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Compact attractors for a Stefan problem with kinetics *

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Abstract

We prove existence of a unique bounded classical solution for a one-phase free-boundary problem with kinetics for continuous initial conditions. The main result of this paper establishes existence of a compact attractor for classical solutions of the problem.

1 Introduction

In this paper we study the asymptotic behavior of solutions of the modified one-phase Stefan problem in one spatial dimension:

$$u_t = u_{xx} - \gamma u, \quad -\infty < x < s(t), \tag{1.1}$$

$$(\partial u/\partial x)|_{x=s(t)} = -V(t), \quad g(u|_{x=s(t)}) = V(t), \tag{1.2}$$

$$u(x,0) = u^0(x). \tag{1.3}$$

Here $u(x,t)$ is the temperature, the damping term is due to the volumetric heat losses, $\gamma \geq 0$. The two boundary conditions overdetermine the problem and allow us to find the free boundary whose position is denoted by $s(t)$, and $V(t) = \dot{s}(t)$ is the velocity.

The free-boundary problem (1.1-1.3) arises naturally as a mathematical model of a variety of exothermic phase transition type processes, such as condensed phase combustion [11] (also known as self-sustained high-temperature synthesis or SHS [12]), solidification with undercooling [10], laser induced evaporation [9], rapid crystallization in thin films [16] etc. These processes are characterized by production of heat at the interface, and their dynamics is determined by the feedback mechanism between the heat release due to the kinetics $g(u|_{x=s(t)})$ and the heat dissipation by the medium. The first boundary condition in (1.2) (the Stefan boundary condition) expresses the balance between the heat produced at the free boundary and the heat diffusion through the adjacent medium. As the problem (1.1)-(1.3) describes propagation of the phase transition front, the second boundary condition in (1.2) is a manifestation of

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the non-equilibrium nature of the transition; its analog for the classical Stefan problem is just $u|_{x=s(t)} = 0$. In the context of condensed phase combustion the kinetic boundary condition expresses the dependence of the propagation velocity on the flame front temperature.

The principal objective of the paper is to prove existence of a compact attractor for the infinite-dimensional dynamical system generated by the one-phase problem (1.1-1.3). In other words, if the problem is viewed as evolution in a functional space then for all the initial conditions, solutions approach a compact set in the functional space. This result is motivated by results of DNS and in particular by numerical experiments described in [2] and [3] (see also [5]). In [2] we performed DNS of dynamics of the system for a variety of kinetic functions (actually, for a parametric family of kinetic functions). We demonstrated that for a wide range of initial conditions the system develops complex thermo-kinetic oscillations. The resulting asymptotic regimes are attained very fast and do not depend on the initial conditions, thus indicating existence of an attractor for the dynamics. As the parameter governing kinetics is varied, the dynamical patterns exhibited by the system include a Hopf bifurcation, period doubling cascades leading to chaotic pulsations, a Shilnikov-Hopf bifurcation etc. Most of these patterns appear to be of a finite-dimensional nature and are well-known for the finite-dimensional dynamical systems.

On the other hand, in [3] we derived and studied a 3×3 system of ODEs which is a three-mode pseudo-spectral approximation to the free-boundary problem: the dynamics of the finite-dimensional dynamical system mimic those of the infinite-dimensional system to an amazing degree. These observations led to the conjecture that the asymptotic dynamics of (1.1-1.3) should be finite-dimensional. The result of this paper is therefore a first step in proving this conjecture. In the sequel to this paper [6] (to be published elsewhere), based on the compactness results we demonstrate that the attractor actually has a finite Hausdorff dimension.

In a nutshell, the idea of the proof of existence of a compact attractor is as follows: we start with initial data inside a ball in a functional space and establish uniform a priori estimates on the solutions and their first spatial derivative that secure the existence of an absorbing compact set. Then we use the simple abstract result from dynamical systems that the ω-limit set of a compact absorbing set is a compact attractor. It should be noted however that the actual proof is not as straightforward as just indicated. If the solution is viewed as a sum of contributions from the free boundary and from initial data, it is easy to see that while the former does compactify with time, the latter cannot compactify. Fortunately the damping guarantees this contribution to decay exponentially. Also, since we are working on an infinite spatial interval, the compactness is predicated on a uniform spatial decay, which *can* be justified for the contribution from the free boundary.

The rest of the paper is organized as follows. In Sec. 2 we present some minimal background information that places the model in the context of condensed phase combustion. A local existence result is obtained in Sec. 3. We are not interested in local existence per se since it is well established in the literature for various versions of the problem, see for example [13, 19, 1].The principal value of this result for our purposes is in the direct construction for a solution that allows us to derive necessary estimates for the solution and its spatial derivative. A solution is found in the form of a single layer potential. We demonstrate that the potential density necessarily should have a one over the square root singularity. We develop a suitable version of potential theory and design a convergent iteration scheme that produces the density and the free boundary velocity.

The single-layer representation for the solution is employed in Sec. 4 to obtain an estimate for the spatial derivative of the solution. This estimate can be proved uniform with respect to the sup norm of initial data, provided the solution itself is uniformly bounded. Such a uniform bound is also presented in Sec. 4: namely, assuming that the kinetic function g satisfies some natural requirements, we demonstrate an a priori estimate on the solutions uniformly with respect to the initial conditions. The estimate is based on a representation of the solution in terms of a combination of single and double layer potentials. The latter representation is instrumental for the argument of Sec. 5, since it gives a natural decomposition of the solution into the two contributions, one from the initial conditions, and another one from the free boundary.

We show that the contributions from the free boundary are uniformly bounded and decaying at infinity. Together with the uniform bound on the spatial derivative, this guarantees that contributions from the free boundary for different initial data from a fixed ball form a precompact set. At this point the presence of the heat loss becomes crucial. Although the contribution from initial data is not compactified by the evolution, it decays exponentially with time due to the damping. Thus the evolution is a combination of a compactifying part and a decaying part. We complete the proof of existence of a compact attractor by incurring an appropriate abstract result from dynamical systems.

2 Motivation of the free boundary model

In this section we sketch a derivation of the free boundary model (1.1)-(1.3). We note that although the derivation is purely heuristic, the model itself is well accepted in the literature. The interpretation of the free boundary problem (1.1)-(1.3) in terms of nonequilibrium phase transition with the interface attachment kinetics g is rather transparent (see e.g. [16]), here we would like to demonstrate its relevance for the condensed phase combustion. For this type of combustion the solid fuel mixture is transformed directly into a solid product. In addition to its theoretical interest, gasless combustion currently finds technological applications as a method of synthesizing certain technologically advanced materials for high-temperature semiconductors, nuclear safety devices, fuel cells etc., see [12], [17] and also [18] for a popular exposition. The process is characterized by highly exothermic reactions propagating through mixtures of fine elemental reactant powders (e.g., Ti + C, Ti +2B), resulting in the synthesis of compounds.

The most primitive model of gasless combustion involves a system of differential equations for the temperature u and the concentration of the fuel C (see Shkadinsky *et al.*, [14]). For appropriately nondimensionalized variables the one-dimensional formulation of the model takes the form:

$$u_t = (\kappa u_x)_x + qW(C,u) - \gamma u \tag{2.1}$$

$$C_t = -W(C,u) \tag{2.2}$$

where κ is the thermal diffusivity, W is the chemical reaction rate, q is the heat release and γu, the volumetric heat loss.

For physically relevant values of parameters, the system is characterized by the strong temperature sensitivity of the rate and by rather sharply defined regions of dramatic change in the field variables that are usually associated with propagating fronts. This suggests an alternative to the model with distributed kinetics (the so-called flame sheet approximation, see [20]): the distributed reaction rate in (2.1)–(2.2) is replaced by the δ-function,

$$W = g(u)\delta(x - s(t))$$

with an appropriate rate $g(u)$ supported at the interface $x = s(t)$ between the fresh ($C = 1$) and burnt ($C = 0$) material (see, Matkowsky & Sivashinsky, [11]). In the case of gaseous combustion with Arrhenius kinetics, the sharp interface model can be obtained as an asymptotic approximation of the distributed kinetics model in the large activation energy limit. In this case the strength of the δ-function $g(u)$ is determined through an asymptotic analysis by matching relevant inner and outer solutions. Of course, all the intricacies of the behavior in the reaction zone are lost in this approximation.

The system (2.1)–(2.2) with the δ-function source is understood in the sense of distributions. This leads to the system of two heat equations coupled at the interface

$$\begin{gathered} u_t^- = (\kappa u_x^-)_x - \gamma u, \quad u_t^+ = (\kappa u_x^+)_x - \gamma u \\ u^-|_{x=s(t)} = u^+|_{x=s(t)}, \quad (\kappa u_x^+ - \kappa u_x^-)_{x=s(t)} = -w(u)_{x=s(t)} \\ \frac{ds}{dt} = -g(u)|_{x=s(t)} \end{gathered} \tag{2.3}$$

where

$$\begin{aligned} u^-(x,t) &= u(x,t) \quad \text{for} \quad x < s(t), \\ u^+(x,t) &= u(x,t) \quad \text{for} \quad x > s(t). \end{aligned}$$

This is the free interface *two-phase* problem of condensed phase combustion. The physical properties of the material such as the heat diffusion coefficient κ may differ substantially ahead and behind the interface. If, for instance, the product is a foam-like substance then $\kappa_{product} \ll \kappa_{fuel}$. By setting $\kappa_{product} = 0$ in the equation and the boundary condition for u^+ in (2.3), we arrive at the *one-phase* model problem in (1.1)-(1.2) for $u \equiv u^+$.

