_0(r)\,, \quad 0<r<1\,. \tag{1.7c}$$

We also consider Neumann or Robin boundary conditions:

$$u_r(1,t)+\beta u(1,t)=0\ , \quad \beta\in[0,\infty). \tag{1.8}$$

Note that $u_r(0,t)=0$ is a consequence of boundedness of solutions rather than a specific constraint upon them.

We note that any solution of (1.1) with $u_0>0$ is positive, moreover if $u_0(x)=u_0(r)$ and Ω is a ball, then $u(x,t)=u(r,t)$ is radially symmetric and satisfies (1.7) with the proper boundary conditions ((1.7b) or (1.8)). Furthermore if $u_0'(r)<0$ then $u_r(r,t)<0$, $0<r<1$, $0<t<T$, i.e. u is radially decreasing. The same properties hold for the steady solutions of problem (1.1), see Gidas et al., [13].

The non-local problem under consideration belongs to the class where the maximum principle holds (due to (1.2)) and comparison with suitable upper and lower solutions is used to prove stabilization or blow-up. In the contrary when $f(s)$ is an increasing function, maximum principle does not hold. Nevertheless for $f(s)=e^s$, stabilization and blow-up can be studied by using a Lyapunov functional, [5, 11].

The present work is organized as follows. In Section 2 the existence and uniqueness of solutions to (1.1) in $\Omega\subset\mathbb{R}^N$ is discussed. In Sections 3, 4, some particular functions, the Heaviside and the exponential are studied, while in Section 5 a general decreasing function is considered. In each of these cases, the critical value λ^* is estimated.

In the rest of this article we mainly follow the ideas and techniques which have been used in the one-dimensional case [18, 19], but have to be modified because of the extra technical difficulties encountered in this two-dimensional problem.

2 Existence, uniqueness and monotonicity

Problem (1.1) in $\Omega\subset\mathbb{R}^N$ $N\geq 1$, for a measurable and bounded $u_0(x)$, can be written in a Green' s integral formulation:

$$u(x,t)=\lambda\int_0^t\int_\Omega g(x,y,t-s)\frac{f(u(y,s))}{(\int_\Omega f(u(y,s))\,dy)^2}\,dy\,ds+\int_\Omega g(x,y,t)u_0(y)\,dy\,. \tag{2.1}$$

Setting now v_n instead of u on the left-hand side and v_{n-1} instead of u on the right-hand side for $n\geq 1$ and taking $v_0\equiv 0$, we find on passing to the limit and following on standard Picard - iteration - type arguments, that if $\lambda > 0$, $f(s) \geq c > 0$ and Lipschitz for $s \in (a, b)$, where $a < \min\{0, \inf u_0\} \leq u \leq \max\{0, \sup u_0\} < b$, then there exists a unique solution u to (1.1) and (2.1). Moreover the solution continues to exist as long as it remains less than or equal to b; this implies that it can only cease to exist due to blow-up.

On the other hand, if we restrict our attention to a decreasing f, positive and Lipschitz, then we have a sort of comparison. In particular $\overline{u}$ is called a strict **upper solution** to (1.1) in $\Omega \subset \mathbb{R}^N$, $N \geq 1$, if it satisfies

$$\overline{u}_t(x,t) > \Delta\overline{u} + \frac{\lambda f(\overline{u})}{(\int_\Omega f(\overline{u})\,dx)^2}\,, \quad \text{in } \Omega\,, \quad t > 0\,, \tag{2.2a}$$

$$\mathcal{B}(\overline{u}) > \mathcal{B}(u) = 0 \quad \text{on} \quad \partial\Omega\,,\ t > 0\,, \tag{2.2b}$$

$$\overline{u}_0(x) > u_0(x)\,,\ \text{ in }\ \Omega\,, \tag{2.2c}$$

while if $\underline{u}$ satisfies the reversed inequalities of (2.2) it is called a strict **lower solution**. Now if we set $v = \overline{u} - \underline{u}$ then there exists $T > 0$ such that

$$\begin{gathered} v_t > \Delta v + \frac{\lambda f'(s)}{(\int_\Omega f(\underline{u})\,dx)^2}\,v\,,\ \ x \in \Omega\,,\ \ 0 < t < T\,, \\ v > 0 \ \text{ at } \ t = 0 \ \text{ in } \Omega \ \text{ and } \ \mathcal{B}(v) > 0\,,\ \text{ on } \ \partial\Omega\,,\ \ 0 < t < T\,, \end{gathered} \tag{2.3}$$

which implies, by the maximum principle, that $v > 0$ at $t = T$. Moreover, if (2.2) holds with $\geq$, then (2.3) also holds with $\geq$ in the place of $>$. As long as $\underline{u}$, $\overline{u}$ exist and $\overline{u} \geq \underline{u}$, with f Lipschitz, we can apply iteration schemes similar to those of Sattinger [21], to show that there exists a unique solution u to (1.1) such that $\underline{u} \leq u \leq \overline{u}$. If now f is increasing then some of the above results can be adapted by using a pair of upper-lower solutions; see [18].

3 The Heaviside function

We consider now $f(s)$ to be the Heaviside function (decreasing), $f(s) = H(1-s)$, then $f(s) = 1$ for $s < 1$, and $f(s) = 0$ for $s \geq 1$, which is neither strictly positive nor Lipschitz continuous. Thus problem (1.7) becomes,

$$u_t = \Delta_r u + \lambda H(1-u)/4\pi^2\Big(\int_0^1 H(1-u) r\,dr\Big)^2, \quad 0 < r < 1\,, \quad t > 0\,, \tag{3.1a}$$

$$u(1,t) = u_r(0,t) = 0\,, \quad t > 0\,, \quad u(r,0) = u_0(r)\,, \quad 0 < r < 1\,, \tag{3.1b}$$

where ($\Delta_r = \partial^2/\partial r^2 + \frac{1}{r}\partial/\partial r$). In particular equation (3.1a) can be written:

$$\begin{gathered} u_t = \Delta_r u\,, \quad \text{where} \quad u \geq 1\,,\ \text{ and} \\ u_t = \Delta_r u + \lambda/m^2(t)\,, \quad \text{where} \quad u < 1\,, \end{gathered}$$

writing $m(t)$ the measure of the subset of the unit ball $B(0,1)$ where $u < 1$. The existence and uniqueness of a “weak” (classical a.e.) solution to (3.1) is obtained by using an approximating regularized version of this problem, see [15] and the references therein. Hence, taking into account this remark, in the following we can use comparison arguments in the classical sense. We take $u_0(r) \le 1$, and for simplicity $u_0'(r) \le 0$ and bounded below. With such initial data $v = 1$ is an upper solution to problem (3.1), hence $u \le 1$. Thus either $u < 1$ for $0 < r < 1$ whereupon (3.1a) becomes

$$u_t = \Delta_r u + \lambda/\pi^2, \quad 0 < r < 1\,, \ \ t > 0, \tag{3.2}$$

or there exists an or some $s = s(t)$, $0 < s(t) < 1$, such that

$$u_t = \Delta_r u + \lambda/\pi^2(1-s^2)^2, \quad 0 \le u < 1, \quad s < r < 1\,, \ \ t > 0\,, \tag{3.3a}$$

$$u = 1\,, \quad u_r = 0, \quad 0 \le r \le s\,, \ \ t > 0\,, \tag{3.3b}$$

where $\pi(1 - s^2(t)) = m(t)$. Note that u is continuous and $u_r \le 0$, the latter follows by using the maximum principle. The corresponding steady state to (3.2) is

$$\Delta_r w + \lambda/\pi^2 = 0\,, \quad 0 < r < 1\,, \quad w'(0) = w(1) = 0\,, \tag{3.4}$$

which for $\lambda < 4\pi^2$, has a solution $w(r) = \frac{\lambda}{4\pi^2}(1 - r^2)$. Also a steady state for (3.3) satisfies:

$$\Delta_r w + \frac{\lambda}{\pi^2(1-S^2)^2} = 0\,, \quad S < r < 1\,, \quad 0 \le w < 1\,, \tag{3.5a}$$

$$w(r) = 1\,,\ w'(r) = 0\,,\ 0 \le r \le S \ \text{ for } \ 0 \le S < 1\,, \quad w(1) = 0\,. \tag{3.5b}$$

Equations (3.5a), (3.5b) give a one-parameter family of steady states of the form:

$$w(r;S) = \frac{\hat\lambda(S)(1 + 2S^2 \ln r - r^2)}{4\pi^2(1-S^2)^2} = \frac{1 + 2S^2 \ln r - r^2}{1 + 2S^2 \ln S - S^2}\,, \quad S < r < 1, \tag{3.6}$$

$$\text{where} \quad \lambda = \hat\lambda(S) = \frac{4\pi^2(1-S^2)^2}{1 + 2S^2 \ln S - S^2}.$$

It is easily seen that $\hat\lambda(S)$ is strictly increasing, $\hat\lambda(1-) = 8\pi^2$ and $\hat\lambda(0+) = 4\pi^2$. If we note by $\|w'\| = \sup|w'|$, then $\|w'\| = -w'(1)$ and the following hold: for $0 < \lambda < 4\pi^2 = \hat\lambda(0+)$ there exists a unique steady state $w(r) = \frac{\lambda}{4\pi^2}(1 - r^2)$, for $4\pi^2 \le \hat\lambda(S) < 8\pi^2$, $S \in [0,1)$, there exists a one-parameter family of steady states given by (3.6), whereas for $\lambda \ge \hat\lambda(1-) = 8\pi^2$ there is no steady solution. Hence we get the diagram of Figure 2.

