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A class of nonlinear elliptic variational inequalities: qualitative properties and existence of solutions *

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Abstract

We study a class of nonlinear elliptic variational inequalities in divergence form. In the recent paper [6], we obtained results on the local control of essential infimum and supremum of solutions of quasilinear elliptic equations, and here we extend this point of view to the case of variational inequalities. It implies a new qualitative property of solutions in $W^{1,p}(\Omega)$ which we call "jumping over the control obstacle." Using the Schwarz symmetrization technique, we give an existence and symmetrization theorems in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ which agree completely with previous qualitative results. Also we consider generating singularities of weak solutions in $W^{1,p}(\Omega)$ of variational inequalities.

1 Problem setting and main results

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, and $p \in (1,\infty)$. We are concerned with the following nonlinear elliptic double obstacle problem: Find $u \in W^{1,p}(\Omega)$ such that $\omega_1 \leq u \leq \omega_2$ in Ω and

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(v-u)\,dx \geq \int_\Omega \big[f(x,u)+g(x,u)|\nabla u|^p\big](v-u)\,dx, \qquad (1.1)$$

for all $v \in W^{1,p}(\Omega)$ such that $v-u \in L^\infty(\Omega)$ and $\omega_1 \leq v \leq \omega_2$ in Ω.

The obstacles ω_1 and ω_2 are two measurable functions without any global regularity, such that $\omega_1 \leq \omega_2$ in Ω. For any measurable set A in $\mathbb{R}^N$ we say that a property holds "in A" if it holds in the a.e. sense. The leading term on the left-hand side is the Carathéodory vector function $a(x,\eta,\xi)$ satisfying general structure conditions of Leray-Lions type, see (1.7)–(1.8). The leading terms on the right-hand side are Carathéodory real functions $f(x,\eta)$ and $g(x,\eta)$ which will essentially influence the main results.

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In order to describe our main goals of this paper, we introduce an additional obstacle ω_c, that we call control obstacle, which is taken to be a measurable real function defined on Ω, satisfying the following natural condition relative to ω_1 and ω_2:

$$\omega_1 \le \omega_c \le \omega_2 \quad \text{in } \Omega\,. \tag{1.2}$$

Furthermore, we assume that there exist two balls B_r and B_ρ in Ω such that $B_{2r} \subseteq \Omega$, $B_{2\rho} \subseteq \Omega$, $B_{2r} \cap B_{2\rho} = \emptyset$,

$$\begin{gathered} m_1 = \operatorname{ess\,inf}_{B_{2r}} \omega_1, \quad M_c = \operatorname{ess\,sup}_{B_{2r}} \omega_c, \quad m_2 = \operatorname{ess\,inf}_{B_{2r}} \omega_2, \\ -\infty < m_1 < M_c < m_2 < \infty, \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} M_1 = \operatorname{ess\,sup}_{B_{2\rho}} \omega_1, \quad m_c = \operatorname{ess\,inf}_{B_{2\rho}} \omega_c, \quad M_2 = \operatorname{ess\,sup}_{B_{2\rho}} \omega_2, \\ -\infty < M_1 < m_c < M_2 < \infty. \end{gathered} \tag{1.4}$$

Relation (1.3) (respectively (1.4)) has the following meaning: the obstacles ω_1 and ω_2 are bounded from below (respectively from above) on the corresponding ball, and the obstacles ω_c and ω_2 (respectively ω_c and ω_1) are strictly separated on the corresponding balls.

In this paper we find sufficient conditions on functions $f(x,\eta)$ and $g(x,\eta)$ which will give us the following three types of results: (i) jumping over a prescribed control obstacle on prescribed balls, (ii) existence of at least one essentially bounded weak solution, (iii) generating of singularities, bumping on the upper obstacle and pushing to the upper obstacle.

(i) We say that a solution u of (1.1) *jumps over the prescribed control obstacle* ω_c if it satisfies

$$\big|\{x \in \Omega : u(x) > \omega_c(x)\}\big| \neq 0, \qquad \big|\{x \in \Omega : u(x) < \omega_c(x)\}\big| \neq 0\,, \tag{1.5}$$

where $|A|$ denotes the Lebesgue measure of a subset A of $\mathbb{R}^N$. Taking B_r, B_ρ, m_1, M_1, m_c, M_c, m_2 and M_2 as in (1.2)–(1.4), and α_0, $a_0(x)$, a_1, a_2 as in structure conditions (1.7)–(1.8), we now impose two crucial sets of hypotheses. First those corresponding to ball B_{2r}:

(H1) $g(x,\eta) \ge 0$ in B_{2r}, for all $\eta \in I_1 = (m_1, M_c)$

(H2) There exists $f_1 \in L^1(B_{2r})$, such that $f(x,\eta) \ge f_1(x)$ in B_{2r}, for all $\eta \in I_1$, $f_1(x) \ge 0$ in $B_{2r} \setminus B_r$, and

$$\int_{B_r} f_1(x)\,dx > D_1 \frac{m_2 - m_1}{m_2 - M_c},$$

where $D_1 = \overline{d} \int_{B_{2r}} [a_0(x) + a_1 \widehat{m}^{p-1}]^{p'}\,dx + \big(\frac{p}{d}\big)^{p-1} \frac{(2^N-1)|B_r|}{r^p}$, $\widehat{m} = \max\{|m_1|, |M_c|\}$, $\overline{d} = \frac{\alpha_0}{a_2^{p'}(m_2 - m_1)}$, $d = \frac{p'}{2^{p'-1}}\overline{d}$.