We note that in the context of solidification of overcooled liquids or the amorphous to crystalline transition the kinetic boundary condition corresponds to the so-called interface attachment kinetics, which are determined by various microscopic mechanisms of incorporating the matter into the crystalline lattice at the interface. Concerning the choice of the kinetic function we remark that this issue is far from settled either theoretically or experimentally. For example, for solid combustion the widely used exponential approximation of Arrhenius kinetics has not been obtained from an analysis of molecular collisions in the spirit of the kinetic theory of gases and, consequentally, asymptotic expansion in transition to the δ-function approximation, but rather "transplanted" from the sharp interface model of gas combustion. There are several types of functions that were suggested for a more realistic description of kinetics in specific chemical and physical settings.

We will assume that $g(u)$ is a monotonically decreasing differentiable function on $[0,\infty]$ with $|g'| \le C$ and satisfying

$$-V_0 \le g(u) \le -v_0 \text{ for some } V_0, v_0 > 0. \tag{2.4}$$

The lower bound is satisfied for the standard Arrhenius kinetics where $V = ce^{-A/u}$ while the upper bound v_0 corresponds to the ignition temperature (in our case, "ignition velocity") kinetics: the model is valid only for moving fronts.

3 Existence of local classical solutions

In order not to clutter formulas with factors of the type $e^{-\gamma t}$, from now on, until Sec. 5 we set the damping coefficient $\gamma = 0$. The modifications to the $\gamma > 0$ case are trivial and will be indicated when needed. A short-time solution of the free boundary problem (1.1-1.3) will be sought in the form of a superposition of heat potentials,

$$u(x,t) = \int_0^t G(x, s(\tau), t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G(x,\xi,t)u^0(\xi)d\xi, \tag{3.1}$$

where G is the fundamental solution of the heat equation,

$$G(x,\xi,t-\tau) = \exp\left\{-\frac{(x-\xi)^2}{4(t-\tau)}\right\}[4\pi(t-\tau)]^{-1/2}$$

The density of the single layer potential φ and the front position $s(t)$ are to be determined.

We will demonstrate a little later that the single-layer potential is continuous up to the boundary and its derivative possesses the standard jump property:

$$\lim_{x\to s(t)-}\frac{\partial}{\partial x}\int_0^t G(x,s(\tau),t-\tau)\varphi(\tau)d\tau = \frac{\varphi(t)}{2} + \int_0^t G_x(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau \tag{3.2}$$

This result is, of course, well-known if φ is continuous. It turns out however, that by the nature of the free-boundary problem at hand, φ must have a $1/\sqrt{t}$ singularity at 0. Thus a justification of (3.2) will require an extra effort. If the jump property in (3.2) holds then for the solution represented by (3.1), the boundary conditions in (1.2) yield the following equations

$$\begin{aligned} u(s(t),t) &= g^{-1}(V(t)) \qquad (3.3)\\ &= \int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi \\ u_x(s(t),t) &= -V(t) \qquad (3.4)\\ &= \frac{\varphi}{2} - \int_0^t G_\xi(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau - \int_{-\infty}^0 G_\xi(s(t),\xi,t)u^0(\xi)d\tau \end{aligned}$$

We will choose the density of the form $\varphi(t) = \psi(t)/\sqrt{t}$,where $\psi(t)$ is continuous on $[0,T]$. To motivate this choice, let us consider asymptotics of (3.4) as $t \to 0$. Let us assume for simplicity of the argument that $u^0 \in C^1$ and V is continuous on $[0,T]$. First we integrate by parts the second integral in (3.4) and note that it has a $1/\sqrt{t}$ singularity:

$$\begin{aligned} &-\int_{-\infty}^0 G_\xi(s(t),\xi,t)u^0(\xi)d\xi \\ &= -u^0(0)\frac{\exp\{-s(t)^2/4t\}}{\sqrt{4\pi t}} + \int_{-\infty}^0 G(s(t),\xi,t)u^0_\xi(\xi)d\xi \qquad (3.5)\\ &\sim -u^0(0)\frac{\exp\{-V(0)^2t/4\}}{\sqrt{4\pi t}} + \frac{u^0_\xi(0)}{2} \end{aligned}$$

As to the first integral in (3.4), for continuous φ it converges to 0 as $t \to 0$:

$$\begin{aligned} \Big|\int_0^t G_\xi(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau\Big| &= \frac{1}{2}\Big|\int_0^t \frac{s(t)-s(\tau)}{t-\tau}G\varphi(\tau)d\tau\Big| \\ &\sim \frac{1}{2}|V(0)|\sup|\varphi|\sqrt{t}, \qquad (3.6) \end{aligned}$$

since $|G(\cdot,\cdot,t-\tau)| \le 1/\sqrt{t-\tau}$. Thus, for a continuous φ the singularities in (3.4) cannot balance.

If φ has a singularity of the type $b/\sqrt{t}$ then the estimate in (3.6) should be augmented by the term

$$\begin{aligned} &\int_0^t G_\xi(s(t),s(\tau),t-\tau)\frac{b}{\sqrt{\tau}}d\tau \\ &= \frac{b}{2}\int_0^t \frac{s(t)-s(\tau)}{t-\tau}\frac{\exp\{-(s(t)-s(\tau))^2/4(t-\tau)\}}{\sqrt{4\pi(t-\tau)\tau}}d\tau \\ &\sim \frac{b}{2}V(0)\exp\{-V(0)^2t/4\}\int_0^t \frac{d\tau}{\sqrt{4\pi(t-\tau)\tau}} \\ &= \frac{b}{4}V(0)\sqrt{\pi}\exp\{-V(0)^2t/4\} \end{aligned}$$

which converges to the finite value. Thus, the only way to balance the singularity (3.5) in the boundary condition in (3.4) is for φ itself to have a singularity. The balance condition then reads:

$$\lim_{t\to 0}\sqrt{t}\varphi(t) = u^0(0)/\sqrt{\pi} \tag{3.7}$$

A similar limit obtained from the first integral equation (3.3) leads to the initial condition for V:

$$V(0) = g(u^0(0)) \tag{3.8}$$

Next we rewrite the integral equations in (3.3)-(3.4) in terms of φ and V:

$$V = K_1(V,\varphi) \tag{3.9}$$

$$\varphi = -2K_1(V,\varphi) + K_2(V,\varphi) \tag{3.10}$$

where the nonlinear operators K_1, K_2 are defined as follows

$$K_1(V,\varphi) = g\{\int_0^t G(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi\} \tag{3.11}$$

$$K_2(V,\varphi) = 2\int_0^t G_\xi(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau + 2\int_{-\infty}^0 G_\xi(s(t),\xi,t)u^0(\xi)d\xi \tag{3.12}$$

Here as usual,

$$s(t) = \int_0^t V(\tau)d\tau. \tag{3.13}$$

The equations are supplemented by the initial conditions:

$$V(0) = g(u^0(0)); \qquad \lim_{t\to 0}\sqrt{t}\varphi(t) = u^0(0)/\sqrt{\pi} \tag{3.14}$$

The principal goal of the present section is the proof of the following local existence result:

Theorem 3.1 *Let $g < 0$ be continuously differentiable, monotone decreasing function, $u^0 \in C(-\infty,0]$, $u^0 > 0$. Then the problem in (3.9)-(3.10) has a unique solution V,φ such that V and $\sqrt{t}\varphi(t)$ are continuous on $[0,\sigma]$ for some $\sigma > 0$, where σ depends only on $\sup u^0$. The solution to the free boundary problem is determined by V,φ via the representation (3.1) with $s(t) = \int_0^t V(\tau)d\tau$.*

The proof of the theorem is given in the next subsections. Its outline is as follows. First of all, we justify the integral equations by establishing the single-layer potential jump property (3.2) for densities with the $1/\sqrt{t}$ singularity. Then we demonstrate that the solution of the system of integral equations (3.9)-(3.10) generates a solution to the free boundary problem via the representation (3.1). After that we concentrate on existence for the system of integral equations. We show that, if $\sigma > 0$ is small enough, the integral operator is a contraction.

It should be noted that the singularity in the potential density precludes a simple-minded iteration scheme from being a contraction. Roughly speaking, the contraction rate for nonsingular densities is on the order of $\sqrt{\sigma}$. The $1/\sqrt{t}$ singularity leads to a "cancelation" (the rate of order one) and prevents us from making the rate coefficient smaller than one. To overcome this difficulty we introduce a two-step iteration scheme.