We wish now to study the stability of the steady solutions for $\lambda < 8\pi^2 = \hat\lambda(1-)$. Therefore we construct an upper solution $\overline{v}$ (lower solution $\underline{v}$) to problem (3.1), decreasing (increasing) in time, of a form similar to the steady state, i.e. $w(r; s(t))$. Namely for $\lambda < 4\pi^2$ we take,

$$\overline{v}(r,t) = 1\,,\ \ \overline{v}_r(r,t) = 0 \ \text{ for } \ 0 \le r \le s(t)\,, \quad \text{and} \tag{3.7a}$$

$$\overline{v}(r,t) = w(r; s(t)) = \frac{\hat\lambda(s)}{4\pi^2(1-s^2)^2}\left(1 + 2s^2 \ln r - r^2\right) = \frac{1 + 2s^2 \ln r - r^2}{1 + 2s^2 \ln s - s^2}\,, \tag{3.7b}$$

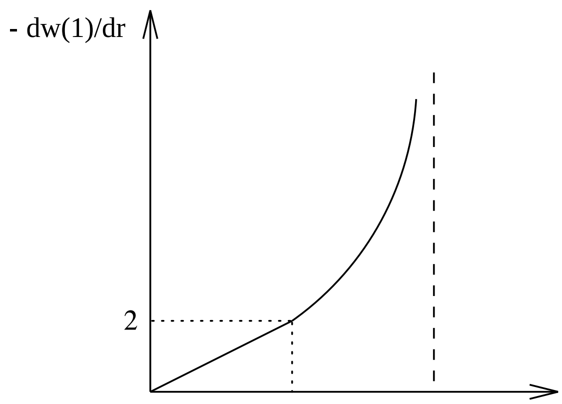

Figure 2: Response diagram for (3.4), (3.5), $f(s) = H(1-s)$.

$s(t) < r < 1\,,\quad 0 \le t < t_1$ where $s = s(t) \in (0,1)\,,\quad s(0) = s_0$. For any initial data $u_0(r) \le 1$, we choose s_0 so that $u_0(r) \le 1$ for $0 \le r \le s_0$ and $u_0(r) \le w(r; s_0)$, $0 < s_0 < r < 1$, i.e. $\overline{v}(r,0) = w(r; s_0) \ge u_0(r)$. Then we have,

$$\begin{aligned}
\mathcal{E}(\overline{v}) \quad &:= \quad \overline{v}_t - \Delta_r \overline{v} - \lambda H(1-\overline{v})/4\pi^2 \Big(\int_0^1 H(1-\overline{v}) r\, dr \Big)^2 \\
&= \quad \begin{cases} 0\,, & 0 \le r \le s(t)\,, \\ \overline{v}_t + \dfrac{\overline{\lambda}(s) - \lambda}{\pi^2(1-s^2)^2} \ge 0\,, & s(t) \le r \le 1\,, \end{cases}
\end{aligned}$$

provided that $s(t)$ satisfies:

$$0 < -\dot{s} = h(s) \equiv \frac{(\overline{\lambda}(s) - \lambda)(1 + 2s^2 \ln s - s^2)}{4\pi^2 (1-s^2)^2 (1-s)}\,,$$

giving $\dot{s}(t) < 0\,,\ \ \dot{\overline{\lambda}} = \overline{\lambda}'(s)\dot{s} < 0$ for $\overline{\lambda}(s) > \lambda$ and $s(t_1) = 0$, for $t_1 < \infty$.

Hence $\overline{v}(0, t_1) = 1$, and $w(r; s(t)) \to 1 - r^2$ as $t \to t_1-$. Again for $t \ge t_1$ we can take

$$\overline{v}(r,t) = a(t)(1-r^2)\,,\quad a_1 = a(t_1) = 1\,,\ \text{for}\ \ t \ge t_1\,, \tag{3.8a}$$
$$\overline{v}_r(0,t) = \overline{v}(1,t) = 0\,,\quad t \ge t_1\,, \tag{3.8b}$$

giving

$$\mathcal{E}(\overline{v}) = \dot{a}(1-r^2) + 4a - \frac{\lambda}{\pi^2} \ge \dot{a} + 4(a - \frac{\lambda}{4\pi^2}) = 0\,,$$

on taking $\dot{a} = -4(a - \frac{\lambda}{4\pi^2}) < 0$, $\frac{\lambda}{4\pi^2} < a < 1$, since $\dot{a}(t) < 0$. Then

$$a(t) = \frac{\lambda}{4\pi^2} + (1 - \frac{\lambda}{4\pi^2}) e^{4(t_1 - t)} \to \frac{\lambda}{4\pi^2} \quad \text{as}\ \ t \to \infty\,.$$

Hence $\overline{v}$ is an upper solution to u-problem which exists for all time, in particular $u \le \overline{v}$ and $\overline{v} \to w = \frac{\lambda}{4\pi^2}(1-r^2)$ as $t \to \infty$. On the other hand for $\underline{v}(r,t)$ we take

$$\begin{gathered}\underline{v}(r,t) = b(t)(1-r^2)\,, \quad 0<r<1\,, \quad t>0\,,\\ b(0)=0\,, \quad \text{where } 0 \le b \le \min\{1, \lambda/4\pi^2\}\,.\end{gathered} \tag{3.9}$$

Then we have,

$$\mathcal{E}(\underline{v}) = \dot{b}(t)(1-r^2) + 4b - \frac{\lambda}{4\pi^2} \le \dot{b}(t) + 4b - \frac{\lambda}{4\pi^2} = 0\,,$$

on taking $\dot{b}(t) + 4b - \frac{\lambda}{4\pi^2} = 0$, since $\dot{b}(t) > 0$, giving $b(t) = \frac{\lambda}{4\pi^2}(1-e^{-4t}) \to \frac{\lambda}{4\pi^2}$ as $t \to \infty$. Thus for $0 < \lambda < 4\pi^2$ we have that $\underline{v}(r,t) \le u \le \overline{v}(r,t)$, $\overline{v}(r,t)$, $\underline{v}(r,t) \to w(r) = \frac{\lambda}{4\pi^2}(1-r^2)$ as $t \to \infty$ uniformly in r, which implies that u is bounded for all time and $u(r,t) \to w(r)$ as $t \to \infty$ (w is the unique steady state for $\lambda < 4\pi^2$). Hence w, is globally asymptotically stable solution [18, 19, 21]. Moreover, if $4\pi^2 \le \lambda < 8\pi^2$, then we can proceed in a similar way. In fact, we construct an upper solution decreasing in time as (3.7), then $\mathcal{E}(\overline{v}) \ge 0$, provided that

$$0 < -\dot{s} = h(s) \equiv \frac{(\overline{\lambda}(s) - \lambda)(1 + 2s^2 \ln s - s^2)}{4\pi^2(1-s^2)^2(1-s)}\,,$$

giving now $\dot{s}(t) < 0$ for $\overline{\lambda}(s) > \lambda$ and $s \to S_0+$, $\overline{\lambda}(s) \to \lambda = \lambda(S_0)$, as $t \to \infty$. Also we construct a lower solution $\underline{v}$ increasing in time, having a similar form to that as in the proof of the blow-up (see below), in particular like (3.9) followed by a complementary version of (3.7). But now $\dot{s}(t) > 0$, $\dot{\underline{\lambda}} = \underline{\lambda}'(s)\dot{s} > 0$ for $t > t_1$, $s(t_1) = 0$ and $s(t) \to S_0-$, $\underline{\lambda}(s) \to \lambda = \lambda(S_0)$, as $t \to \infty$. Hence $\underline{v} \le u \le \overline{v}, u$ exists for all time and $u \to w(r; S_0)$ the unique solution for $4\pi^2 \le \lambda = \lambda(S_0) < 8\pi^2$, which is globally asymptotically stable.

We show now that the solution u, "blows up" (it ceases to be less than 1 in $[0,1)$, we recall that $u \le 1$ in $(0,1)$ as long as $u_0(r) \le 1$) in the sense that it becomes 1 in $[0,1)$ in finite time, for $\lambda > 8\pi^2$. Therefore we get a lower solution $\underline{v}(r,t)$ of the form: $\underline{v}(r,t) = b(t)(1-r^2)$, $0 < r < 1$, $0 \le t \le t_1$, which satisfies (3.9) (note that $u_0(r) \le 1$), $b(t_1) = 1$, $u(r,t_1) \ge \underline{v}(r,t_1) = 1 - r^2$, provided that u still exists ($u \le 1$) up to t_1. Also we take $\underline{v}(r,t)$ to satisfy:

$$\begin{gathered}\underline{v}(r,t) = 1\,, \quad \underline{v}_r(r,t) = 0 \quad \text{for } 0 \le r \le s(t)\,, \quad t > t_1\,, \quad \text{and}\\ \underline{v}(r,t) = \frac{1 + 2s^2 \ln r - r^2}{1 + 2s^2 \ln s - s^2}\,, \quad s(t) < r < 1\,, \ \ t > t_1\,,\end{gathered}$$

$b(t_1) = 1$, $s(t_1) = 0$, $\underline{v}(r,t_1) = 1 - r^2$. Then we have,

$$\mathcal{E}(\underline{v}) = \begin{cases} 0\,, & 0 \le r \le s(t)\,,\ t \ge t_1\,,\\ \underline{v}_t + \dfrac{\underline{\lambda} - \lambda}{\pi^2(1-s^2)^2} \le 0\,, & s(t) \le r \le 1\,, \ \ t > t_1\,,\end{cases}$$

provided that $s(t)$ satisfies

$$0 < \dot{s} = -h(s) \equiv \frac{(\lambda - \underline{\lambda}(s))(1 + 2s^2 \ln s - s^2)}{4\pi^2(1-s^2)^2(1-s)}\,,$$

for $\underline{\lambda}(s) < 8\pi^2 < \lambda$. This implies that $\dot{\underline{\lambda}} = \underline{\lambda}'(s)\dot{s} > 0$, and $s(t) \to 1-$, $\underline{\lambda}(s) \to 8\pi^2-$, as $t \to T^* < \infty$. Hence $\underline{v}$ becomes 1 in $[0,1)$ at T^* and a form of " blow-up" ($u \to 1$ in $[0,1)$ as $t \to t^*-$, $t^* \leq T^*$) has been established for u, with derivative $u_r(1,t)$ becoming unbounded as $t \to t^*-$.