Now we impose the dual hypotheses corresponding to ball $B_{2\rho}$.

(H3) $g(x,\eta) \le 0$ in $B_{2\rho}$, for all $\eta \in I_2 = (m_c, M_2)$

(H4) There exists $f_2 \in L^1(B_{2\rho})$, such that $f(x,\eta) \le f_2(x)$ in $B_{2\rho}$ for all $\eta \in I_2$, $f_2(x) \le 0$ in $B_{2\rho} \setminus B_\rho$,
$$\int_{B_\rho} f_2(x)\,dx < -D_2 \frac{M_2 - M_1}{m_c - M_1},$$
where $D_2 = \overline{d} \int_{B_{2\rho}} [a_0(x) + a_1 \widehat{m}^{p-1}]^{p'}\,dx + \left(\frac{p}{d}\right)^{p-1} \frac{(2^N-1)|B_\rho|}{\rho^p}$, $\widehat{m} = \max\{\,|m_c|, |M_2|\,\}$, $\overline{d} = \frac{\alpha_0}{a_2^{p'}(M_2-M_1)}$, $d = \frac{p'}{2^{p'-1}}\overline{d}$, with p' satisfying $1/p + 1/p' = 1$.

Two complementary situations occur: the hypotheses (H1)–(H2) require that the function $f(x,\eta)$ be sufficiently large and positive in the strip $B_{2r} \times I_1$, and that $g(x,\eta)$ be non-negative in the same strip (respectively, the hypotheses (H3)–(H4) require that $f(x,\eta)$ be sufficiently large and negative in the strip $B_{2\rho} \times I_2$, and that $g(x,\eta)$ be non-positive in the strip). These conditions will imply, see Theorem 1.1, that each solution u of (1.1) satisfy
$$\big|\{x \in B_{2r} : u(x) > \omega_c(x)\}\big| \ne 0, \qquad \big|\{x \in B_{2\rho} : u(x) < \omega_c(x)\}\big| \ne 0\,, \tag{1.6}$$
that is to say, there are two measurable sets $E_r \subseteq B_{2r}$ and $E_\rho \subseteq B_{2\rho}$, $|E_r| \ne 0$, $|E_\rho| \ne 0$, satisfying $u(x) > \omega_c(x)$ for each $x \in E_r$ and $u(x) < \omega_c(x)$ for each $x \in E_\rho$.

Since $B_{2r} \cap B_{2\rho} = \emptyset$, both pairs of hypotheses (H1)–(H2) and (H3)–(H4) are independent of each other, which allows us to combine them and derive the main result of this paper:

Theorem 1.1 (Jumping over the Control obstacle in $W^{1,p}(\Omega)$) *Under assumptions (1.2)–(1.4), let the Carathéodory vector function $a(x,\eta,\xi)$ satisfy:*
$$\exists \alpha_0 > 0,\ a(x,\eta,\xi)\cdot\xi \ge \alpha_0 |\xi|^p \quad \text{in } \Omega,\ \eta \in \mathbb{R},\ \xi \in \mathbb{R}^N, \tag{1.7}$$
$$\begin{gathered} \exists a_0 = a_0(x) \ge 0, \quad a_0 \in L^{p'}(\Omega), \quad \exists a_1 \ge 0, \quad \exists a_2 > 0, \\ |a(x,\eta,\xi)| \le a_0(x) + a_1|\eta|^{p-1} + a_2|\xi|^{p-1} \quad \text{in } \Omega,\ \eta \in \mathbb{R},\ \xi \in \mathbb{R}^N. \end{gathered} \tag{1.8}$$
If the Carathéodory functions $f(x,\eta)$ and $g(x,\eta)$ satisfy the hypotheses (H1)–(H4), then for each solution $u \in W^{1,p}(\Omega)$ of (1.1) satisfies (1.6).

To prove (1.6) we argue by contradiction. First we choose appropriate test functions in order to localize the balls in $\mathbb{R}^N$, then integrate over level sets of the form $\{u > t\}$ and $\{u < t\}$, and then use several elementary inequalities in $\mathbb{R}$ in order to obtain contradiction. We want to indicate that the proof of property (1.6) is not difficult. As pointed out in the abstract, the same method has been exploited in [6] in order to obtain the local control of essential infimum and supremum of solutions of elliptic equations, and in [7] to obtain some new qualitative properties of solutions. In (H2) and (H4) we have to impose slightly different conditions on nonlinear term $f(x,\eta)$ than we did in [6, Theorem 5], in order to have the desired control effect.