Another standard precaution should be taken for the proof to proceed. Because of the nonlinearity of the problem the contraction rate depends on the size of $\{V, \varphi\}$. Thus to guarantee that the iteration sequence does not deteriorate the contraction rate and therefore requires smaller and smaller σ, we need to secure the existence of a ball in the functional space which is mapped by the operator into itself.

All the results of the section hold without the basic assumption on the kinetic function in (2.4). Nonetheless, we do not hesitate to assume it whenever it leads to a substantial simplification of the presentation.

3.1 Two lemmas on the single-layer potential

In this section we study properties of the single-layer potential whose density has a one over square root singularity. For our purposes it is convenient to introduce a norm which is appropriate for functions with this singularity:

$$\|\varphi\|_\sigma = \sup_{0\le\tau\le\sigma} \sqrt{\tau}|\varphi(\tau)| \tag{3.15}$$

Obviously, if $\varphi(t) = \psi(t)/\sqrt{t}$, where $\psi(t)$ is continuous , then $\|\varphi\|_\sigma = \|\psi\|_{C[0,\sigma]}$.

Specifically we are interested in the behavior of the spatial derivative of the potential and its limit at the boundary.

Lemma 3.2 *Let $\varphi(t) = \psi(t)/\sqrt{t}$, where $\psi(t)$ is a continuous function on $[0, T]$ and let $s(t)$ be Lipschitz continuous on $[0, T]$ and non-increasing. Then for every $0 < t \le T$, and $x < s(t)$*

$$|\Phi(x, t)| = |\frac{\partial}{\partial x}\int_0^t G(x, s(\tau), t-\tau)\varphi(\tau)d\tau| \le \text{const} \tag{3.16}$$

Proof The lemma holds if monotonicity condition for s is dropped, but in our case s is monotone which simplifies the proof. It is convenient to consider separately the two cases: $|s(t) - x| > 1$ and $|s(t) - x| < 1$.

For the case $|s(t)-x|>1$

$$
\begin{aligned}
|\Phi(x,t)| =& \Big|\int_0^t \frac{x-s(\tau)}{2(t-\tau)}\frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}}\varphi(\tau)d\tau\Big| \\
=& \Big|\int_0^t \frac{(x-s(\tau))^2}{2(t-\tau)(x-s(\tau))}e^{-(x-s(\tau))^2/8(t-\tau)} \\
& \times \exp\{-\frac{(x-s(t))^2+2(x-s(t))(s(t)-s(\tau))+(s(t)-s(\tau))^2}{8(t-\tau)}\} \\
& \times \frac{\psi(\tau)}{\sqrt{4\pi\tau(t-\tau)}}\,d\tau\Big| \\
\le& \frac{C\,\|\varphi\|_t}{|s(t)-x|}e^{-v_0|x-s(t)|/4}\int_0^t \frac{e^{-v_0^2(t-\tau)/8}}{\sqrt{4\pi\tau(t-\tau)}}d\tau \le \frac{Ce^{-v_0|x-s(t)|/4}}{|s(t)-x|}\,\|\varphi\|_t
\end{aligned}
$$

In the last estimate we used the following simple observations: $\eta e^{-\eta}\le$const, for $\eta=\dfrac{(x-s(\tau))^2}{4(t-\tau)}>0$, $|s(\tau)-x|>|s(t)-x|$ and

$$
\int_0^t 1/\sqrt{\tau(t-\tau)}d\tau=\pi. \tag{3.17}
$$

Remark 3.3 *Thus the proof above shows that if $|s(t)-s(\tau)|\ge v_0|t-\tau|$ which holds if the basic assumption on the kinetics in (2.4) is satisfied, then the derivative decays exponentially*

$$
|\Phi(x,t)|\le \frac{Ce^{-v_0|x-s(t)|/4}}{|s(t)-x|}\,\|\varphi\|_t \tag{3.18}
$$

The exponent $-v_0/4$ can be improved to $-v_0/(2+\varepsilon)$ (at the price of increasing C).

For the case $|s(t)-x|<1$ we split the integral into the two parts

$$
\begin{aligned}
&\frac{\partial}{\partial x}\int_0^t G(x,s(\tau),t-\tau)\varphi(\tau)d\tau \\
&\qquad\qquad = -\Big[\int_0^{t-\delta}+\int_{t-\delta}^t\Big]\frac{x-s(\tau)}{2(t-\tau)}G(x,s(\tau),t-\tau)\varphi(\tau)d\tau,
\end{aligned}
$$

where $0<\delta<t$ to be chosen later on. In the estimates below we follow rather closely the argument from Friedman [8] (inequality (1.18), p.219):

$$
\begin{aligned}
&\int_{t-\delta}^t \frac{x-s(\tau)}{2(t-\tau)}G(x,s(\tau),t-\tau)\varphi(\tau)d\tau \\
&= \int_{t-\delta}^t \frac{x-s(t)}{2(t-\tau)}G(x,s(\tau),t-\tau)\varphi(\tau)d\tau \\
&\quad + \int_{t-\delta}^t \frac{s(t)-s(\tau)}{2(t-\tau)}G(x,s(\tau),t-\tau)\varphi(\tau)d\tau \\
&= I_1+I_2
\end{aligned}
$$

We shall estimate the difference between I_1 and

$$J_1 = \int_{t-\delta}^{t} \frac{x-s(t)}{2(t-\tau)} G(x,s(t),t-\tau)\varphi(\tau)d\tau$$

as follows

$$|I_1 - J_1| = \Big| \int_{t-\delta}^{t} \frac{x-s(t)}{2(t-\tau)} G(x,s(t),t-\tau)\Big\{1 - e^{\frac{(x-s(t))^2-(x-s(\tau))^2}{4(t-\tau)}}\Big\}\varphi(\tau)d\tau\Big| \quad (3.19)$$

Since obviously $1-\exp(-\eta) < \eta$ for any $\eta > 0$, the expression in the braces is estimated:

$$\begin{aligned} 0 &< 1-\exp[\frac{(x-s(t))^2-(x-s(\tau))^2}{4(t-\tau)}] < \frac{s(\tau)-s(t)}{4(t-\tau)}[s(t)-x+s(\tau)-x] \\ &= \frac{s(\tau)-s(t)}{4(t-\tau)}[2(s(t)-x)+s(\tau)-s(t)] \le \frac{V_0}{4}[2(s(t)-x)+s(\tau)-s(t)] \end{aligned}$$

here V_0 is the Lipschitz constant for $s(t)$ (the maximal velocity). We note now that

$$\sup_{t-\delta\le\tau\le t} |\varphi(\tau)| = \sup_{t-\delta\le\tau\le t} (|\varphi(\tau)|\sqrt{\tau})|/\sqrt{\tau} \le \|\varphi\|_t/\sqrt{t-\delta}$$

and continue (3.19):

$$\begin{aligned} &|I_1 - J_1| \\ &\le \int_{t-\delta}^{t} \frac{s(t)-x}{2(t-\tau)} G(x,s(t),t-\tau)\frac{V_0}{4}[2(s(t)-x)+s(\tau)-s(t)]\frac{\|\varphi\|_t}{\sqrt{t-\delta}}d\tau \\ &= \frac{V_0}{4}\frac{\|\varphi\|_t}{\sqrt{t-\delta}} \int_{t-\delta}^{t} \Big\{\frac{[s(t)-x]^2}{(t-\tau)} + \frac{s(\tau)-s(t)}{2(t-\tau)}[s(t)-x]\Big\} e^{-\frac{(x-s(t))^2}{4(t-\tau)}} \frac{d\tau}{\sqrt{4\pi(t-\tau)}} \\ &\le \frac{V_0}{4}\frac{\|\varphi\|_t}{\sqrt{t-\delta}}\sqrt{\delta/\pi}\Big\{C + \sqrt{\delta}\frac{V_0}{2}[s(t)-x]\Big\}. \end{aligned}$$

In the last inequality we have used $\eta^p\exp(-\eta) \le C$, for any $p > 0$.