For the critical value $\lambda = 8\pi^2$, again we construct an upper solution $\overline{v}$ but now increasing in time; indeed $\mathcal{E}(\overline{v}) \geq 0$, provided that $s(t)$ satisfies

$$0 < \dot{s} = -h(s) \equiv \frac{(8\pi^2 - \overline{\lambda}(s))(1 + 2s^2 \ln s - s^2)}{4\pi^2(1-s^2)^2(1-s)}\,.$$

For $4\pi^2 < \overline{\lambda}(s) < 8\pi^2$, we get $s \to 1$ as $t \to \infty$, which implies that u exists for all time but becomes 1 in $[0,1)$ as $t \to \infty$.

4 The exponential function

4.1 Stationary solutions

We now consider $f(s) = e^{-s}$, so $f(s) > 0$, $f'(s) < 0$ for $s \geq 0$ and $\int_0^\infty f(s)\,ds = 1$. The corresponding steady problem to (1.7) for $f(s) = e^{-s}$ is

$$w''(r) + \frac{1}{r}w'(r) + \mu e^{-w(r)} = 0\,, \quad 0 < r < 1\,, \quad w(1) = w'(0) = 0, \tag{4.1}$$

where

$$\mu = \frac{\lambda}{4\pi^2(\int_0^1 e^{-w(r)} r\,dr)^2}\,. \tag{4.2}$$

The solution of (4.1) is

$$w(r) = 2\ln[\alpha(1-r^2) + r^2]\,, \quad \mu = 8\alpha(\alpha - 1)\,, \tag{4.3}$$

where $\alpha > 1$, $M = \sup\|w\| = w(0) = 2\ln\alpha$. The parameter λ is given by

$$\lambda = 4\pi^2\mu\Big(\int_0^1 e^{-w} r\,dr\Big)^2 = 8\big(1 - \frac{1}{\alpha}\big)\pi^2 = 8\pi^2\left(1 - e^{-M/2}\right) < 8\pi^2\,, \tag{4.4}$$

so $\alpha = \frac{\lambda^*}{\lambda^* - \lambda}$, $\lambda = \lambda(M) \to 8\pi^2 = \lambda^*$ as $M \to \infty$, or equivalently as $\alpha \to \infty$, and $\lambda'(M) > 0$. For each M there is a corresponding unique solution $w(r)$(this follows from a shooting argument). Finally from the above we get the diagram of Figure 3.

If $\lambda \to \lambda^*-$, which implies that $\alpha \to \infty$, then the solution $w(r) = 2\ln[\alpha(1 - r^2) + r^2)] \to \infty$, for every compact subset of $[0,1)$. We see below that this also holds for a general decreasing f.

4.2 Stability for $\lambda < \lambda^*$

To study the stability, we use upper solutions which are decreasing in time and lower solutions which are increasing in time to problem (1.7) with $f(s) = \exp(-s)$.

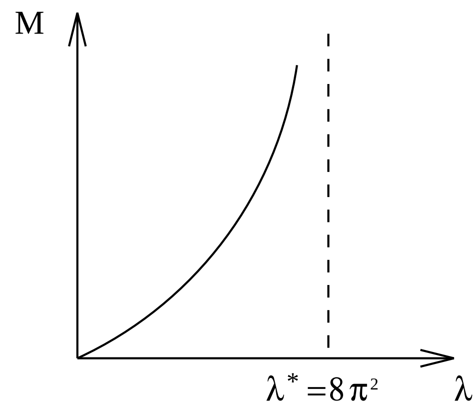

Figure 3: Response diagram for (4.1), $f(s) = \exp(-s)$.

We first note that $v(r,t) = w(r;\mu(t)) = 2\ln[\alpha(t)(1-r^2)+r^2]$ is an upper solution, provided that

$$\dot\alpha + 2\frac{\lambda^*-\lambda}{\lambda^*}\alpha - 2 = 0\,, \quad \alpha(0)=\alpha_0 \tag{4.5}$$

and α_0 is sufficiently large. Hence for $\lambda \in (0,\lambda^*)$ the solution of (4.5) is

$$\alpha(t) = \frac{\lambda^*}{\lambda^*-\lambda} + \big[\alpha_0 - \frac{\lambda^*}{\lambda^*-\lambda}\big]\exp\big(-2\frac{\lambda^*-\lambda}{\lambda^*}\,t\big), \tag{4.6}$$

and $\dot\alpha(t) < 0$ for $\alpha_0 > \frac{\lambda^*}{\lambda^*-\lambda}$. Furthermore we require $v(r,0) = 2\ln[\alpha_0(1-r^2)+r^2] \ge u_0(r)$. It is sufficient to choose

$$\alpha_0 = \max\{\frac{\lambda^*}{\lambda^*-\lambda}\,,\,\sup_r \frac{\exp(u_0(r)/2)-r^2}{1-r^2}\}\,.$$

Also $\mathcal{B}(v) \ge \mathcal{B}(u)$ on $\partial\Omega$, in fact it is $v(1,t) = u(1,t) = 0$. The calculations are like these of the one-dimensional case [18, 19]; we find an upper solution v decreasing in time, $v \ge u$ and $v(r,t) \to 2\ln[A(1-r^2)+r^2] = w(r;\lambda)$ as $t\to\infty\,,\ \alpha(t)\to A = \frac{\lambda^*}{\lambda^*-\lambda}$ as $t\to\infty$ (see (4.6)).

In a similar way we construct a lower solution $z(r,t)$ increasing in time. Again $z(r,t) = 2\ln[\alpha(t)(1-r^2)+r^2]$ is a lower solution provided that $\alpha(t)$ satisfies (4.5) and $\alpha_0 - \frac{\lambda^*}{\lambda^*-\lambda} < 0$, $\alpha(t)$ is of the form of (4.6). Also we require $z(r,0) \le u_0(r)$. It is sufficient to choose $\alpha_0 = \min\{\frac{\lambda^*}{\lambda^*-\lambda}\,,\inf_r \frac{\exp(u_0(r)/2)-r^2}{1-r^2}\}$. But on $\partial\Omega$ $z(r,t) = u(r,t) = 0$, which finally implies that $z \le u$. Hence for $0<\lambda<\lambda^* = 8\pi^2$ we find an upper solution v and a lower solution z such that $z \le u \le v$ with $v(r,t)\to w+\,,\quad z(r,t)\to w-$, as $t\to\infty$. Thus the solution u is global and $u(r,t) \to w(r;\lambda) = 2\ln\Big[\frac{\lambda^*}{\lambda^*-\lambda}(1-r^2)+r^2\Big]$ as $t\to\infty$, where $w(r;\lambda)$

is the unique steady state. The above procedure holds for any (admissible) initial data $u_0(r)$, from which it follows that the solution w is globally asymptotically stable.

4.3 Blow-up for $\lambda > \lambda^*$

To prove that the solution $u(r,t)$ blows up for $\lambda > \lambda^* = 8\pi^2$, we construct a lower solution which blows up. Again we take as a lower solution a function with a similar form to the steady state $w(r) : z(r,t) = w(r;\mu(t)) = 2\ln[\alpha(t)(1-r^2)+r^2]$. We first note that if $\alpha(t)$ satisfies (4.5) and $\alpha_0 < \frac{\lambda^*}{\lambda^*-\lambda}$, then $\dot\alpha(t) > 0$, moreover $z(r,t)$ is an unbounded lower solution to (1.7) and $z(r,t) \to \infty$ as $t \to \infty$ for any $r \in [0,1)$. This implies that $u(r,t)$ is unbounded, more precisely $\limsup_{t\to t^*} \|u(\cdot,t)\| \to \infty\,, \quad t^* \le \infty$. To prove that $t^* < \infty$ we take a modified comparison function, $Z(r,t) = p\ln[\alpha(t)(1-r^2)+r^2]$. We show that $Z(r,t)$ is a lower solution to (1.7) and blows up for a certain value of p. Thus we have

$$\begin{aligned}
\mathcal{E}(Z) \;&:=\; Z_t - \Delta_r Z - \lambda e^{-Z}/4\pi^2\Big(\int_0^1 e^{-Z} r\,dr\Big)^2 \\
&\le\; \frac{p}{(\alpha-\beta r^2)}\Big\{\dot\alpha(1-r^2)(\alpha-\beta r^2) \\
&\quad -4(\alpha-\beta r^2)^{2-p}\Big[\frac{\lambda}{\lambda^*}\frac{2(p-1)^2}{p\,k^2}-1\Big]\alpha^2\Big\}
\end{aligned} \tag{4.7}$$

where $\beta(t) = \alpha(t)-1$, $\dot\alpha(t) > 0$, $0 < p < 2$, $k > 1$ and $\alpha - 1 \ge \alpha/k$. The last condition is satisfied for $t \ge t_1$, for some t_1 since the use of the lower solution z above guarantees unboundedness of u and allows Z to be large for $t \ge t_1$. For $\lambda > \lambda^*$ and $1 < p < 2$, we have $\lambda/\lambda^* > 1$, $2(p-1)^2/p < 1$, while $\frac{2(p-1)^2}{p} \to 1$ as $p \to 2-$, so we can choose $p \in (1,2)$:

$$\frac{\lambda}{\lambda^*}\frac{2(p-1)^2}{p} > 1\,. \tag{4.8}$$

Now for a fixed $\lambda > \lambda^*$ we can choose suitable p and k so that both (4.8) and the following hold:

$$\frac{\lambda}{\lambda^*}\frac{2(p-1)^2}{p} > k^2 > 1 \quad\text{or}\quad \Lambda = \frac{\lambda}{\lambda^*}\frac{2(p-1)^2}{pk^2} > 1\,. \tag{4.9}$$