(ii) The second type of result is the existence of at least one solution of (1.1) in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ that satisfy jumping condition (1.5). This condition is fulfilled if $|f(x,\eta)|$ is uniformly bounded in $\Omega\times\mathbb{R}$ and $|g(x,\eta)|$ is uniformly small enough in $\Omega\times\mathbb{R}$, or conversely, $|g(x,\eta)|$ is uniformly bounded and $|f(x,\eta)|$ uniformly small enough. Precisely, we will impose one more hypotheses on these functions, which does not contradict (H1)–(H4):

(H5) There exist $f_0>0$ and $g_0\geq 0$ such that $|f(x,\eta)|\leq f_0$, $|g(x,\eta)|\leq g_0$, in $\Omega\times\mathbb{R}$, and
$$f_0^{p'-1}g_0<\Big(\frac{\alpha_0 N C_N^{1/N}}{2\mid\Omega\mid^{1/N}}\Big)^{p'}\frac{p'}{N(p'+1)}.$$

Then we have the following result on existence and symmetrization of solutions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$).

Theorem 1.2 *Under the structure assumptions (1.2)–(1.4) where $\omega_1\leq 0\leq\omega_2$ in Ω and $\omega_1,\omega_2\in L^p(\Omega)$, let the Carathéodory vector function $a(x,\eta,\xi)$ satisfy the hypotheses (1.7), (1.9), and*
$$(a(x,\eta,\xi)-a(x,\eta,\xi^*))\cdot(\xi-\xi^*)>0\quad in\ \Omega,\ \eta\in\mathbb{R},\ \xi,\xi^*\in\mathbb{R}^N,\ \xi\neq\xi^*. \tag{1.9}$$
Assume that Carathéodory functions $f(x,\eta)$ and $g(x,\eta)$ satisfy conditions (H1)–(H5). Then there exists a solution $u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ of (1.1) satisfying (1.5). Moreover,
$$\begin{gathered}u^\#(x)\leq v^\#(x)=v(x)\quad in\ \Omega^\#,\\ \|u\|_{L^\infty(\Omega)}\leq\|v\|_{L^\infty(\Omega^\#)}\quad and\quad \|\nabla u\|_{L^p(\Omega)}\leq\|\nabla v\|_{L^p(\Omega^\#)},\end{gathered}\tag{1.10}$$
where $u^\#$ is the Schwarz symmetrization of u, and v is the unique solution of the symmetrized problem
$$\begin{gathered}-\operatorname{dvi}(\alpha_0|\nabla v|^{p-2}\nabla v)=f_0+g_0|\nabla v|^p\quad in\ \Omega^\#,\\ v\in W_0^{1,p}(\Omega^\#)\cap L^\infty(\Omega^\#),\ v\ is\ positive\ and\ spherically\ symetric.\end{gathered}\tag{1.11}$$
Here $\Omega^\#$ is a ball in $\mathbb{R}^N$ centered at the origin, with the same volume as Ω.

Applications of the Schwartz symmetrization to partial differential equations can be seen for instance in [1, 4, 14], while applications to variational inequalities are treated in [2, 3, 5]. Let us mention that the additional condition (H5), that is to say, the "smallness condition" on the data $f(x,\eta)$ and $g(x,\eta)$ is used only sufficient for existence of a solution v of the symmetrized equation (1.11). This problem is treated in detail in [12, 8, 15].

In contrast to the proof of qualitative property (1.6), the proof of existence result requires a more complicated procedure. Here we exploit the method of penalty functions as approximation step, and the method of Schwartz symmetrization of penalty equation in order to derive a priori estimates which are independent on the approximative process. This construction has already been announced in the recent paper [10], but without proof and details.

(iii) The third type of results concerns the possibility to generate singularities of solutions in a given point. In particular, this enables to obtain nonexistence result for essentially bounded weak solutions.

It will be convenient to define essential supremum u^* and essential infimum u_* of a measurable function $u\,\Omega \to \overline{\mathbb{R}}$ in the point $x_0 \in \Omega$:

$$u^*(x_0) = \lim_{r\to 0} \operatorname{ess\,sup}_{B_r(x_0)} u(x), \quad u_*(x_0) = \lim_{r\to 0} \operatorname{ess\,inf}_{B_r(x_0)} u(x). \tag{1.12}$$

We say that u has singularity at x_0 if $u^*(x_0) = +\infty$. The following theorem shows that it is possible to generate a singularity of all solutions of variational problem (1.1) in a given point $x_0 \in \Omega$. Of course, the upper obstacle w_2 also has to be singular in this point.

Theorem 1.3 (Generating singularities of solutions) *Assume that $a(x,\eta,\xi)$ satisfies conditions (1.7) and (1.8). Let there exist $x_0 \in \Omega$ and $\beta \in \mathbb{R}$ such that:*

$$\int_{B_r} [a_0(x) + a_1 \widehat{m}^{p-1}]^{p'}\,dx = O(r^\beta), \quad \text{as } r \to 0, \tag{1.13}$$

where $B_r = B_r(x_0)$. Assume that there exist positive constants α, γ, R, C_1, C_2, such that for a.e. $x \in B_R$ and $\eta \geq \operatorname{ess\,inf}_{B_R} \omega_1$,

$$g(x,\eta) \geq 0, \quad f(x,\eta) \geq \frac{C_1}{|x-x_0|^\gamma}, \quad w_2(x) \geq \frac{C_2}{|x-x_0|^\alpha}. \tag{1.14}$$

If

$$\alpha < \frac{\gamma - p}{p-1}, \tag{1.15}$$

and

$$\alpha + \beta + \gamma > N, \tag{1.16}$$

then any solution u of problem (1.1) is singular at x_0, that is, $u^*(x_0) = \infty$. In particular, problem (1.1) has no solutions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark Note that conditions (1.14), (1.15) and $\alpha > 0$ imply that $p < \gamma < N$. This shows that Theorem 1.3 is in accordance with the Sobolev imbedding theorem, since we have $p < N$ and singular solutions.