The integral J_1 can be reduced via a substitution $4(t-\tau)/[s(t)-x]^2 = z$ as follows

$$\begin{aligned} |J_1| &= \int_{t-\delta}^{t} \frac{s(t)-x}{4\sqrt{\pi}(t-\tau)^{3/2}} e^{-\frac{(x-s(t))^2}{4(t-\tau)}}|\varphi(\tau)|d\tau \\ &\le \frac{\|\varphi\|_t}{\sqrt{t-\delta}}\frac{1}{2\sqrt{\pi}}\int_0^{\delta/[s(t)-x]^2} z^{-3/2}e^{-1/z}dz \end{aligned}$$

Since $\frac{1}{\sqrt{\pi}}\int_0^\infty z^{-3/2}e^{-1/z}dz = 1/2$ and the integrand is positive we have

$$|J_1| < \frac{1}{2}\frac{\|\varphi\|_t}{\sqrt{t-\delta}}.$$

Now we need to estimate I_2.

$$\begin{aligned} |I_2| &= \Big|\int_{t-\delta}^{t} \frac{s(t)-s(\tau)}{2(t-\tau)} G(x,s(\tau),t-\tau)\varphi(\tau)d\tau\Big| \\ &\le \frac{V_0}{2}\frac{\|\varphi\|_t}{\sqrt{t-\delta}}\int_{t-\delta}^{t} G(x,s(\tau),t-\tau)d\tau \le \frac{V_0}{2}\frac{\|\varphi\|_t}{\sqrt{t-\delta}}\sqrt{\delta/\pi} \end{aligned}$$

Finally, for $|x-s(t)|<1$ we get that on the interval $t-\delta\le\tau\le t$

$$\begin{aligned} &\Big|\int_{t-\delta}^{t} \frac{x-s(\tau)}{2(t-\tau)} G(x,s(\tau),t-\tau)\varphi(\tau)d\tau\Big| \\ &\le |I_1|+|I_2| \le |I_1-J_1|+|J_1|+|I_2| \\ &\le \frac{V_0}{4}\frac{\|\varphi\|_t}{\sqrt{t-\delta}}\sqrt{\delta/\pi}\Big\{C+\sqrt{\delta}\frac{V_0}{2}[s(t)-x]\Big\} + \frac{1}{2}\frac{\|\varphi\|_t}{\sqrt{t-\delta}} + \frac{V_0}{2}\frac{\|\varphi\|_t}{\sqrt{t-\delta}}\sqrt{\delta/\pi} \\ &= \frac{\|\varphi\|_t}{\sqrt{t-\delta}}\Big[\frac{1}{2}+C_1\sqrt{\delta/(t-\delta)}\Big]. \end{aligned}$$

As for the estimate on the interval $0\le\tau\le t-\delta$ for $|x-s(t)|<1$ we get

$$\Big|\int_0^{t-\delta} \frac{x-s(\tau)}{2(t-\tau)}\frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}}\varphi(\tau)d\tau\Big| \le C_2\frac{\|\varphi\|_t}{\delta}\sqrt{t-\delta}$$

Now, by combining the estimates above

$$|\Phi(x,t)| \le C_2\frac{\|\varphi\|_t}{\delta}\sqrt{t-\delta} + \frac{\|\varphi\|_t}{\sqrt{t-\delta}}\Big[\frac{1}{2}+C_1\sqrt{\delta/(t-\delta)}\Big]$$

we conclude the proof of the lemma for $|x-s(t)|<1$. It is possible to optimize the above estimate by choosing an appropriate δ. However for our purposes it will suffice to set $\delta = ct$ that results in

$$|\Phi(x,t)| \le C\,\|\varphi\|_t/\sqrt{t} \tag{3.20}$$

$\diamondsuit$

Remark 3.4 The above estimate for the derivative Φ is obtained for the density $\varphi = \psi(t)/\sqrt{t}$. If φ itself is a continuous function then the above estimate becomes

$$|\Phi(x,t)| \le C\,\|\varphi\|_t/\sqrt{t} = C\sup_{0\le\tau\le t}|\varphi(\tau)\sqrt{\tau}|/\sqrt{t} \le C\sup_{0\le\tau\le t}|\varphi(\tau)| \tag{3.21}$$

The next lemma presents a version of the classical jump property for the single-layer potential with singularity.

Lemma 3.5 *Let $\varphi(t)=\psi(t)/\sqrt{t}$, where $\psi(t)$ is a continuous function on $[0,T]$ and let $s(t)$ be Lipschitz continuous on $[0,T]$ and non-increasing. Then for every $0<t\le T$*

$$\lim_{x\to s(t)-}\Phi(x,t) = \frac{1}{2}\varphi(t) + \int_0^t G_x(s(t),s(\tau),t-\tau)\varphi(\tau)d\tau$$

Proof As in the previous lemma the result holds if the monotonicity condition for s is dropped. For $\varphi(t)$ continuous the result is contained in Friedman [8]. The proof for our case follows the continuous case with relatively minor modifications.

It is easy to see that for any fixed $\delta > 0$

$$\int_0^{t-\delta} \frac{x-s(\tau)}{2(t-\tau)} \frac{e^{-\frac{(x-s(\tau))^2}{4(t-\tau)}}\varphi(\tau)d\tau}{\sqrt{4\pi(t-\tau)}} \to \int_0^{t-\delta} \frac{s(t)-s(\tau)}{2(t-\tau)} \frac{e^{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}}\varphi(\tau)d\tau}{\sqrt{4\pi(t-\tau)}}$$

as $x \to s(t)_-$ since the singularity at $\tau = 0$ is integrable. On the other hand, on the interval $[t-\delta, t]$ the density $\varphi(\tau)$ is nonsingular and the classical argument shows that

$$\int_{t-\delta}^{t} \frac{x-s(\tau)}{2(t-\tau)} \frac{e^{-\frac{(x-s(\tau))^2}{4(t-\tau)}}\varphi(\tau)d\tau}{\sqrt{4\pi(t-\tau)}} \to \int_{t-\delta}^{t} \frac{s(t)-s(\tau)}{2(t-\tau)} \frac{e^{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}}\varphi(\tau)d\tau}{\sqrt{4\pi(t-\tau)}} + \frac{\varphi(\xi)}{2}$$

where $t-\delta \le \xi \le t$. By passing to the limit $\delta \to 0$ one obtains the result of the lemma. $\diamondsuit$

Now consider the integral representation (3.1) with φ, V ($s(t) = \int_0^t V(\tau)d\tau$) being a solution of the system of integral equations (3.9)-(3.10). Since G is a fundamental solution of the heat equation, for $x < s(t)$, $t > 0$, $u(x,t)$ solves the heat equation. Similar to the argument in the proof of the lemma, it is easy to show that $\lim_{x\to s(t)-} u(x,t)$ exists and is equal to the right hand side of the integral equation (3.3). Thus, by the virtue of the integral equation the kinetic boundary conditions is satisfied. The Stefan boundary conditions is nothing else than the integral equation in (3.4) which is justified through the lemma. Finally, for $x < 0$ it is easily seen that $\lim_{t\to 0} u(x,t) = u^0(x)$.

3.2 Iteration scheme

The system of integral equations in (3.9)-(3.10) will be solved iteratively. Given $\phi = (\varphi, V)$ we define the operator $K : (\varphi, V) \to \omega = (\chi, v)$ through the following two-stage procedure. First we define

$$\begin{aligned}\chi &= -2V + K_2(V,\varphi)\\ &= -2V + 2\int_0^t G_\xi(S(t), S(\tau), t-\tau)\varphi(\tau)d\tau + 2\int_{-\infty}^0 G_\xi(S(t), \xi, t)u^0(\xi)d\xi\end{aligned}$$

where $S(t) = \int_0^t V(\tau)d\tau$. Then, on the base of the just found χ we compute v:

$$v = K_1(V,\chi) = g\{\int_0^t G(S(t), S(\tau), t-\tau)\chi(\tau)d\tau + \int_{-\infty}^0 G(S(t), \xi, t)u^0(\xi)d\xi\}$$

We will show that K has a fixed point, which obviously provides a solution to the original integral equations (3.9)-(3.10), (3.13)-(3.14).

3.3 Invariant ball

We start with the following remark. Based on its physical interpretation, the kinetic function $g(u)$ is defined for $0 < u < \infty$ and varies in the interval $-V_0 < V < -v_0 = g(0)$. It is not clear a priori, whether the integral operator K preserves the "physical" cone of positive temperatures. To avoid complications caused by using an iteration scheme in the set $\{g(K_1(V,\varphi)) > 0\}$, we extend the function g to the interval $(-\infty, 0)$ as $g(u) \equiv -v_0$. We abuse the notation slightly using the same letter for the extension (which has the same Lipschitz constant as the original g).

In the space of pairs $\Xi = \{\phi = (\varphi, V) : \varphi(.)\sqrt{.}, V \in C[0,\sigma]\}$ we define the norm

$$\|\phi\| = \max\{\|\varphi\|_\sigma, \|V\|_{C[0,\sigma]}\} = \max\{\|\varphi(.)\sqrt{.}\|_{C[0,\sigma]}, \|V\|_{C[0,\sigma]}\},$$

that makes Ξ a Banach space. The fixed point will be sought in the closed set $B_{M,\sigma} = \{\phi = (\varphi, V) : -V_0 \le V \le -v_0, \|\varphi\|_\sigma \le M\}$ with M and σ to be determined.