The inequalities (4.7), (4.9) imply

$$\mathcal{E}(Z) \le \frac{p(1-r^2)}{(\alpha-\beta r^2)}\left[\dot\alpha - 4(\alpha-\beta r^2)^{1-p}(\Lambda-1)a^2\right] \le 2\left[\dot a - 4(\Lambda-1)\alpha^{3-p}\right] = 0\,,$$

by taking $\alpha(t)$ to satisfy,

$$\dot\alpha - 4(\Lambda-1)\alpha^{3-p} = 0\,, \quad \alpha(0) = \alpha_0 > 0\,. \tag{4.10}$$

We also require $Z(r,0) \le u_0(r)$, for which it is sufficient to take

$$\alpha_0 \le \inf_r \frac{\exp(u_0(r)/p) - r^2}{1-r^2}\,, \quad 1 < p < 2\,,$$

and $Z(1,t) = u(1,t) = 0$ holds on $\partial\Omega$. Hence $Z(r,t)$, is a lower solution to (1.7) i.e. $Z(r,t) \le u(r,t)$ and $Z(r,t)$ is increasing in time since $\dot\alpha(t) > 0$. Furthermore from (4.10) we obtain,

$$4(\Lambda-1)(t-t_1) = \int_{\alpha(t_1)}^{\alpha(t)} s^{p-3}\,ds < \int_{\alpha_1}^{\infty} s^{p-3}\,ds = \frac{\alpha_1^{p-2}}{2-p} < \infty\,, \tag{4.11}$$

where $\alpha_1 = \alpha(t_1) = k/(k-1) < \alpha(t)$, since we have used that $\alpha - 1 > \alpha/k$. The relation (4.11) implies that $\alpha(t)$ blows up at

$$T^* = t_1 + \frac{\alpha_1^{p-2}}{4(\Lambda-1)(2-p)} < \infty$$

and, since $Z(r,t)$ is a lower solution for u, this means that u blows up at $t^* \le T^* < \infty$. This completes the proof of the blow-up of u. In the next section, for general decreasing functions, we shall show that this blow-up is global, this means that $u(r,t) \to \infty$ as $t \to t^*-$ for every $r \in [0,1)$.

5 General decreasing functions

5.1 Stationary Solutions

We consider an arbitrary decreasing function f satisfying (1.2). Again we may use comparison techniques due to the monotonicity of f as in Section 2. We follow the same procedure as in the previous section; see also [5, 6]. For the moment we suppose that $\int_0^\infty f(s)\,ds < \infty$, unless otherwise stated. The corresponding steady problem of (1.7) is

$$w''(r) + \frac{1}{r}w'(r) + \mu f(w(r)) = 0\,, \quad 0 < r < 1\,, \tag{5.1a}$$

$$w(1) = w'(0) = 0\,, \tag{5.1b}$$

where $\lambda = 4\pi^2\mu(\int_0^1 f(w)r\,dr)^2$. Multiplying (5.1a) by r and integrating,

$$\lambda = \frac{4\pi^2}{\mu}(w'(1))^2\,. \tag{5.2}$$

Again multiplying (5.1a) by w' and integrating as before we get

$$\frac{(w'(1))^2}{2} + \int_0^1 \frac{(w'(r))^2}{r}\,dr - \mu\int_0^M f(s)\,ds = 0\,, \tag{5.3}$$

which implies $\frac{(w'(r))^2}{\mu} < 2\int_0^M f(s)\,ds$. By rescaling the problem we may assume that

$$\int_0^\infty f(s)\,ds = 1, \tag{5.4}$$

then from (5.2) and (5.3) we get

$$\lambda < 8\pi^2 \int_0^M f(s)\,ds < 8\pi^2 \quad \text{and} \quad \frac{(w'(r))^2}{\mu} < 2\,.$$

Lemma 5.1 *For the Dirichlet problem (5.1), if (5.4) holds, then* $\frac{(w'(1))^2}{\mu} \to 2$ *as* $\mu \to \infty$.

Proof: We consider the auxiliary problem:

$$z''(r) + \mu g(z(r)) = 0\,, \quad 1-\delta < r < 1, \tag{5.5a}$$

$$z(r) = \sup_r z(r) = M\,, \quad z'(r) = 0\,, \quad 0 \le r \le 1-\delta\,, \quad z(1) = 0\,, \tag{5.5b}$$

where $0 < g(s) < f(s)$, and z, z_r, are continuous at $1-\delta$. Multiplying (5.5a) by z' and integrating we obtain

$$(z'(r))^2 = 2\mu \int_{z(r)}^M g(s)\,ds = 2\mu[G(z) - G(M)]\,, \tag{5.6}$$

where $G(z) = \int_z^\infty g(s)\,ds$. Then

$$\int_0^M [G(z) - G(M)]^{-1/2}\,dz = \delta\sqrt{2\mu}\,, \tag{5.7}$$

since $z'(r) < 0$, and $(z'(1))^2 = 2\mu[G(0) - G(M)] = 2\mu\int_0^M g(s)\,ds$. We prove now that the solution to problem (5.5) is a lower solution to problem (5.1). Indeed $z''(r) + \frac{1}{r}z' + \mu f(z) = \mu f(z) > 0$ in $0 \le r \le 1-\delta$. Also taking into account (5.5)-(5.7),

$$\begin{aligned} & z''(r) + \frac{1}{r}z' + \mu f(z) \\ = \quad & \frac{z'}{r} + \mu(f(z) - g(z)) > \frac{z'(r)}{1-\delta} + \mu(f(z) - g(z)) \\ = \quad & -\frac{\sqrt{2\mu}\,[G(z) - G(M)]^{1/2}}{1-\delta} + \mu(f(z) - g(z))\,, \ \text{ in } \ 1-\delta < r < 1\,. \end{aligned} \tag{5.8}$$

Now choosing μ large enough such that

$$\mu \ge \mu_0 = \sup_{z\in(0,M)} \frac{2[G(z) - G(M)]}{(1-\delta)^2\,[f(z) - g(z)]^2}\,, \tag{5.9}$$

and $\delta < 1$, relations (5.8), (5.9) give

$$z''(r) + \frac{1}{r}z' + \mu f(z) > \mu(f(z) - g(z)) - \mu(f(z) - g(z)) = 0\,.$$

In addition $z'(0) = z(1) = w'(0) = w(1) = 0$, hence z is a lower solution to w-problem. This implies

$$z(r) \le w(r) \ \text{ and } \ w'(1) \le z'(1) < 0\,, \tag{5.10}$$

(if the latter inequality were $w'(1) > z'(1)$ it would give $z(r) > w(r)$ for some r, which would be a contradiction).
Now taking:

(a) g, such that $0 < g(s) < f(s)$ and $1 - \epsilon < G(0) = \int_0^\infty g(s)\,ds \le 1$

(b) M such that $[G(0) - G(M)] > 1 - 2\epsilon$, $\epsilon > 0$, from the definition of G,

(c) μ to satisfy (5.9).

Note that $G'(z) = -g(z) < 0$, $G(z)$ is decreasing and $G(0) \le 1$); from (5.3), (5.6) and (5.10) we obtain

$$2 > \frac{(w'(1))^2}{\mu} \ge \frac{(z'(1))^2}{\mu} = 2[G(0) - G(M)] > 2(1 - 2\epsilon)\,.$$

This relation holds for every $\epsilon > 0$, as far as $\mu \gg 1$, hence this proves the lemma. □

Proposition 5.2 *If (5.4) holds then* $\lambda < \lambda^* = 8\pi^2$ *and* $\lambda \to 8\pi^2-$ *as* $M \to \infty$ ($\lambda \to \lambda^* \int_0^\infty f(s)\,ds = \lambda^*$ *as* $M \to \infty$).

Proof: The first relation is obtained by (5.3), (5.4). For the second, using (5.2) and Lemma 5.1 we obtain,

$$\lambda = 4\pi^2 \frac{(w'(1))^2}{\mu} \to 8\pi^2 \quad \text{as } \mu \to \infty \ \text{(or equivalently as } M \to \infty).$$

□

Also from Lemma 5.1 and relation (5.3) we deduce that

$$\lim_{\mu\to\infty} \frac{2}{\mu} \int_0^1 \frac{(w'(r))^2}{r}\,dr = 0 \ \text{ and}$$
$$\Big[\int_0^1 \frac{(w'(r))^2}{r}\,dr\Big]/(w'(1))^2 = \Big[\frac{4\pi^2}{\lambda} \int_0^M f(s)\,ds - 1/2\Big] \to 0 \ \text{ as } \ \mu \to \infty\,.$$

Now we assume that

$$\int_0^\infty f(s)\,ds = \infty \tag{5.11}$$

holds instead of (5.4), so we have the following statement.