Remark If $a_0(x) \equiv 0$ and $a_1 = 0$, then condition (1.16) can be dropped, since then we can take β arbitrarily large, in order to ensure (1.16). Also, if $a_0(x) = O(r^{\overline{\beta}})$ for some $\overline{\beta} \geq 0$, or $a_1 > 0$, then condition (1.16) is fulfilled with $\beta = N$ or $\beta < \frac{\alpha+\gamma}{p}$ respectively. It is also possible to consider the case of $\alpha = (\gamma-1)/(p-1)$ in Theorem 1.3.

Under simple additional conditions we can ensure that the solution will bump on the upper obstacle infinitely many times near its singular point x_0, that is, along an infinite sequence converging to x_0.

Theorem 1.4 (Bumping on the Upper Obstacle near the Singularity) *Assume that $a(x,\eta,\xi) = |\xi|^{p-2}\xi$, and let there exist positive constants α, γ, R, C_1, C_2, such that for a.e. $x \in B_R$ and $\eta \geq \operatorname{ess\,inf}_{B_R(x_0)}\omega_1$,*

$$g(x,\eta) \geq 0, \quad f(x,\eta) \geq \frac{C_1}{|x-x_0|^\gamma}, \tag{1.17}$$

$$\omega_2(x) \leq C_2|x-x_0|^{-\alpha} \text{ a.e. on } B_R(x_0). \tag{1.18}$$

Let $\alpha < \frac{\gamma-p}{p-1}$ and let a solution u of (1.1) and lower obstacle w_1 satisfy the condition

$$u_*(x) > w_1^*(x) \quad \text{for all } x \in B_R \text{ for some } R > 0, \tag{1.19}$$

$$u(x) \geq 0 \quad \text{a.e. on } B_R, \tag{1.20}$$

Then for any $r > 0$ there exists $x_r \in B_r(x_0) \setminus \{x_0\}$, such that

$$u^*(x_r) = w_{2*}(x_r). \tag{1.21}$$

Theorem 1.5 (Pushing to the upper obstacle) *Assume that $a(x,\eta,\xi) = |\xi|^{p-2}\xi$. Let x_0 be a given point in Ω such that $w_{2*}(x_0) < \infty$. Assume that*

$$f(x,\eta) \geq C \cdot |x-x_0|^{-\gamma} \quad \text{for a.e. } x \in B_R = B_R(x_0),\ \eta \in (\overline{m}_1, \overline{m}_2),$$

where $\overline{m}_1 = \operatorname{ess\,inf}_{B_R} w_1$ and $\overline{m}_2 = \operatorname{ess\,sup}_{B_R} w_2$. If $p < c < N$, then for any solution u of variational problem (1.1), we have $u^*(x_0) = w_{2*}(x_0)$.

2 Proof of results

Proof of Theorem 1.1 As we have described in Introduction, we will restrict our attention to the first case in (1.6) only, because the dual case can be obtain in analogous way. We start by an easy localization fact from harmonic analysis (see for instance [7]) that we state in the following form. For any $c_0 > 1$ and $r > 0$ there exists a function $\Phi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \Phi \leq 1$ in $\mathbb{R}^N$ such that

$$\begin{gathered} \Phi(x) = 1 \quad \text{for } x \in B_r \quad \text{and} \quad \Phi(x) = 0 \quad \text{for } x \in \mathbb{R}^N \setminus B_{2r} \\ \Phi(x) > 0 \quad \text{for } x \in B_{2r} \quad \text{and} \quad |\nabla\Phi| \leq c_0/r \quad \text{in } \mathbb{R}^N\,. \end{gathered} \tag{2.1}$$

Taking B_r, m_1, M_c and m_2 such that conditions (1.2)–(1.4) are satisfied, let us choose for any $c_0 > 1$ an appropriate test function φ defined by

$$\varphi = (u-t)^-\Phi^p + u \quad \text{with } t \in (M_c, m_2]\,. \tag{2.2}$$

Here and in the sequel u is a solution of (1.1) and $\eta^- = \max\{0, -\eta\}$. The basic step it is to check that φ has the properties

$$\varphi \in W^{1,p}(\Omega), \quad \varphi - u \in L^\infty(\Omega), \quad \text{and} \quad \omega_1 \leq \varphi \leq \omega_2 \quad \text{in}\Omega\,. \tag{2.3}$$