First we note that the velocity component of the operator automatically remains in $B_{M,\sigma}$ by virtue of the definition of g:

$$-V_0 \le g\{\int_0^t G(s(t), s(\tau), t-\tau)\varphi(\tau)d\tau \ + \int_{-\infty}^0 G(s(t), \xi, t)u^0(\xi)d\tau\} \le -v_0$$

In a similar fashion, for the φ-component of $K\phi$ we obtain:

$$\begin{aligned} \|\chi\|_\sigma &= \sup_{0\le t\le\sigma} \sqrt{t}\Big(2V + 2|\int_0^t G_\xi(S(t), S(\tau), t-\tau)\varphi(\tau)d\tau| \\ &\quad +2|\int_{-\infty}^0 G_\xi(S(t), \xi, t)u^0(\xi)d\xi|\Big) \end{aligned}$$

To estimate the first integral we again use (3.17),

$$\begin{aligned} &\Big|\int_0^t G_\xi(S(t), S(\tau), t-\tau)\varphi(\tau)d\tau\Big| \\ &= \Big|\int_0^t \frac{S(t)-S(\tau)}{2(t-\tau)} G(S(t), S(\tau), t-\tau)\varphi(\tau)d\tau\Big| \\ &\le \int_0^t \frac{1}{2}|V(\theta)| \frac{1}{\sqrt{4\pi(t-\tau)}\sqrt{\tau}} |\varphi(\tau)|\sqrt{\tau}d\tau \le \frac{\sqrt{\pi}}{2} V_0 M \end{aligned}$$

The second integral is treated as follows:

$$\begin{aligned} &\Big|\int_{-\infty}^0 \frac{\xi - S(t)}{2t} \frac{e^{-(\xi - S(t))^2/4t}}{\sqrt{4\pi t}} u^0(\xi)d\xi\Big| \\ &= \frac{1}{\sqrt{t}}\Big|\int_{-\infty}^0 \sqrt{8}\frac{\xi - S(t)}{\sqrt{8t}} e^{-(\xi-S(t))^2/8t}\sqrt{2}\frac{e^{-(\xi-S(t))^2/8t}}{\sqrt{8\pi t}} u^0(\xi)d\xi\Big| \\ &\le \frac{4}{e\sqrt{t}}\|u^0\|. \end{aligned}$$

Thus

$$\|\chi\|_\sigma \le \frac{\sqrt{\pi}}{2} V_0 M \sqrt{\sigma} + \frac{4}{e}\|u^0\|$$

If the right side of the above inequality is less or equal than M then K will map $B_{M,\sigma}$ into itself. This is insured by choosing $\sqrt{\sigma} < 2/(V_0\sqrt{\pi})$ and consequently

$$M \ge \frac{8\|u^0\|}{e(2 - V_0\sqrt{\pi\sigma})}$$

3.4 Iteration for density

Now we will prove that for a sufficiently small σ, K is a contraction in the density component. Let $\omega = K\phi$, $\omega' = K\phi'$. For the χ-component of $\omega-$ ω' the estimates are as follows,

$$\begin{aligned}
&|\chi - \chi'| \\
&\quad \le \; 2\|V - V'\| \qquad\qquad (3.25)\\
&\qquad +2\Big|\int_{-\infty}^{0} G_\xi(S(t),\xi,t)u^0(\xi)d\xi - \int_{-\infty}^{0} G_\xi(S'(t),\xi,t)u^0(\xi)d\xi\Big| \\
&\qquad +2\Big|\int_0^t G_\xi(S(t),S(\tau),t-\tau)\varphi(\tau)d\tau - \int_0^t G_\xi(S'(t),S'(\tau),t-\tau)\varphi'(\tau)d\tau\Big| \\
&\quad = \; 2\|V - V'\| + 2\,|w_1| + 2|w_2|
\end{aligned}$$

First we estimate w_1. Suppose $S(t) < S'(t) < 0$ and split the integral for w_1 into three integrals:

$$w_1 = \int_{-\infty}^{S} \delta G_\xi u^0 d\xi + \int_{S}^{S'} \delta G_\xi u^0 d\xi + \int_{S'}^{0} \delta G_\xi u^0 d\xi \qquad (3.26)$$

where

$$\delta G_\xi = G_\xi(S(t),\xi.t) - G_\xi(S'(t),\xi,t).$$

By the mean value theorem,

$$\begin{aligned}
|\delta G_\xi| &= \Big|(S - S')\frac{\partial^2 G}{\partial x^2}(\tilde{s}-\xi,0,t)\Big| \\
&= \frac{1}{\sqrt{4\pi}}\,|S(t) - S'(t)|\,\Big|\frac{(\tilde{s}-\xi)^2}{4t^{5/2}} - \frac{1}{2t^{3/2}}\Big|\, e^{-\frac{(\tilde{s}-\xi)^2}{4t}}
\end{aligned}$$

where $\tilde{s}$ is an intermediate value:

$$\tilde{s} = \tilde{s}(t,\xi), \quad S(t) \le \tilde{s} \le S'(t).$$

Thus for the first integral in (3.26) we have

$$
\begin{aligned}
&|\int_{-\infty}^{S} \delta G_\xi u^0 d\xi| \\
\le\ & \frac{|S(t)-S'(t)|}{t}\int_{-\infty}^{S}\Big|\frac{(\tilde{s}-\xi)^2}{4t}-\frac{1}{2}\Big|e^{-\frac{(\tilde{s}-\xi)^2}{8t}}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(\tilde{s}-\xi)^2}{8t}}|u^0(\xi)|d\xi \\
\le\ & \|V-V'\|\int_{-\infty}^{0} C\frac{1}{\sqrt{4\pi t}}\ e^{-\frac{(\tilde{s}-\xi)^2}{8t}}|u^0(\xi)|d\xi \le C\sup|u^0(\xi)|\,\|V-V'\|.
\end{aligned}
$$

The integral over $(S',0)$ in (3.26) is estimated similarly. As to the second integral in (3.26) , it is even simpler:

$$
\begin{aligned}
&|\int_{S}^{S'} \delta G_\xi u^0 d\xi| \\
=\ & |\int_{S}^{S'}\frac{1}{\sqrt{4\pi t}}(\frac{\xi-S}{2t}e^{-\frac{(S-\xi)^2}{4t}}-\frac{\xi-S'}{2t}e^{-\frac{(S'-\xi)^2}{4t}})u^0(\xi)d\xi| \\
\le\ & \int_{S}^{S'}\frac{1}{\sqrt{4\pi t}}(\frac{\xi-S}{2t}+\frac{S'-\xi}{2t})u^0(\xi)d\xi\frac{(S'-S)}{2t}\frac{1}{\sqrt{4\pi t}}2(S'-S)\|u^0\| \\
\le\ & C\ t^{1/2}\|V-V'\|\,\|u^0\|
\end{aligned}
$$

Finally, by combining the preceding estimates we get:

$$
w_1 \le (C + C_3\ t^{1/2})\|V-V'\|\ \|u^0\|. \tag{3.27}
$$

Next we estimate the free boundary contribution w_2:

$$
w_2 \le \int_0^t |\Delta G_\xi|\ |\varphi|\ d\tau + \int_0^t |G_\xi(S'(t),S'(\tau),t-\tau)|\,\|\varphi-\varphi'|\ d\tau \tag{3.28}
$$

For $|\Delta G_\xi|$ we get:

$$
\begin{aligned}
&|\Delta G_\xi| \equiv |G_\xi(S(t),S(\tau),t-\tau)-G_\xi(S'(t),S'(\tau),t-\tau)| \\
&= |S(t)-S(\tau)-(S'(t)-S'(\tau))|\,|\frac{\partial^2 G}{\partial x^2}(\tilde{s},0,t-\tau)| \\
&= \frac{1}{\sqrt{4\pi}}\,|S(t)-S'(t)-(S(\tau)-S'(\tau))|\ |\frac{\tilde{s}^2}{4(t-\tau)^{5/2}}-\frac{1}{2(t-\tau)^{3/2}}|\ e^{-\frac{\tilde{s}^2}{4(t-\tau)}} \\
&\le \frac{1}{(t-\tau)^{1/2}}(\frac{1}{e}+\frac{1}{2})\frac{1}{\sqrt{4\pi}}\Big|\frac{dS}{dt}(\tilde{\tau})-\frac{dS'}{dt}(\tilde{\tau})\Big| \le C\frac{\|V-V'\|}{(t-\tau)^{1/2}}
\end{aligned}
$$

where $\tilde{s}$ is between $S(t)-S(\tau)$ and $S'(t)-S'(\tau)$. Therefore

$$
\begin{aligned}
\int_0^t |\Delta G_\xi|\ |\varphi|\ d\tau \ &\le\ C\|V-V'\|\int_0^t \frac{\sqrt{\tau}|\varphi|}{\sqrt{\tau}\sqrt{t-\tau}}\ d\tau \\
&\le\ C\pi\|V-V'\|\,\|\varphi\|_t = C_1\|V-V'\|\,\|\varphi\|_t\,.
\end{aligned}
$$

Meanwhile,

$$\begin{aligned}
&\int_0^t |G_\xi(S'(t),S'(\tau),t-\tau)|\,|\varphi-\varphi'|\;d\tau \\
&= \int_0^t \Big|\frac{S'(t)-S'(\tau)}{2(t-\tau)}\Big|\,|G(S'(t),S'(\tau),t-\tau)|\;\frac{1}{\sqrt{\tau}}\sqrt{\tau}\;|\varphi-\varphi'|\;d\tau \\
&\le C_2\|V'\|\,\|\varphi-\varphi'\|_t
\end{aligned} \tag{3.29}$$

Together (3.28)-(3.29) yield

$$w_2 \le C_1\|V-V'\|\,\|\varphi\|_t + C_2\;\|V'\|\,\|\varphi-\varphi'\|_t$$

Thus

$$\begin{aligned}
|\chi-\chi'| &\le 2\|V-V'\| + 2\;|w_1| + 2|w_2| \\
&= \big(2 + (C+C_3\;t^{1/2})\|u^0\| + C_1\|\varphi\|_t\big)\|V-V'\| + C_2\;\|V'\|\,\|\varphi-\varphi'\|_t
\end{aligned}$$

and we observe that although the densities χ and χ' both have singularities at zero, their difference is bounded. This rather remarkable result demonstrates that the integral operator is a contraction. Indeed,

$$\|\chi-\chi'\|_\sigma = \sup_{0\le t\le\sigma}\sqrt{t}|\chi(t)-\chi'(t)| \le \sqrt{\sigma}[c_1\|V-V'\| + c_2\;\|\varphi-\varphi'\|_\sigma] \tag{3.30}$$

where the constants c_1 and c_2 depend explicitly on M and $\|u^0\|$. By taking σ sufficiently small, one can make K a contraction (in the φ-component).