Proposition 5.3 *Let (5.11) hold and* w *is the solution to problem (5.1) then* $(w'(1))^2/\mu \to \infty$ *as* $\mu \to \infty$ *and* $\lambda(M) \to \infty$ *as* $M \to \infty$.

Proof: Again the solution $z(r)$ to problem (5.5) is a lower solution to w-problem, provided that (5.9) holds. Thus we have $z(r) \leq w(r)$ which implies that $w'(1) \leq z'(1) < 0$, or $(w'(1))^2 \geq (z'(1))^2$, but now from (5.6) at $r = 1$ we get,

$$\frac{(w'(1))^2}{\mu} \geq \frac{(z'(1))^2}{\mu} = 2\int_0^M g(s)\,ds \to \infty \quad \text{as} \quad M \to \infty\,,$$

provided that we take g such that $0 < g(s) = \gamma f(s) < f(s)$, for $0 < \gamma < 1$, then $\int_0^\infty g(s)\,ds = \infty$. □

We now obtain a uniqueness result for the steady problem.

Proposition 5.4 *Let f, satisfy*

$$-sf'(s) < f(s)\,, \quad s > 0\,, \tag{5.12}$$

then problem (5.1) has a unique solution.

Proof: From (5.2) we get $\lambda(\mu) = 4\pi^2 \frac{(w'(1))^2}{\mu} = 4\pi^2 (W'(1))^2$ by writing $w(r) = \nu W(r)$, $\nu = \sqrt{\mu}$. Then (5.1) gives $W_\nu'(0) = W_\nu(1) = W'(0) = W(1) = 0$ and

$$W'' + \frac{W'}{r} + \nu f(w) = 0 \quad \text{or} \quad W_\nu'' + \frac{W_\nu'}{r} + \nu^2 f'(w) W_\nu = -f(w) - f'(w)w\,.$$

If (5.12) holds, then using maximum principle and Hopf's boundary lemma, we get that $W_\nu(r) > 0$ and $W_\nu'(1) < 0$ or $\frac{d}{d\nu}(W'(1))^2 > 0$, since also $W'(1) < 0$. Then $\lambda'(\mu) = \frac{2\pi^2}{\nu}\frac{d}{d\nu}(W'(1))^2 > 0$ which implies uniqueness. □

Remark: Proposition 5.4 is also true for a general domain Ω. The relation (5.12) implies (5.11).

Finally we obtain the response diagram of Figure 4.

5.2 Stability where a unique steady state exists

We follow the same procedure as in the previous section, we seek for a decreasing (increasing)-in-time upper (lower) solution to problem (1.7). We first look for an upper solution of a form similar to the steady state: $v(r,t) = w(r;\overline{\mu}(t)) = \overline{w}$, where $\overline{w}$ is a steady state. Then

$$\mathcal{E}(v) = \overline{w}_{\overline{\mu}}\,\dot{\overline{\mu}} + [4\pi^2\overline{\mu}\, I^2(\overline{w}) - \lambda] f(\overline{w})/4\pi^2 I^2(\overline{w})\,, \tag{5.13}$$

where $I(w) = \int_0^1 f(w) r\,dr$ and $-\Delta_r w = \mu f(w)$ ($\Delta_r = \partial^2/\partial r^2 + \frac{1}{r}\partial/\partial r$). Also $\overline{w}_{\overline{\mu}} = \frac{\partial \overline{w}}{\partial \overline{\mu}} > 0$, $w(r;\overline{\mu}(t))$ is increasing with respect to $\overline{\mu}$, by using the maximum principle. Taking now any $\lambda > 0$, so that $w(x;\lambda)$ is the unique stationary solution, we can choose $\overline{\mu}(0)$ such that $w(r;\overline{\mu}(0)) = v(r,0) \geq u_0(r)$; this can be done since $u_0(r)$, $u_0'(r)$ are bounded. Furthermore, since a unique steady state exists (see Proposition 5.4) for these values of λ, there exists a μ such

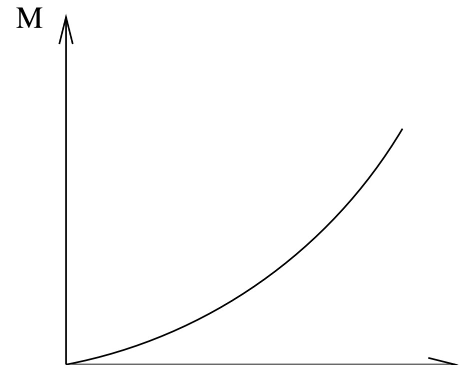

Figure 4: Response diagram for the Dirichlet problem, $\int_0^\infty f(s)\,ds = \infty$.

that $\lambda = 4\pi^2\mu I^2(w)$, $M = w(0;\mu) = w(0)$, where $w(r) = w(r;\mu)$ is the unique steady state of problem (5.1), and as long as $\overline{\mu} > \mu$ then $\overline{w} > w$ and $\overline{\lambda} = \overline{\lambda}(t) = 4\pi^2\overline{\mu}(t)I^2(\overline{w}) > \lambda$. As before $w(r;\overline{\mu})$ is an upper solution, decreasing in time, which tends to the stationary solution w provided that

$$0 < -\dot{\overline{\mu}} = h(\overline{\mu}) \equiv \left(\overline{\lambda}(t) - \lambda\right)\ I^{-2}(\overline{w})\ \inf_r\{\frac{f(\overline{w})}{\overline{w}_{\overline{\mu}}}\}, \tag{5.14}$$

and $\overline{\mu}(0) > \mu$ (note that $f(s)$ is bounded away from zero and w_μ is also finite). Hence $u = u(r,t) \le v(r,t) = w(r;\overline{\mu}(t))$ and $\dot{\overline{\mu}} < 0$ which implies that $v_t = \overline{w}_{\overline{\mu}}\dot{\overline{\mu}} < 0$. In a similar way we can construct a lower solution $z(r,t) = w(r;\underline{\mu}(t))$ which is increasing in time and tends to the steady state w. Finally we obtain, $z(r,t) = w(r;\underline{\mu}(t)) \le u(r,t) \le v(r,t) = w(r;\overline{\mu}(t))$, and $\overline{\mu}(t) \to \mu+$, $\underline{\mu}(t) \to \mu-$, $v(r,t) \to w+$, $z(r,t) \to w-$, as $t \to \infty$. This implies that $u(\cdot,t) \to w(\cdot)$ uniformly as $t \to \infty$, and that w is globally asymptotically stable.

For the case of $\lambda \ge \lambda^*$ and $\int_0^\infty f(s)\,ds = 1$, there is no steady solution to (5.1), then $\underline{\lambda} = \underline{\lambda}(t) = 4\pi^2\underline{\mu}(t)I^2(\underline{w}) < \lambda$ for every $\underline{\mu} > 0$. Taking the above lower solution $z(r,t)$ we obtain that $\underline{\mu}(t) \to \infty$ ($\dot{\underline{\mu}} > 0$) and $u(r,t) \ge z(r,t) = w(r;\underline{\mu}(t)) \to \infty$ as $t \to \infty$ (note that $w_\mu > 0$ and $\mu(t) \to \infty$ as $t \to \infty$ then $z \to \infty$ as $t \to \infty$). In particular $\sup_r u(r,t) \to \infty$ as $t \to t^* \le \infty$, hence u is unbounded.

5.3 Blow-up for $\lambda > \lambda^*$

We can now prove that the solution u to problem (1.7) blows up in finite time if $\lambda > \lambda^* = 8\pi^2$. To prove this we use similar methods to those in the previous sections, (or see [15, 18, 19]). We look for a lower solution $z(r,t)$ to the u-problem which itself blows up. We try to find a lower solution with a similar form to the steady state. We take into account the form of blow-up in the one-dimensional case, therefore we consider $z(r,t)$ to satisfy:

$$z(r,t) = M(t) = \sup_r z(r,t)\,, \quad z_r(r,t) = 0\,, \quad 0 \le r \le 1-\delta(t)\,, \;\; t>0\,, \tag{5.15a}$$

$$z_{rr} + \mu(t) g(z) = 0\,, \quad 1-\delta(t) \le r \le 1\,, \;\; z(1,t) = 0\,, \;\; t>0\,, \tag{5.15b}$$

where $0 < g(s) = \gamma f(s) < f(s)$, $0<\gamma<1$, and z, z_r are continuous at $1-\delta(t)$. Multiplying (5.15b) by z_r, and integrating in $(1-\delta, r)$ we obtain

$$\frac{z_r^2}{2} + \mu(t) \int_M^z g(s)\, ds = 0\,, \tag{5.16}$$

which gives

$$z_r^2 = 2\mu(t)\,[G(z) - G(M)] \quad \text{and} \quad \frac{z_r^2}{\mu} < 2\,, \tag{5.17}$$

on writing $G(z) = \int_z^\infty g(s)\, ds$ (then $G'(z) = -g(z)$ and $G(0) = \gamma$)). The relation (5.17) implies