Arguing by contradiction, we assume that there holds the opposite of (1.6), say $|\{x \in B_{2r} : u > \omega_c\}| = 0$. In other words, $u \le \omega_c$ in B_{2r}. Remark that we already have $\omega_1 \le u$ in Ω. Taking one-sided supremum and infimum over B_{2r} in the previous two inequalities, we deduce:

$$m_1 \le u \le M_c \qquad \text{in } B_{2r}\,. \tag{2.4}$$

Since

$$\nabla(u-t)^- = \begin{cases} -\nabla u & \text{in } \{u < t\}, \\ 0 & \text{in } \{u \ge t\}, \end{cases}$$

substituting test function φ from (2.2) into (1.1), and using the structure condition (1.7), we obtain:

$$\begin{aligned} \alpha_0 &\int_{\{u<t\}_{2r}} |\nabla u|^p \Phi^p \, dx \\ &\le \; p\int_{\{u<t\}_{2r}} |a(x,u,\nabla u)|\Phi^{p-1}(t-u)|\nabla\Phi| \, dx \\ &\quad - \int_{\{u<t\}_{2r}} f(x,u)(t-u)\Phi^p \, dx - \int_{\{u<t\}_{2r}} g(x,u)|\nabla u|^p (t-u)\Phi^p \, dx. \end{aligned} \tag{2.5}$$

where $\{u<t\}_{2r} = \{u<t\} \cap B_{2r}$. It is interesting to mention that in the light of (2.4) the level set $\{u<t\}_{2r}$ satisfies:

$$\{u<t\}_{2r} = \{u \le M_c\}_{2r} \cup \{M_c < u < t\}_{2r} = \{u \le M_c\}_{2r} = B_{2r}.$$

Thus, we are able to reduce all integrations over $\{u<t\}_{2r}$, appearing in (2.5), to B_{2r}.

In the sequel, due to (H1) we can drop the last term on the right-hand side of (2.5). This together with the structure condition (1.8) and $t-u = (t-u)^{\frac{1}{p'}}(t-u)^{\frac{1}{p}}$ leads us to the inequality

$$\begin{aligned} \alpha_0 &\int_{B_{2r}} |\nabla u|^p \Phi^p \, dx \\ &\le \int_{B_{2r}} \Big[\big(a_0(x) + a_1|u|^{p-1} + a_2|\nabla u|^{p-1}\big)\Phi^{p-1}(t-u)^{\frac{1}{p'}}\Big]\cdot\Big[p|\nabla\Phi|(t-u)^{\frac{1}{p}}\Big] \, dx \\ &\quad - \int_{B_{2r}} f(x,u)(t-u)\Phi^p \, dx\,. \end{aligned} \tag{2.6}$$

Now we consider the product under the first integral on the right-hand side of (2.6). Applying the following two elementary inequalities, $a(pb) \le \frac{d}{p'}a^{p'} + (\frac{p}{d})^{p-1}b^p$ and $(a+b)^{p'} \le 2^{p'-1}(a^{p'} + b^{p'})$ for all $a \ge 0$, $b \ge 0$, and $d > 0$, we obtain

$$\begin{aligned} 0 \;&=\; \big[\alpha_0 - a_2^{p'}\overline{d}(m_2 - m_1)\big]\int_{B_{2r}} |\nabla u|^p \Phi^p \, dx \\ &\le\; \overline{d}\int_{B_{2r}} \big(a_0(x) + a_1|u|^{p-1}\big)^{p'}\Phi^p(t-u)\, dx + \big(\frac{p}{d}\big)^{p-1}\int_{B_{2r}} |\nabla\Phi|^p(t-u)\, dx \\ &\quad - \int_{B_{2r}} f(x,u)(t-u)\Phi^p \, dx, \end{aligned} \tag{2.7}$$

where the numbers d and $\overline{d}$ are defined in (H2).

Now we are in the position to exploit properties of the localization function Φ in (2.7). First, since $f(x,\eta) \geq f_1(x)$ in B_{2r} for all $\eta \in I_1$ (see $(H2)$), then using (2.1) and (2.4) we derive

$$\begin{aligned} 0 \quad \leq \quad & (t-m_1)\overline{d}\int_{B_{2r}} [a_0(x)+a_1\widehat{m}^{p-1}]^{p'}\,dx \\ & +(t-m_1)\big(\frac{p}{d}\big)^{p-1}|B_{2r}\setminus B_r|\big(\frac{c_0}{r}\big)^p-(t-M_c)\int_{B_r} f_1(x)\,dx. \end{aligned}$$

Setting $s=t-M_c$, using $|B_{2r}\setminus B_r|=(2^N-1)|B_r|$, and passing to the limit as $c_0\to 1$, we obtain

$$\int_{B_r} f_1(x)\,dx \leq D_1\frac{\big[s+(M_c-m_1)\big]}{s}, \qquad \text{for all } s\in(0,m_2-M_c].$$

Since in the previous inequality the function appearing on the right-hand side is decreasing with respect to s, we can set $s=m_2-M_c$ to obtain

$$\int_{B_r} f_1(x)\,dx \leq D_1\frac{m_2-m_1}{m_2-M_c}.$$

However, this contradicts (H2), and the theorem is proved. $\diamondsuit$

Sketch of the proof of Theorem 1.2 The penalty method is carried out in the following three steps (see for instance [9] for the case of $Au=f\in V'$).