3.5 Iteration for velocity

For the V-component of $\omega-\;\omega'$ we have

$$\begin{aligned}
|v-v'| &\le L\Big\{\Big|\int_{-\infty}^0 [G(S(t),\xi,t)u^0(\xi) - G(S'(t),\xi,t)]u^0(\xi)d\xi\Big| \\
&\qquad + \Big|\int_0^t [G(S(t),S(\tau),t-\tau)\chi - G(S'(t),S'(\tau),t-\tau)\chi']d\tau\Big|\Big\} \\
&= L\{W_1+W_2\}
\end{aligned} \tag{3.31}$$

where L is the Lipschitz constant for g. This estimate is very similar to (3.25), the principal difference being that the integrand there contains G_ξ instead of G. The estimates for separate terms are quite elementary and are based on the mean value theorem. To estimate W_2 we will need a bound for the difference,

$$\begin{aligned}
|\Delta G| &\equiv |G(S(t)-S(\tau),t-\tau,0) - G(S'(t)-S'(\tau),t-\tau,0)| \\
&= |S(t)-S(\tau)-(S'(t)-S'(\tau))|\,\Big|\frac{\partial G}{\partial x}(\tilde{s},t-\tau,0)\Big| \\
&= \Big|\frac{S(t)-S'(t)-(S(\tau)-S'(\tau))}{2(t-\tau)}\Big|\,|\tilde{s}G(\tilde{s},t-\tau,0)| \\
&= \frac{1}{2}\Big|\frac{dS}{dt}(\tilde{\tau}) - \frac{dS'}{dt}(\tilde{\tau})\Big|\,|\tilde{s}G(\tilde{s},t-\tau,0)| \\
&\le \frac{1}{2}\|V-V'\|\,|\tilde{s}G(\tilde{s},t-\tau,0)|
\end{aligned}$$

where $\tau \le \tilde\tau \le t$ and $\tilde s$ are intermediate values. Since $\tilde s$ is between $S'(t) - S'(\tau)$ and $S(t) - S(\tau)$,

$$|\tilde s| \le \max\{|S'(t) - S'(\tau)|, |S(t) - S(\tau)|\} \le V_0(t-\tau)$$

Taking into account $|G| \le [4\pi(t-\tau)]^{-1/2}$ we get the estimate

$$|\Delta G| \le C_1\|V - V'\|\ (t-\tau)^{1/2}. \tag{3.32}$$

where $C_1 = V_0/4\sqrt{\pi}$ is an absolute constant. For the term W_2 we obtain

$$\begin{aligned}
W_2 &\le \int_0^t |\Delta G|\ \chi\ d\tau + \int_0^t G(S'(t), S'(\tau), t-\tau)\ |\chi - \chi'|\ d\tau \\
&\le C_1\|V - V'\| \int_0^t \frac{\sqrt{(t-\tau)}}{\sqrt{\tau}} \sqrt{\tau}\ \chi\ d\tau \\
&\quad + \int_0^t \frac{e^{-(S'(t)-,S'(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)\tau}} \sqrt{\tau}|\chi - \chi'|\ d\tau \\
&\le C_1\frac{t\pi}{2}\|V - V'\|\ M + \|\chi - \chi'\|_t \frac{\sqrt{\pi}}{2} \\
&\le (C_1\frac{t\pi}{2} + C\sqrt{t})\|V - V'\|\ M + C_0\sqrt{t}\|\varphi - \varphi'\|_t \frac{\sqrt{\pi}}{2}
\end{aligned} \tag{3.33}$$

The estimation for the W_1 in (3.31) is a little different. Suppose $S(t) < S'(t) < 0$ and split the integral for W_1 into three integrals:

$$W_1 = \int_{-\infty}^{S} \delta G u^0 d\xi + \int_S^{S'} \delta G u^0 d\xi + \int_{S'}^0 \delta G u^0 d\xi \tag{3.34}$$

where $\delta G = G(S(t), t, \xi) - G(S'(t), t, \xi)$. By the mean value theorem,

$$\delta G = (S - S')\frac{\partial G}{\partial x}(\tilde s - \xi, t, 0) = (S - S')\frac{\tilde s - \xi}{2t} G(\tilde s - \xi, t, 0)$$

where $\tilde s$ is an intermediate value: $\tilde s = \tilde s(t, \xi)$, $S(t) \le \tilde s \le S'(t)$.

If $\xi < S < \tilde s < S'$ then

$$\begin{aligned}
&(\tilde s - \xi)G(\tilde s - \xi, t, 0) \\
&= (4\pi)^{-1/2}(\tilde s - \xi)t^{-1/2}e^{-(\tilde s-\xi)^2/4t} \\
&= (4\pi)^{-1/2}(8t)^{1/2}\frac{\tilde s - \xi}{(8t)^{1/2}} e^{-(\tilde s-\xi)^2/8t} 2^{1/2}(2t)^{-1/2}e^{-(\tilde s-\xi)^2/8t} \\
&\le 4t^{1/2}c_1 G(\tilde s - \xi, 2t, 0) \le 4c_1 t^{1/2} G(\tilde s - \xi, 2t, 0)
\end{aligned}$$

where $c_1 = \max(xe^{-x^2})$. Thus for the first integral in (3.34) we have

$$\begin{aligned}
\Big|\int_{-\infty}^{S} \delta G u^0 d\xi\Big| &\le \frac{S - S'}{2t} 4c_1 t^{1/2} \int_{-\infty}^{S} G(S - \xi, 2t, 0)|u^0(\xi)| d\xi \\
&\le 2c_1 t^{1/2}\frac{1}{t}\int_0^t [V(\tau) - V'(\tau)]\, d\tau \int_{-\infty}^0 G(S - \xi, 2t, 0)|u^0(\xi)| d\xi \\
&\le 2c_1 t^{1/2} \sup|u^0(\xi)|\ \|V - V'\|.
\end{aligned}$$

The integral over $(S', 0)$ in (3.34) is estimated similarly. As to the second integral in (3.34), it is even simpler:

$$\begin{aligned} \Big|\int_S^{S'} \delta G u^0 d\xi\Big| &= \Big|\int_S^{S'} [G(S(t),t,\xi) - G(S'(t),t,\xi)]\, d\xi\Big| \\ &\le (S'-S)2\sup G \cdot \|u^0\| \\ &= (S'-S)2(4\pi)^{-1/2}t^{-1/2}\|u^0\| \\ &\le 2(4\pi)^{-1/2}\, t^{1/2}\|V-V'\|\,\|u^0\| \end{aligned}$$

Finally, by combining the preceding estimates we get:

$$W_1 \le C_3\, t^{1/2}\|V-V'\|\ \|u^0\|. \tag{3.35}$$

Thus, the estimates in (3.33), (3.35) yield

$$\begin{aligned} \|v-v'\| &= \sup_{0\le t\le\sigma} |v-v'| \\ &\le C_3\,\sigma^{1/2}\|V-V'\|\ \|u^0\| + \ C_1\frac{\sigma\pi}{2}\|V-V'\|\, M \\ &\quad + C_0\sqrt{\sigma}\|\varphi-\varphi'\|_\sigma \frac{\sqrt{\pi}}{2} \\ &\le (A_1\sigma^{1/2} + A_2\sigma)\|V-V'\| + A_3\sigma^{1/2}\,\|\varphi-\varphi'\|_\sigma, \end{aligned} \tag{3.36}$$

where the constants depend only on M and $\|u^0\|$. It is clear from (3.30) and (3.36) that the map K is a contraction for a sufficiently small σ.

This completes the proof of Theorem 3.1 since by the contraction mapping principle, the system of integral equations has a unique solution. As we saw at the end of Sec. 3.1 it produces the solution for the problem (1.1)-(1.3).

4 A priori bounds and global existence

In this section we present a priori bounds for solutions of the free-boundary problem and their first spatial derivatives. The a priori bound for the solutions was established in our prior work [4] (where a more involved case of g sublinear is also considered); its proof is included here for reader's convenience.