$$\delta\sqrt{2\mu} = \int_0^M [G(s) - G(M)]^{-1/2}\; ds\,. \tag{5.18}$$

Again integrating (5.17) we obtain

$$(1-r)\sqrt{2\mu} = \int_0^z [G(s) - G(M)]^{-1/2}\; ds\,. \tag{5.19}$$

On the other hand, we can get

$$\begin{aligned}
\int_0^1 g(z) r\, dr &= \int_0^{1-\delta} g(z) r\, dr + \int_{1-\delta}^1 g(z) r\, dr \le g(M)\frac{(1-\delta)^2}{2} - \frac{1}{\mu} z_r(1,t) \\
&= g(M)\frac{(1-\delta)^2}{2} + \sqrt{\frac{2}{\mu}}\Big(\int_0^M g(s)\, ds\Big)^{1/2} \\
&\sim g(M)/2 + \sqrt{2\gamma/\mu} \quad \text{for} \quad \delta \ll 1 \ll M \;\; \text{and} \;\; 1 \ll \mu\,,
\end{aligned}$$

by using (5.17) at $r=1$, $\int_0^M g(s)\, ds \sim \gamma$, $1-\delta \sim 1$, for $\delta \ll 1 \ll M$. Finally we get

$$\int_0^1 g(z) r\, dr \;\lesssim\; \frac{g(M)}{2}(\alpha+1)\,, \qquad \delta \ll 1 \ll M\,, \tag{5.20a}$$

on taking

$$\sqrt{2\gamma/\mu} = \alpha g(M)/2, \tag{5.20b}$$

where α is a suitable chosen constant; in particular choose $\alpha > 1/[(\lambda/8\pi^2)^{1/2} - 1]$ for $\lambda > \lambda^* = 8\pi^2$. Such an α gives $\Lambda = \frac{1}{3}[\lambda/\pi^2(1+\alpha)^2 - \frac{8}{\alpha^2}] > 0$.

Lemma 5.5 *$M\,f(M) \to 0$ as $M \to \infty$.*

For the proof of this lemma, see [19].

Lemma 5.6 *$\delta \to 0$ as $\mu \to \infty$.*

Proof: From the previous lemma we have that $f(M) \to 0$ and $g(M) \to 0$ as $M \to \infty$ and for fixed α, (5.20) implies that $\mu \to \infty$ as $M \to \infty$. For $0 < s < M$, we have, $(M-s)g(M) \le G(s) - G(M) \le (M-s)g(s)$ and (5.18), (5.20) give:

$$\begin{aligned} \delta &\le \frac{\alpha g(M)}{4\sqrt{\gamma}} \int_0^M [(M-s)g(M)]^{-1/2}\, ds = \frac{\alpha g^{1/2}(M)}{4\sqrt{\gamma}} \int_0^M (M-s)^{-1/2}\, ds \\ &= \frac{\alpha}{2\sqrt{\gamma}}[g(M)M]^{1/2}, \quad for\ \ \delta \ll 1 \ll M\ . \end{aligned}$$

The above relation implies, $0 < \delta \to 0$ since $g(M)M < f(M)M \to 0$, as $M \to \infty$. $\square$

Proposition 5.7 *The solution z, to (5.15) is a lower solution to the u-problem; moreover the solution u blows up in finite time.*

Proof: For $0 < r < 1 - \delta(t)$,

$$\begin{aligned} \mathcal{E}(z) &= \dot{M} - \lambda f(M)/4\pi^2 \Big(\int_0^1 f(z) r\, dr \Big)^2 = \dot{M} - \frac{\lambda g(M)/\gamma}{4\pi^2 (\int_0^1 g(z) r/\gamma\, dr)^2} \\ &\lesssim \dot{M} - \lambda\gamma/\pi^2(\alpha+1)^2 g(M) \le \dot{M} - \gamma\Lambda/g(M) = 0\,, \qquad (5.21) \end{aligned}$$

on choosing $\dot{M} - \gamma\Lambda/g(M) = 0$, where Λ is taken as $3\Lambda = (\lambda/\pi^2(\alpha+1)^2) - 8/\alpha^2 < (\lambda/\pi^2(\alpha+1)^2)$. Hence $\mathcal{E}(z) \lesssim 0$ for $M \gg 1$.

For the interval $1 - \delta < r < 1$, we first differentiate (5.19) with respect to t and get

$$\begin{aligned} z_t &= (1-r)\frac{\dot{\mu}}{\sqrt{2\mu}}[G(z) - G(M)]^{1/2} \\ &\quad + \frac{1}{2} g(M)\dot{M}[G(z) - G(M)]^{1/2} \int_0^z [G(s) - G(M)]^{-3/2}\, ds \\ &= A + B. \end{aligned}$$

For A we have:

$$\begin{aligned} A &= (1-r)\frac{\dot{\mu}}{\sqrt{2\mu}}[G(z) - G(M)]^{1/2} \\ &\le -\frac{g'(M)\dot{M}}{g(M)}[g(z)(M-z)]^{1/2} \int_0^z [G(s) - G(M)]^{1/2}\, ds \\ &\le \frac{-g'(M)\dot{M} g^{1/2}(z) M}{2g^{3/2}(M)} \le \frac{\gamma\Lambda g(z)}{g^2(M)} \quad \text{for } \ M \gg 1\,, \end{aligned}$$

provided that

$$\dot{M} \le -\frac{\gamma\Lambda g^{1/2}(z)}{M g'(M) g^{1/2}(M)}$$

which certainly holds if $0 \le \dot{M} \le -\gamma\Lambda/Mg'(M)$ since $g'(s) \le 0$ so $g(z)/g(M) \ge 1$ for $z \le M$.

For B we have:

$$\begin{aligned} B &= \frac{1}{2} g(M)\dot{M}[G(z)-G(M)]^{1/2}\int_0^z [G(s)-G(M)]^{-3/2}\,ds \le \frac{\dot{M} g^{1/2}(z)}{g^{1/2}(M)} \\ &\le \frac{\gamma\Lambda g(z)}{g^2(M)} \quad \text{for } M\gg 1\,, \end{aligned}$$

provided that $\dot{M}\le \frac{\gamma\Lambda g^{1/2}(z)}{g^{3/2}(M)}$, which holds if $0\le \dot{M}\le \frac{\gamma\Lambda}{g(M)}$, since $g(z)/g(M)\ge 1$ for $z\le M$.
Also, on using (5.17), we have the estimate,

$$-\frac{z_r}{r}\le \frac{4\sqrt{\gamma}}{\alpha}(g(M)M)^{1/2}\frac{g(z)}{g^2(M)} \lesssim \gamma\Lambda\frac{g(z)}{g^2(M)} \text{ since } g(M)M\to 0 \text{ for } M\gg 1\,.$$

Thus for $1-\delta<r<1$ if $0\le \dot{M}=\min\{\gamma\Lambda/g(M)\,, -\gamma\Lambda/Mg'(M)\}$ and using the previous estimate we obtain,

$$\begin{aligned} \mathcal{E}(z) &= A+B-\frac{z_r}{r}+\mu g(z)-\lambda f(z)/4\pi^2\Big(\int_0^1 f(z) r\,dr\Big)^2 \\ &= \frac{2\gamma\Lambda g(z)}{g^2(M)}-\frac{z_r}{r}+\mu g(z)-\frac{\lambda g(z)/\gamma}{4\pi^2(\int_0^1 g(z)r/\gamma\,dr)^2} \\ &\lesssim 2\gamma\Lambda\frac{g(z)}{g^2(M)}+\gamma\Lambda\frac{g(z)}{g^2(M)}+\mu g(z)-\frac{\lambda\gamma g(z)}{\pi^2(a+1)^2 g^2(M)} \\ &= \big[3\gamma\Lambda+8\gamma/\alpha^2-\gamma\lambda/\pi^2(a+1)^2\big]\,\frac{g(z)}{g^2(M)}=(3\gamma\Lambda-3\gamma\Lambda)=0\,, \end{aligned}$$

for $M\gg 1$. Also $z(1,t)=u(1,t)=z_r(0,t)=u_r(0,t)=0$ on the boundary and taking $z(r,0)\ge u_0(r)$, the function $z(r,t)$ is a lower solution to the u-problem, hence $u(r,t)\ge z(r,t)$, for M, large enough (after some time at which u, is sufficiently large).

Now we show that u blows up. Indeed

$$\dot{M}=\min\{\Lambda\gamma/g(M), -\Lambda\gamma/Mg'(M)\}$$

which implies

$$\Lambda\gamma\frac{dt}{dM}=\max\{g(M)\,, -Mg'(M)\}\le g(M)-Mg'(M) \quad \text{or}$$

$$\begin{aligned} \Lambda\gamma t &\le \int^M (g(s)-sg'(s))\,ds \\ &= -Mg(M)+\int^M g(s)\,ds<\int^M f(s)\,ds<\infty\,, \end{aligned}$$

since $Mg(M)\to 0$ as $M\to\infty$, and (5.4) holds. Hence, z, blows up at $T^*<\infty$, and u, must also blows up at some $t^*\le T^*<\infty$. This completes the proof of the proposition. □

As in the one-dimensional case [18, 19], the blow-up is global, i.e. $u(r,t) \to \infty$ as $t \to t^*-$ for all $r \in [0,1)$. Since $f(u)$ is bounded, then u blows up only by having $\int_0^1 f(u)r\,dr \to 0$ as $t \to t^*-$. Indeed,

$$\dot{M} \le \frac{\lambda f(0)}{4\pi^2(\int_0^1 f(u)r\,dr)^2} = h(t)\,,$$

giving

$$M(t) - M(0) \le \int_0^t h(s)\,ds \to \infty \quad \text{as } t \to t^*\,.$$

This implies $\int_0^1 f(u)rdr \to 0$ as $t \to t^*$, but then u blows up globally and $u_r(1,t) \to \infty$ as $t \to t^*-$.