Firstly, we associate to (1.1) an ε-problem, the so called penalty equation

$$\begin{gathered} -\operatorname{dvi} a(x,u_\varepsilon,\nabla u_\varepsilon)+\frac{1}{\varepsilon}\beta(x,u_\varepsilon)=f(x,u_\varepsilon)+g(x,u_\varepsilon)|\nabla u_\varepsilon|^p \quad \text{in } D'(\Omega), \\ u_\varepsilon\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \end{gathered} \tag{2.8}$$

where the penalty function $\beta(x,\eta)$ is a Carathéodory function defined by

$$\beta(x,\eta)=((\eta-\omega_2(x))^+)^{p-1}-((\eta-\omega_1(x))^-)^{p-1}, \qquad \text{in } \Omega\times\mathbb{R}. \tag{2.9}$$

Since $\omega_1(x)\leq 0\leq\omega_2(x)$ in Ω, the penalty function β has the following three important properties:

$$\beta(x,v)=0 \quad \text{in } \Omega \quad \text{if and only if} \quad \omega_1\leq v\leq\omega_2 \quad \text{in } \Omega, \tag{2.10}$$
$$(\beta(x,\eta_1)-\beta(x,\eta_2))(\eta_1-\eta_2)>0 \quad \text{in } \Omega,\ \eta_1\neq\eta_2\in\mathbb{R}, \tag{2.11}$$
$$\beta(x,\eta)\operatorname{sgn}(\eta)\geq 0 \quad \text{in } \Omega,\ \eta\in\mathbb{R}. \tag{2.12}$$

In the first step, by means of the Schwartz symmetrization we derive some basic a priori estimates for u_ε in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Proposition 2.1 *Under the assumptions of Theorem 1.2, for each $\varepsilon > 0$ there exist a solution u_ε of (2.8), and two constants C_1 and C_2 which are independent on ε, such that*

$$\begin{gathered} u_\varepsilon^\#(x) \le v^\#(x) = v(x) \quad \text{in } \Omega^\#, \\ \|u_\varepsilon\|_{L^\infty(\Omega)} \le C_1 = \|v\|_{L^\infty(\Omega^\#)}, \\ \|\nabla u_\varepsilon\|_{L^p(\Omega)} \le C_2 = \|\nabla v\|_{L^p(\Omega^\#)}, \end{gathered} \tag{2.13}$$

where $u_\varepsilon^\#$ is the Schwarz symmetrization of u, and v is the unique solution of (1.11).

Having in mind the sign condition (2.12) for the penalty function $\beta(x, \eta)$, the proof of Proposition 1 is very similar to the proofs of [11, Theorems 2, 3, 4].

Next, we consider the relative compactness of the sequence u_ε. According to previous estimates and the relative compactness results from [2], one can show the following proposition.

Proposition 2.2 *Under the assumptions of Theorem 1.2, let u_ε be a solution of (2.8). Then there exist a subsequence of u_ε, still denoted by u_ε, and a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that as $\varepsilon \to 0$,*

$$\begin{gathered} u_\varepsilon \to u \quad \text{strongly in } W_0^{1,p}(\Omega), \\ a(x, u_\varepsilon, \nabla u_\varepsilon) \to a(x, u, \nabla u) \quad \text{weakly in } L^{p'}(\Omega), \\ f(x, u_\varepsilon) \to f(x, u), \quad g(x, u_\varepsilon)|\nabla u_\varepsilon|^p \to g(x, u)|\nabla u|^p \quad \text{weakly in } L^1(\Omega). \end{gathered} \tag{2.14}$$

Proof According to (2.13), and using the reflexivity of $W_0^{1,p}(\Omega)$ and compactness of imbedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, we immediately conclude that there exist a subsequence of u_ε, still denoted by u_ε, and a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that

$$\begin{gathered} u_\varepsilon \to u \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega), \\ \text{a.e. in } \Omega \text{ and weak}^* \text{ in } L^\infty(\Omega). \end{gathered} \tag{2.15}$$

By means of the monotonicity assumption (1.9) we are able to repeat all steps from the proof of [2, Lemma 4, p. 189], and to derive:

$$\int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon) - a(x, u_\varepsilon, \nabla u)] \cdot \nabla(u_\varepsilon - u)\, dx \to 0. \tag{2.16}$$

Now, with the help of the compactness result from [2, Lemma 5, p. 190], and using the convergence result from [9, Lemma 3.2], together with (2.15) and (2.16), we derive all claims in (2.14). ♢

Finally, as a consequence of two previous propositions, we obtain the following statement.