4.1 A priori estimate for the solution

The derivation of the bound is based upon an integral representation of the solution that we will obtain next. If G is the fundamental solution of the heat equation and $u(x,t)$, $s(t)$ is a classical solution of (1.1)-(1.3), then by integrating Green's identity

$$\frac{\partial}{\partial\xi}\Big(G\frac{\partial u}{\partial\xi} - u\frac{\partial G}{\partial\xi}\Big) - \frac{\partial}{\partial\tau}(Gu) = 0 \tag{4.1}$$

over the domain $\xi < s(\tau)$, $0 < \tau < t$ and using the Stokes formula, we obtain the integral representation for the solution:

$$\begin{aligned} u(x,t) &= \int_0^t G(x,s(\tau),t-\tau)\,[-V(\tau)+U(\tau)V(\tau)]\,d\tau \\ &\quad - \int_0^t \frac{\partial G}{\partial \xi}(x,s(\tau),t-\tau)U(\tau)d\tau + \int_{-\infty}^0 G(x,\xi,t)u^0(\xi)d\xi \end{aligned} \tag{4.2}$$

which employs both the single-layer and double-layer potentials. In the limit $x \to s(t)-$ the integral representation yields the following equation:

$$\begin{aligned} \frac{1}{2}U(t) &= \int_0^t G(s(t),s(\tau),t-\tau)[-V(\tau)]d\tau + \int_0^t G(s(t),s(\tau),t-\tau)U(\tau)V(\tau)d\tau \\ &\quad - \int_0^t \frac{\partial G}{\partial \xi}(s(t),s(\tau),t-\tau)U(\tau)d\tau + \int_{-\infty}^0 G(x,\xi,t)u^0(\xi)d\xi \end{aligned} \tag{4.3}$$

where

$$U(t) = g^{-1}(V(t)), \quad s(t) = \int_0^t V(\tau)d\tau. \tag{4.4}$$

Note that the factor $\frac{1}{2}$ arises from the jump relation for the second integral in (4.2).

We will show first that the temperature at the front, $U(t)$, is uniformly bounded. The proof consists of separate estimates for different terms in (4.3). For the first integral we obtain:

$$\begin{aligned} \int_0^t G(s(t),s(\tau),t-\tau)[-V(\tau)]d\tau &\le V_0 \int_0^t \frac{e^{-v_0^2(t-\tau)/4}}{\sqrt{4\pi(t-\tau)}}d\tau \\ &\le V_0 \int_0^\infty \frac{e^{-v_0^2\tau/4}}{\sqrt{4\pi\tau}}d\tau = V_0/v_0. \end{aligned} \tag{4.5}$$

In the above estimate we have used the obvious bound:

$$\frac{(s(t)-s(\tau))^2}{t-\tau} = \big[\frac{s(t)-s(\tau)}{t-\tau}\big]^2(t-\tau) \ge v_0^2(t-\tau).$$

We combine together the two subsequent integrals with respect to τ from (4.3):

$$\Phi(t) = \int_0^t G(s(t),s(\tau),t-\tau)U(\tau)\big[V(\tau) - \frac{1}{2}\frac{s(t)-s(\tau)}{t-\tau}\big]d\tau, \tag{4.6}$$

note that the term with the average velocity,

$$\frac{s(t)-s(\tau)}{t-\tau},$$

arises from the explicit differentiation in $\partial G/\partial\xi$. The estimate is based on the observation that the integrand in (4.6) is *negative* for the velocity of large magnitude.

Lemma 4.1 *Let $g(U) \ge -V_0$ for $U \ge 0$ then the function $\Phi(t)$ defined by (4.6) is bounded uniformly in t.*

Proof We split the domain of integration in (4.6) into the two sets:

$$[0,t] = \{\tau : V(\tau) < -V_0/2\} \cup \{\tau : V(\tau) \geq -V_0/2\} = B_- \cup B_+. \tag{4.7}$$

Note that on B_-

$$V(\tau) - \frac{1}{2}\frac{s(t)-s(\tau)}{t-\tau} = V(\tau) - \frac{1}{2}V(\sigma) \leq -V_0/2 + V_0/2 = 0, \tag{4.8}$$

where $\tau \leq \sigma \leq t$. On B_+ the absolute value of the difference in (4.8) is bounded by $\sup_{B_+}|V| = V_0/2$, therefore, since $U \geq 0$ and $G > 0$, we obtain the estimate:

$$\begin{aligned} \Phi(t) &\leq \int_{B_+} G(s(t),s(\tau),t-\tau)U(\tau)\left[V(\tau) - \frac{1}{2}\frac{s(t)-s(\tau)}{t-\tau}\right] d\tau, \\ &\leq \frac{g^{-1}(-V_0/2)V_0}{2}\int_{B_+}\frac{e^{-v_0^2(t-\tau)/4}}{\sqrt{4\pi(t-\tau)}}d\tau \leq \frac{g^{-1}(-V_0/2)V_0}{2v_0}. \end{aligned} \tag{4.9}$$

□

Therefore for the interface temperature U we obtained the bound:

$$\begin{aligned} U(t) &= 2\{\int_0^t G(s(t),s(\tau),t-\tau)[-V(\tau)+U(\tau)V(\tau)]d\tau \\ &\quad - \int_0^t \frac{\partial G}{\partial \xi}(s(t),s(\tau),t-\tau)U(\tau)d\tau + \int_{-\infty}^0 G(s(t),\xi,t)u^0(\xi)d\xi\} \\ &\leq \frac{[g^{-1}(-V_0/2)+2]V_0}{v_0} + 2\|u^0\| \equiv R_{fb} + 2\|u^0\|. \end{aligned}$$

We have shown that the solution on the free boundary is bounded. In combination with the boundedness of the initial data it yields boundedness of the solution everywhere:

Theorem 4.2 *Let the kinetic function g satisfy the kinetic condition in 2.4. If $u(x,t), V(t)$ is a solution of the free boundary problem (1.1)-(1.3) then the functions u, V are bounded,*

$$0 \leq u(x,t) \leq R_{fb} + 2\|u^0\|, \tag{4.10}$$

where R_{fb} is an "absolute" constant determined by the kinetics.

The proof is extremely simple. We ignore the boundary condition on u_x in (1.2) and note that a solution $u(x,t)$ of the free boundary problem solves the initial value problem for the heat equation with the given Dirichlet boundary conditions $U(t) = g^{-1}((\dot{s}(t))$ at the free boundary. Since both initial data and the boundary conditions are bounded, $u(x,t)$ is also bounded by the maximum principle.

As a corollary we note here *the global existence result* that follows from the local existence and from the a priori bound (cf. Sec. 5 of [1]).

Remark 4.3 Bounds V_0 and v_0 play very different roles in the previous results. It can be shown that a version of the a priori estimate (4.10) holds even if the condition $|g| \leq V_0$ is relaxed to $g(u)/u^{1+\varepsilon} \to 0$ as $u \to \infty$ (see [4] for details).

4.2 A priori estimate for the derivative

Theorem 4.4 *Consider the ball $\|u^0\| \le R$. There exists $\sigma > 0$ depending on R such that for any fixed t, $0 < t \le \sigma$, the derivative of the solutions of the free boundary problem with the initial data from the ball is uniformly bounded. More specifically*

$$|u_x(x,t)| \le \frac{C}{1 + |s(t) - x|}$$

where C is determined by R and t.

Proof Consider the solution $u(x,t)$ given by the single-layer integral representation (3.1). By Theorem3.1 the representation is valid on some life span σ that is completely determined by R. We differentiate u with respect to x

$$u_x(x,t) = \frac{\partial}{\partial x}\int_0^t G(x, s(\tau), t-\tau)\varphi(\tau)d\tau + \int_{-\infty}^0 \frac{\partial}{\partial x} G(x,\xi,t)u^0(\xi)d\xi$$

It was shown in Sec. 3 for $|s(t) - x| > 1$, see (3.18), that

$$\frac{\partial}{\partial x}\int_0^t G(x, s(\tau), t-\tau)\varphi(\tau)d\tau \le \frac{C\|\varphi\|_\sigma}{|s(t) - x|} \tag{4.12}$$

and for $|s(t) - x| \le 1$, [see (3.20)]

$$\frac{\partial}{\partial x}\int_0^t G(x, s(\tau), t-\tau)\varphi(\tau)d\tau \le C\,\|\varphi\|_\sigma / \sqrt{t} \tag{4.13}$$

Now we need the estimate for the integral of the initial data

$$\begin{aligned}\Big|\int_{-\infty}^0 \frac{\partial}{\partial x} G(x,\xi,t)u^0(\xi)d\xi\Big| &= \int_{-\infty}^0 \frac{1}{\sqrt{4\pi t}}\frac{|\xi - x|}{2t} e^{-\frac{(x-\xi)^2}{4t}} \left|u^0(\xi)\right| d\xi \\ &= \frac{1}{\sqrt{t}}\int_{-\infty}^0 \frac{2}{\sqrt{8\pi t}} e^{-\frac{(x-\xi)^2}{8t}} \frac{|\xi - x|}{\sqrt{8t}} e^{-\frac{(x-\xi)^2}{8t}} \left|u^0(\xi)\right| d\xi \\ &\le \frac{2}{e\sqrt{t}}\left\|u^0\right\|\end{aligned} \tag{4.14}$$

It was shown in the proof of the local existence of solutions that, given a bound on the initial data u^0, the density φ belongs to the invariant ball of radius M and therefore is uniformly bounded for all initial conditions within the bound. Thus, (4.12)-(4.14) yield the result of the theorem.