5.4 The Robin Problem

We consider again u to satisfy (1.7a), (1.7c) but now we take boundary conditions of Robin type,

$$u_r(1,t) + \beta u(1,t) = 0\,, \quad t > 0\,,\ \ \beta > 0\,. \tag{5.22}$$

The corresponding steady problem is

$$w'' + \frac{1}{r}w' + \mu f(w) = 0\,, \quad 0 < r < 1\,, \tag{5.23a}$$

$$w'(1) + \beta w(1) = 0\,, \quad \beta > 0\,, \quad w'(0) = 0\,. \tag{5.23b}$$

Multiplying again by w' and integrating we obtain

$$\frac{(w'(1))^2}{2} + \int_0^1 \frac{(w'(r))^2}{r}\,dr - \mu \int_m^M f(s)\,ds = 0\,, \tag{5.24}$$

where $0 < m = w(1) < M = \max w = w(0)$, ($m = \min w$, by using the maximum principle). We also consider the auxiliary problem,

$$\begin{gathered} z(r) = \sup_r z(r) = N\,, \quad z'(r) = 0 \quad 0 \le r \le 1-\delta\,, \\ z''(r) + \mu g(z(r)) = 0\,, \quad 1-\delta < r < 1\,, \\ z'(1-\delta) = 0 \quad z'(1) + \beta z(1) = 0\,, \end{gathered} \tag{5.25}$$

where $0 < g(s) < f(s)$, and z, z_r, are continuous at $1-\delta$.

The following result is similar to the one of Lemma 5.1.

Lemma 5.8 *Let (5.4) hold, then the solution to (5.25) is a lower solution to the Robin problem (5.23); moreover* $\frac{(w'(1))^2}{\mu} \to 0$ *as* $\mu \to \infty$.

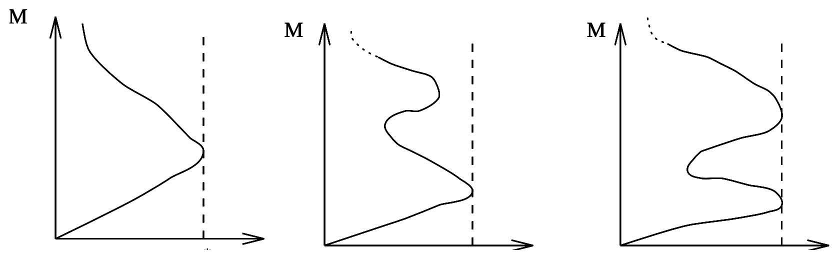

Figure 5: Possible non-local response diagrams for the Robin problem.

Proof: As in the proof of Lemma 5.1 we again have $(z'(r))^2 = 2\mu[G(z) - G(N)]$, and $z'(r) < 0$, where $G(z) = \int_z^\infty g(s)\,ds$. This implies

$$\delta = \frac{1}{2\mu}\int_{z(1)}^{N}[G(s) - G(N)]^{-1/2}\,ds < \frac{1}{2\mu}\int_0^N [G(s) - G(N)]^{-1/2}\,ds\,.$$

Taking $\mu \geq \mu_0$, where

$$\mu_0 = \sup_{z\in(0,N)} \frac{2[G(z) - G(N)]}{(1-\delta)^2[f(z) - g(z)]^2}\,,$$

z is a lower solution to problem (5.23), $z(r) < w(r)$ and $N \leq M$. Note that $N \to \infty$, and $M \to \infty$, as $\mu \to \infty$. Moreover $z''(r) < 0$ in $(1-\delta, 1)$, then $0 < -z'(r) < -z'(1)$, $z(r) < z(1)[1 + \beta(1-r)]$, and for $r = 1-\delta$, $N < z(1)(1+\beta\delta) < w(1)(1+\beta\delta) = m(1+\beta\delta)$, which implies that $m \to \infty$ as $N \to \infty$. Since $\int_0^\infty f(s)\,ds < \infty$, $\int_{m(\mu)}^{M(\mu)} f(s)\,ds \to 0$ as $\mu \to \infty$. From (5.24) we get $\frac{(w'(1))^2}{\mu} \to 0$ as $\mu \to \infty$ and $\frac{1}{\mu}\int_0^1 \frac{(w'(r))^2}{r}\,dr \to 0$ as $\mu \to \infty$. $\square$

Now multiplying by r, (5.23a) gives, $(\int_0^1 f(w) r\,dr)^2 = \frac{(w'(1))^2}{\mu^2}$ and $\lambda = 4\pi^2 \frac{(w'(1))^2}{\mu}$. From Lemma 5.8 we obtain that $\lambda(M) \to 0$ as $M \to \infty$ and from (5.24) that $\frac{(w'(1))^2}{\mu} < 2\int_0^M f(s)\,ds \to 2$ as $M \to \infty$. Hence $\lambda(M) < 8\pi^2$ which means that there exists a $0 < \lambda^* \leq 8\pi^2$ such that for $0 < \lambda < \lambda^*$ problem (5.23) has at least two solutions whereas it has no solutions for $\lambda > \lambda^*$. Thus we have the diagrams of Figure 5.

5.5 Stability

We consider, as in the Dirichlet problem, an upper solution $v(r,t) = w(r; \overline{\mu}(t))$ which is decreasing in time and a lower solution $z(r,t) = w(r; \underline{\mu}(t))$ which is increasing in time, to the u-problem. More precisely we have $\mathcal{E}(v) \geq 0$, and

$\mathcal{E}(z) \le 0$ provided that

$$0 < -\dot{\overline{\mu}} = h(\overline{\mu}) \equiv [4\pi^2\overline{\mu}\, I^2(\overline{w}) - \lambda] \inf_{(0,1)} \{\frac{f(\overline{w})}{\overline{w}_\mu}\}/I^2(\overline{w})\,,$$
$$0 < \dot{\underline{\mu}} = h(\underline{\mu}) \equiv [\lambda - 4\pi^2\underline{\mu}\, I^2(\underline{w})] \inf_{(0,1)} \{\frac{f(\underline{w})}{\underline{w}_\mu}\}/I^2(\underline{w})\,,$$

respectively, with $\overline{\mu}(0)$, and $\underline{\mu}(0)$ chosen so that $w(r;\overline{\mu}(0)) > u_0(r)$, $w(r;\underline{\mu}(0)) < u_0(r)$, and $\lambda = \lambda(\mu) = 4\pi^2\mu(\int_0^1 f(w) r\,dr)^2 = 4\pi^2\mu I^2(w)$, where $w(r;\mu)$ is the steady solution. To each μ corresponds a unique M but to each $\lambda \in (0,\lambda^*)$ corresponds more than one M and hence many solutions $w(r;\lambda)$, see Figure 5. Following the same procedure as in the one-dimensional case we know that the quantity $\Phi(\hat{\mu},\lambda) = 4\pi^2\hat{\mu} I^2(w) - \lambda = \hat{\lambda}(\hat{\mu}) - \lambda$, is either greater than, or equal to, or less than zero, as this is the key term for the construction of upper and lower solutions.

Thus if we consider the left response diagram of Figure 5, then $\Phi(\overline{\mu},\lambda) = \overline{\lambda} - \lambda > 0$, if $\mu_1 < \overline{\mu} < \mu_2$, $\Phi(\underline{\mu},\lambda) = \underline{\lambda} - \lambda < 0$ if $\underline{\mu} < \mu_1$ or $\underline{\mu} > \mu_2$ and $\Phi(\mu,\lambda) = 0$ if $\mu = \mu_1$ or $\mu = \mu_2$.

For $\lambda = \lambda^*$, $\Phi(\underline{\mu},\lambda) < 0$, hence $z_1(r,t) \to w^*$ as $t \to \infty$, provided that $w_1(r;\underline{\mu}(0)) < w^*$ and $z_2(r,t) \to \infty$ as $t \to \infty$ provided now that $w_2(r;\underline{\mu}(0)) > w^*$. This means that w^* is unstable. More precisely it is unstable from above and stable from below. For $\lambda > \lambda^*$ again $\Phi(\underline{\mu},\lambda) < 0$, $z(r,t) \to \infty$ as $t \to t^* \le \infty$, hence u is unbounded for any initial data; this also holds even for $\lambda < \lambda^*$ provided that the initial data are greater than the largest steady state. The above procedure can also be applied to the rest of the response diagrams of Figure 5.