Proposition 2.3 *Under the assumptions of Theorem 1.2, let u_ε be a solution of* (2.8) *satisfying (2.13), and let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a function satisfying (2.14). Then we have:*

(i) $\omega_1 \le u \le \omega_2$ *in* Ω

(ii) u *is a solution of (1.1).*

Proof Using (2.10) we see that in order to prove (i) it suffices to check that $\beta(x,u) = 0$ in Ω. Let us remark that with the help of (1.8), $(H5)$ and (2.13) we obtain the existence of three positive constants c_1, c_2 and c_3 such that

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx < c_1, \quad \Big| \int_\Omega f(x, u_\varepsilon) u_\varepsilon \, dx \Big| < c_2,$$
$$\Big| \int_\Omega g(x, u_\varepsilon) |\nabla u_\varepsilon|^p u_\varepsilon \, dx \Big| < c_3.$$

Furhtermore, testing (2.8) with the function $\varphi = \varepsilon u_\varepsilon$, we obtain

$$\begin{aligned} &\varepsilon \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx + \int_\Omega \beta(x, u_\varepsilon) u_\varepsilon \, dx \\ &= \varepsilon \int_\Omega f(x, u_\varepsilon) u_\varepsilon \, dx + \varepsilon \int_\Omega g(x, u_\varepsilon) |\nabla u_\varepsilon|^p u_\varepsilon \, dx. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, from the previous estimates we obtain

$$\int_\Omega \beta(x, u) u \, dx = 0,$$

which together with the "sign" condition (2.12) implies that $\beta(x, u(x))u(x) = 0$ in Ω. Since $\beta(x, 0) = 0$, we obtain that also $\beta(x, u(x)) = 0$ in Ω.

Now we proceed with the proof of the second claim of the proposition. Testing the penalty equation (2.8) with the function $\varphi = v - u_\varepsilon$, where $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\omega_1 \le v \le \omega_2$ in Ω, we have

$$\begin{aligned} &\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla (v - u_\varepsilon) \, dx + \frac{1}{\varepsilon} \int_\Omega \beta(x, u_\varepsilon)(v - u_\varepsilon) \, dx \\ &= \int_\Omega f(x, u_\varepsilon)(v - u_\varepsilon) \, dx + \int_\Omega g(x, u_\varepsilon) |\nabla u_\varepsilon|^p (v - u_\varepsilon) \, dx. \end{aligned} \tag{2.17}$$

Furthermore, by means of (2.10) and (2.11) we also have

$$\frac{1}{\varepsilon} \int_\Omega \beta(x, u_\varepsilon)(v - u_\varepsilon) \, dx = -\frac{1}{\varepsilon} \int_\Omega (\beta(x, v) - \beta(x, u_\varepsilon))(v - u_\varepsilon) \, dx \le 0. \tag{2.18}$$

From (2.17) and using (2.18) we derive

$$\begin{aligned} &\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla (v - u_\varepsilon) \, dx \\ &\ge \int_\Omega f(x, u_\varepsilon)(v - u_\varepsilon) \, dx + \int_\Omega g(x, u_\varepsilon) |\nabla u_\varepsilon|^p (v - u_\varepsilon) \, dx. \end{aligned} \tag{2.19}$$

Finally, according to (2.14)–(2.15), and passing to the limits in (2.19), we deduce that the function u is a solution of the equation (1.1). ♢

Proof of Theorem 1.3 We assume without loss of generality that $x_0 = 0$, and denote $r = |x|$. Let us fix a ball B_{2r}, $r < R/2$, and let us define

$$\begin{aligned} f_1(x) &= C_1(2r)^{-\gamma}, \quad \overline{m}_1 = \operatorname{ess\,inf}_{B_R}\omega_1, \\ m_2(r) &= C_1(2r)^{-\alpha}, \quad M_c(r) = L \cdot m_2(r), \end{aligned}$$

where $L \in (0,1)$ is a given fixed number and $C_1 > 0$. If we show that for any $r > 0$ sufficiently small condition

$$\int_{B_r} f_1(x)\,dx > D_1(r) \cdot \frac{m_2(r) - \overline{m}_1}{m_2(r) - M_c(r)}, \tag{2.20}$$

is satisfied, see (H2), than the claim will follow from Theorem 1.1, since $M_c(r) \to \infty$ as $r \to 0$, and $\cap_{r>0} B_{2r} = \{0\}$. Denoting the left-hand side of (2.20) by $F(r)$, we have (note that $\gamma < N$ since $f_1(x) \le f(x,u) \in L^1(B_{2r})$):

$$F(r) = \frac{C_1\omega_N N}{2^\gamma(N-\gamma)} r^{N-\gamma}, \tag{2.21}$$

where $\omega_N = |B_1(0)|$. To estimate the right-hand side of (2.20), that we denote by $G(r)$, note that for a given and fixed $k > 1$ we have that $m_2(r) - \overline{m}_1 \le k^{1/p} \cdot m_2(r)$ for all r small enough. Also, the left-hand side of (1.13) can be estimated by $C \cdot r^\beta$, where C is a positive constant. Hence,

$$\begin{aligned} &G(r) \\ &\le [\alpha_0 a_2^{-p'} C \cdot r^\beta + 2k\left(\frac{p-1}{\alpha_0}\right)^{p-1} a_2^p (2^N - 1)\omega_N m_2(r)^p \cdot r^{N-p}] \frac{2^\alpha r^\alpha}{(1-L)C_1} \\ &= D_1 r^{\alpha+\beta} + D_2 r^{-\alpha(p-1)+N-p}, \end{aligned} \tag{2.22}$$

where $D_1 \ge 0$ and $D_2 > 0$ are explicit positive constants independent of r. In order to ensure $F(r) > G(r)$ for all $r > 0$ small enough, see (2.20), it suffices have:

$$\frac{C_1\omega_N}{2^\gamma(N-\gamma)} r^{N-\gamma} > D_1 r^{\alpha+\beta} + D_2 r^{-\alpha(p-1)+N-p},$$

that is,

$$\frac{C_1\omega_N}{2^\gamma(N-\gamma)} r^{\alpha(p-1)+p-\gamma} > D_1 r^{\alpha+\beta+\alpha(p-1)-N+p} + D_2.$$

This inequality holds for all $r > 0$ small enough due to $\alpha(p-1) + p - \gamma < 0$ and $\alpha(p-1) + p - \gamma < \alpha + \beta + \alpha(p-1) - N + p$. $\diamondsuit$

Proof of Theorem 1.4 Assume, contrary to claim of the theorem, that there exists a ball $B_r = B_r(x_0)$ such that $u^*(x) < w_{2*}(x)$ for all $x \in B_r$. Since we have strict inequalities

$$w_1^*(x) < u_*(x), \quad u^*(x) < w_{2*}(x), \quad \forall x \in B_r,$$

it is easy to see that solution u of variational inequality (1.1) is in fact a solution of quasilinear elliptic equation on B_r in the sense of distributions:

$$-\Delta_p u = f(x,u) + g(x,u)\cdot|\nabla u|^p, \quad \text{in } \mathcal{D}'(B_r), \tag{2.23}$$

since for any test function $\varphi \in \mathcal{D}(B_r)$ the function $v = u + t\varphi$ is admissible in (1.1) for t small enough. Therefore,

$$-\Delta_p u \geq \frac{C_1}{|x-x_0|^\gamma}, \quad \text{in } \mathcal{D}'_+(B_r).$$

Since $u \geq 0$ on ∂B_r, then using [16, Theorem 3] we conclude that solution u has singularity at least of order $\frac{\gamma-p}{p-1}$ at x_0. However, this is a impossible, since $u(x)$ is dominated from above by obstacle $w_2(x)$, which has singularity at x_0 at most of order equal precisely to α, see (1.18), such that $\alpha < \frac{\gamma-1}{p-1}$. ♢

Remark Using the same method it is possible to show that Theorem 1.4 holds also for p-Laplace like operators, that is for $a(x,\eta,\xi) = A(x,\xi)$ satisfying Leary–Lions conditions with $|A(x,\xi)| \leq a_2|\xi|^{p-1}$.

Proof of Theorem 1.5 Let us assume that $x_0 = 0$ and denote $r = |x|$. Here we define $f_1(x) = C\cdot|x|^{-\gamma}$, $m_2(r) = \operatorname{ess\,inf}_{B_{2r}} w_2$, $M_c(r) = m_2(r) - r^\varepsilon$, where we take $\varepsilon > 0$ small enough. Similarly as in the proof of Theorem 1.3 we obtain that condition (2.20) is satisfied for all $r > 0$ small enough if we have $r^{N-\gamma} > a\cdot r^{N-p-\varepsilon}$, where a is a positive constant independent of r. Thus we have to secure that $r^{\gamma-p-\varepsilon} < 1/a$ for $r > 0$ small, and this is possible by taking $\varepsilon \in (0,\gamma-p)$. The claim follows from Theorem 1.1, since $M_c(r) \to w_{2*}(x_0)$ as $r \to 0$. ♢

As a final remark, we note that our main Theorem 1.1 can be formulated in a much more general context.

Theorem 2.4 *Assume that $a(x,\eta,\xi)$ satisfies conditions (1.7) and (1.8). Let A be a measurable subset of Ω, such that $A_r \subseteq \Omega$ and $|A_r \setminus A| < \infty$. Let M_c be a given number such that*

$$M_c \in (m_1, m_2), \quad m_i = \operatorname{ess\,inf}_{A_r} \omega_i(x), \quad i = 1,2. \tag{2.24}$$

Assume that

$$g(x,\eta) \geq 0 \quad \text{for a.e. } x \in A_r \quad \text{and} \quad \eta \in I_1 = (m_1, M_c), \tag{2.25}$$

$$\exists f_1 \in L^1(A_r), \quad f(x,\eta) \geq f_1(x) \quad \text{for a.e. } x \in A_r,\ \eta \in I_1, \tag{2.26}$$

$$f_1(x) \geq 0 \quad \text{on } A_r \setminus A. \tag{2.27}$$

Furthermore, assume that

$$\int_A f_1(x)\,dx > D_1 \cdot \frac{m_2 - m_1}{m_2 - M_c}, \tag{2.28}$$

where

$$D_1 = \overline{d} \int_{A_r} [a_0(x) + a_1 \widehat{m}^{p-1}]^{p'} dx + \Big(\frac{p}{d}\Big)^{p-1} \frac{|A_r \setminus A|}{r^p}, \tag{2.29}$$

with $\widehat{m}$, $\overline{d}$ *and* d *defined in the same way as in Theorem 1.1. Then for any solution* u *of (1.1) we have*

$$|\{x \in A_r \, u(x) > M_c\}| \neq 0. \tag{2.30}$$

Proof. The proof is the same as the proof of Theorem 1.1. We only have to change B_r to A, B_{2r} to A_r, and to use our general result about localization of measurable sets stated in [7, Lemma 5].

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