5.6 Blow-up of unbounded solutions

We consider now the unbounded solutions appearing for $\lambda > \lambda^*$ or for $\lambda \le \lambda^*$ but with initial conditions larger than the greatest steady state. Following the same method as in the one-dimensional case [19], if u fails to blow-up, then for any given k, there must be a $t_k > 0$ such that $u \ge k$, for $t_k \ge t$ (this is due to the use of the lower solutions, note that $m = \min w(r) = w(1) \to \infty$ as $M \to \infty$). Then we consider the problem,

$$v_t = \Delta_r v + \lambda f_k(v)/4\pi^2(\int_0^1 f_k(v) r\,dr)^2\,,\ \ 0 < r < 1\,,\ \ t > 0$$
$$v(1,t) = v_r(0,t) = 0\,,\ \ t \ge t_k$$
$$v(r,t_k) = 0\,,\ \ 0 < r < 1\,,$$

where $f_k(s) = f(s+k)$. Thus it can easily seen that $v + k$ is a lower solution to the u-problem, hence $u \ge v + k$ for $t \ge t_k$. But from the Dirichlet problem (see Proposition 5.2, we have that if $\lambda > \lambda^* \int_0^\infty f_k(s)\,ds = \lambda^* \int_0^\infty f(s+k)\,ds = \lambda^* \int_k^\infty f(S)\,dS$, then v blows up at finite time. Hence choosing k sufficiently

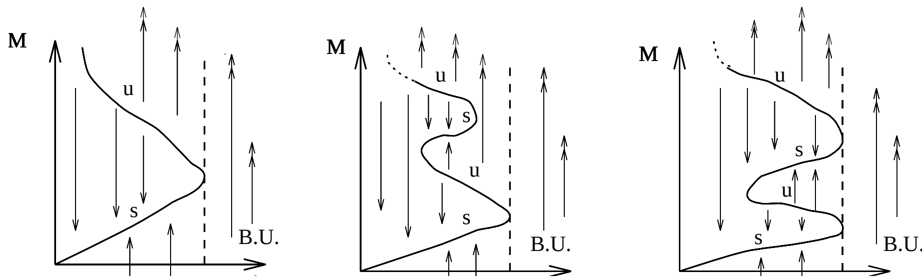

Figure 6: Stability and blow-up of solutions for the Robin problem.

large so that the previous inequality holds, we get that u blows up. This blow-up is global.

Finally, carrying over the analysis similar to the Dirichlet problem and the one-dimensional case [18, 19], we obtain Figure 6. We use the notation: $(\rightarrow \cdot \leftarrow)$, for stable stationary solutions, $(\leftarrow \cdot \rightarrow)$ for unstable, and the double arrows $(\rightarrow\rightarrow)$ for solutions u which blow up. If we lie in the region where $\Phi(\overline{\mu}(t),t) > 0$ then the arrows point downwards while where $\Phi(\underline{\mu}(t),t) < 0$ the arrows point upwards.

Any solution which corresponds to a point of the curves of this type $(\rightarrow \cdot \leftarrow)$ is stable while all others are unstable. More precisely this $(\rightarrow \cdot \rightarrow)$, is stable from one side and unstable from the other whereas this $(\leftarrow \cdot \rightarrow)$, is unstable from both sides.

For the Neumann problem (the boundary conditions are $u_r(0,t) = u_r(1,t) = 0$) there is no positive steady state for any $\lambda > 0$. Concerning the solution $u(r,t)$, this behaves as in the one-dimensional case [18, 19]; if $\int_0^\infty f(s)\,ds < \infty$ then u, blows up globally at $t^* = \frac{2\pi^2}{\lambda}\int_0^\infty f(s)\,ds < \infty$ for $u(0,t) = 0$, whereas if $\int_0^\infty f(s)\,ds = \infty$, then blow-up does not occur but the solution tend to ∞, uniformly as $t \to \infty$,

6 Discussion

In the present work we have studied the non-local, two-dimensional, radially symmetric problem of the form:

$$u_t = \Delta_r u + \lambda f(u)/4\pi^2\Big(\int_0^1 f(u)r\,dr\Big)^2$$

where $f(u) > 0$, $f'(u) < 0$, u, represents the temperature which is produced in a conductor having fixed electric current I, i.e. $\lambda = I^2/\pi^2$. The function f represents electrical conductivity ($f(u) = \sigma(u)$), in contrast to the one-dimensional model where it represents electrical resistivity ($f(u) = \rho(u) = 1/\sigma(u)$). This work extends the results of the one-dimensional problem, and the methods used are similar to the one-dimensional case and are based on comparison techniques.

We find similar behaviour in both problems, and it is rather like that of the standard reaction-diffusion model, $u_t = \Delta u + \lambda f(u)$, $f(u) > 0$, $f'(u) > 0$, $f''(u) > 0$, see [17] and the references therein. More precisely, for the Dirichlet problem, if $\int_0^\infty f(s)\,ds < \infty$ we find a critical value λ^* such that for $\lambda > \lambda^*$ there is no steady state and u, blows up globally. Also in case we have a unique steady solution, this solution is asymptotically stable.

For the Robin problem, provided that $\int_0^\infty f(s)\,ds < \infty$, again there exists a critical value $\lambda^* \leq \lambda_D^*$ (λ_D^* refers to the Dirichlet problem). If $0 < \lambda < \lambda^*$ then there exists at least one stationary solution and no solution for $\lambda > \lambda^*$. Concerning the stability, the minimal solution is stable, the next greater one is unstable, the next stable and so on. On the other hand, if $\lambda > \lambda^*$ then u blows up; u also blows up for $\lambda \in (0, \lambda^*)$ and for sufficiently large initial data . The solution(s) at $\lambda = \lambda^*$ is(are) unstable.
For the Neumann problem there is no steady state and the solution u blows up in finite time if $\int_0^\infty f(s)\,ds < \infty$, whereas $u \to \infty$ uniformly as $t \to \infty$ if $\int_0^\infty f(s)\,ds = \infty$.

It is an interesting question whether or not a similar behaviour occurs for asymmetric problems for dimensions greater than or equal to two.

Acknowledgments The author would like to thank A. A. Lacey for several fruitful discussions. Part of this work was done during the author's visit to the Departments of Mathematics of Heriot-Watt University, Edinburgh, and Universidad Complutense, Madrid.

References

[1] S.N. Antontsev & M. Chipot, *The thermistor problem: Existence, smoothness, uniqueness, blow-up*, SIAM J. Math. Anal., Vol. 25, No. 4, (1994), pp. 1128-1156.

[2] S.N. Antontsev & M. Chipot, *Analysis of blowup for the thermistor problem*, Siberian Mathematical J. Vol. 38, No. 5, (1997), pp. 827-841.

[3] W. Allegretto and H. Xie, *A non-local thermistor problem*, Euro. J. Applied Math., (1995), pp. 83–94.

[4] A. Barabanova, *The blow–up of solutions of a nonlocal thermistor problem*, Appl. Math. Lett. 9, No. 1, (1996), pp. 59–63.

[5] J.W. Bebernes and A.A. Lacey, *Global existence and finite-time blow-up for a class of nonlocal problems*, Adv. Diff. Equns, 2, (1997), pp. 927–953.

[6] J.A. Carrillo, *On a non-local elliptic equation with decreasing nonlinearity arising in plasma physics and heat conduction*, Nonlinear Analysis TMA., 32, (1998), pp. 97–115.

[7] N. Chafee, *The electric ballast resistor: homogeneous and nonhomogeneous equilibria*, In Nonlinear differential equations: invariance stability and bifurcations, P. de Mottoni & L. Salvatori Eds., Academic Press, New York., (1981), pp. 97–127.

[8] G. Cimatti, *Remarks on existence and uniqueness for the thermistor problem under mixed boundary conditions*, Quart. J. Appl. Math., 47, (1989), pp. 117–121.

[9] G. Cimatti, *The stationary thermistor problem with a current limiting device*, Proc. Roy. Soc. Edinburgh, 116A, (1990) pp. 79–84.

[10] G. Cimatti, *On the stability of the solution of the thermistor problem*, Appl. Anal. 73 (1999), no. 3-4, pp. 407–423.

[11] M. Fila, *Boundedness of global solutions of nonlocal parabolic equations.* Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996). Nonlinear Anal. 30 (1997), no. 2, pp. 877–885.

[12] A.C. Fowler, I. Frigaard, and S.D. Howison, *Temperature surges in current-limiting circuit devices*, SIAM J. Appl. Math., 52, (1992), pp. 998–1011.

[13] B. Gidas, W.-M. Ni and L. Nirenberg, *Summetry and related properties via the maximum principle*, Comm. Math. Phys., 68, (1979), pp. 209–243.

[14] S.D. Howison, *A note on the thermistor problem in two space dimensions*, Quart. Appl. Math., 47, (1989), pp. 509–512.

[15] N.I. Kavallaris and D.E. Tzanetis, *An Ohmic heating non-local diffusion-convection problem for the Heaviside function*, preprint.

[16] N.I. Kavallaris and D.E. Tzanetis, *Global existence and divergence of critical solutions of some non-local parabolic problems in the Ohmic heating process*, preprint.

[17] A.A. Lacey, *Mathematical analysis of thermal runaway for spatially inhomogeneous reactions*, SIAM J. Appl. Math., 43, (1983), pp. 1350–1366.

[18] A.A. Lacey, *Thermal runaway in a non-local problem modelling Ohmic heating: Part I: Model derivation and some special cases.*, Euro. J. Applied Math., 6, (1995), pp. 127–144.

[19] A.A. Lacey, *Thermal runaway in a non-local problem modelling Ohmic heating: Part II: General proof of blow-up and asymptotics of runaway*, Euro. J. Applied Math. 6, (1995), pp. 201–224.

[20] J. Lopez–Gomez, *On the structure and stability of the set of solutions of a non–local problem modelling Ohmic heating*, J. Dynam. Diff. Eqns 10, No. 4, (1998), pp. 537–559.

[21] D.H. Sattinger, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J., 21, (1972), pp. 979–1000.

[22] P. Souplet, *Blow–up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal. 29, No. 6, (1998), pp. 1301–1334.

[23] P. Souplet, *Uniform blow–up profiles and boundary behavior for diffusion equations with nonlocal source*, J. Diff. Eqns, 153, (1999), pp. 374–406.

Dimitrios E. Tzanetis
Department of Mathematics, Faculty of Applied Sciences,
National Technical University of Athens,
Zografou Campus, 157 80 Athens, Greece
e-mail: dtzan@math.ntua.gr