Corollary 4.5 *For all $t \ge t_0$, where $t_0 \le \sigma$ the derivative is uniformly bounded: $|u_x(x,t)| \le C$.*

For the proof we note that u_x solves the heat equation in the domain $\{(x,t) : t > t_0,\ x < s(t)\}$ with the initial data $u_x(x,t_0)$ and boundary conditions $V(t)$ which are both bounded. This yields a bound in the interior of the domain by the maximum principle.

5 Absorbing set and attractor

In this section we use the estimates obtained above to establish the existence of a bounded absorbing set and of the attractor which is compact in the space of continuous functions. In order to establish compactness of the attractor we need to reinstitute the heat losses. It can be easily verified that all the analytical properties of the solutions and estimates established above can be only improved by introducing the heat losses. On the other hand the problem with the heat losses exhibits uniform exponential decay in time of the contribution of initial data that is necessary for the proof of compactness of the attractor.

With the heat loss, the integral representation of the solution in terms of mixed potential (4.2) takes the form

$$\begin{aligned} u(x,t) & \int_0^t e^{-\gamma(t-\tau)} G(x, s(\tau), t-\tau)\,[-V(\tau) + U(\tau)V(\tau)]\,d\tau \\ & - \int_0^t e^{-\gamma(t-\tau)} \frac{\partial G}{\partial \xi}(x, s(\tau), t-\tau) U(\tau) d\tau + e^{-\gamma t} \int_{-\infty}^0 G(x,\xi,t) u^0(\xi) d\xi \end{aligned}$$

The representation describes globally in time the evolution of the initial temperature distribution u^0: $u(t) = T(t)u^0$. We think of the evolution as taking place for the functions on the fixed interval $(-\infty, 0)$. To achieve this we introduce the moving coordinate system attached to the free boundary $x' = x - s(t)$.

We split the semigroup operator T into two parts: the contribution of the free boundary

$$\begin{aligned} T_1(t)u^0(x') &= \int_0^t e^{-\gamma(t-\tau)} G(x', s(\tau)-s(t), t-\tau)\,[-V(\tau) + U(\tau)V(\tau)]\,d\tau \\ & \quad - \int_0^t e^{-\gamma(t-\tau)} \frac{\partial G}{\partial \xi}(x', s(\tau)-s(t), t-\tau) U(\tau) d\tau \qquad (5.1) \end{aligned}$$

and that of the initial data

$$T_2(t)u^0 = e^{-\gamma t} \int_{-\infty}^0 G(x', \xi - s(t), t) u^0(\xi) d\xi \qquad (5.2)$$

As a basic metric space we choose a ball in the space $C(-\infty, 0]$:

$$X = \{u \in C(-\infty, 0]; \quad \|u\| = \sup |u(x')| \le N\}$$

where the radius N is large enough (it suffices to take $N > R_{fb} + 2R_{abs}$ where R_{abs} is the radius of the absorbing ball which is estimated in the following proposition). Note that by Theorem 4.2, the evolution of any ball B_R of radius $R \le (N - R_{fb})/2$ stays in X for all time.

The following result establishes existence of an absorbing set for the evolution.

Proposition 5.1 *(i) The semigroup T_2 is uniformly contracting:*

$$r_X(t) = \sup_{u^0 \in X} \|T_2(t)u^0\| \to 0 \quad \text{as} \quad t \to \infty.$$

(ii) There exists a constant, R_{abs}, totally determined by the kinetics such that any ball $B_a = \{u \in X : \|u\| \le a\}$, where $a = R_{abs} + \varepsilon < N$, is an absorbing set for any ball B_R (where $R \le (N - R_{fb})/2$) with respect to the evolution by T_1 (and T).

Proof As is easily seen from (5.2) the contribution of the initial data decays uniformly $\|T_2(t)u^0\| \le e^{-\gamma t}\|u^0\|$. The contribution from the free boundary is represented through the mixed potential integral with the densities U and $-V + UV$. Both densities are bounded: $|V(\tau)| \le V_0$ and U was estimated above in (4.10). In the presence of the heat losses the estimate is modified

$$|U(\tau)| \le R_{fb} + 2e^{-\gamma t}\|u^0\|$$

(the value of R_{fb} is even slightly less than in (4.10): v_0 should be replaced by $v_0 + \sqrt{\gamma}$). Next we estimate contributions of both potentials in the expression for T_1. The estimates are very similar to the estimates for the derivative of the single layer potential in Lemma 3.2:

$$\begin{aligned}
&\int_0^t e^{-\gamma(t-\tau)} \frac{e^{-(x-s(\tau))^2/4(t-\tau)}}{\sqrt{4\pi(t-\tau)}} \left[-V(\tau) + U(\tau)V(\tau)\right] d\tau \\
&\le \int_0^t V_0(R_{fb} + 2e^{-\gamma\tau}N + 1)e^{-(x-s(\tau))^2/8(t-\tau)} \\
&\quad \times \exp\{-\frac{(x')^2 + 2x'(s(t)-s(\tau)) + (s(t)-s(\tau))^2}{8(t-\tau)}\} \frac{e^{-\gamma(t-\tau)}d\tau}{\sqrt{4\pi(t-\tau)}} \\
&\le e^{-v_0|x'|/4} \int_0^t V_0(R_{fb} + 2e^{-\gamma\tau}N + 1)e^{-(s(t)-s(\tau))^2/8(t-\tau)} \frac{e^{-\gamma(t-\tau)}d\tau}{\sqrt{4\pi(t-\tau)}} \\
&\le e^{-v_0|x'|/4} \int_0^t V_0(R_{fb} + 2e^{-\gamma\tau}N + 1)e^{-v_0^2(t-\tau)/8} \frac{e^{-\gamma(t-\tau)}d\tau}{\sqrt{4\pi(t-\tau)}} \\
&\le \frac{\sqrt{2}V_0(R_{fb} + 2e^{-\gamma t}N + 1)}{v_0} e^{-v_0|x'|/4}.
\end{aligned} \tag{5.3}$$

The estimation for the double layer potential term from (5.1) is almost identical to the corresponding estimate in Lemma 3.5. For $|x'| > 1$, it produces the bound:

$$\Big|\int_0^t e^{-\gamma(t-\tau)} \frac{\partial G}{\partial \xi}(x', s(\tau) - s(t), t-\tau)U(\tau)d\tau\Big| \le \frac{c_1(R_{fb} + 2e^{-\gamma t}N)}{v_0(1 + |x'|)} e^{-v_0|x'|/4}, \tag{5.4}$$

while for $|x'| < 1$ it is bounded by $c_2(R_{fb} + 2e^{-\gamma t}N)/v_0$. Both c_1 and c_2 are explicit, order one constants.

If now we take R_{abs} equal to the sum of the constants in the above estimates (5.3)-(5.4) then

$$|T_1(t)u^0| \le R_{abs}e^{-v_0|x'|/4} + Ce^{-\gamma t}e^{-v_0|x'|/4}N \tag{5.5}$$

if $\|u^0\| \le N$. By choosing t_1 such that $Ce^{-\gamma t_1}N < a - R_{abs}$ we ensure that the orbit of any bounded subset of X enters B_a and remains there after that time t_1, which means that B_a is absorbing for X. □

Next we prove that the boundary contribution to the evolution, i.e. the operators $T_1(t)$ are *uniformly compact*. Namely, the following proposition holds:

Proposition 5.2 *There exists $t_0 > 0$ such that $\cup_{t\ge t_0}T_1(t)X$ is relatively compact in X.*

Proof The proof of the proposition contains the following two basic ingredients: We establish certain estimates on the functions $T_1(t)u$, and their first spatial derivatives, uniformly in $u \in X$, that are valid for any $t \ge t_0 > 0$, next we demonstrate that the set determined by the estimates is relatively compact.

First we recall that by Corollary 4.5 for sufficiently small $t_0 > 0$ and any $u^0 \in X$,

$$|(T(t)u^0)_x| \le C \text{ for } t \ge t_0,\ x \in (-\infty, 0]$$

On the other hand the contribution from the free boundary

$$\begin{aligned} |(T_1(t)u^0)_x| &= |(T(t)u^0)_x - (T_2(t)u^0)_x| \\ &\le |(T(t)u^0)_x| + |(T_2(t)u^0)_x| \\ &\le C + C\|u^0\|/\sqrt{t} \le C \end{aligned}$$

since the contribution of the initial conditions is also uniformly bounded $|(T_2(t)u^0)_x| \le C\|u^0\|/\sqrt{t}$, see (4.14). Therefore the family $\cup_{t\ge t_0}T_1(t)X$ is equicontinuous.

For the version of Arzela-Ascoli theorem appropriate for $(-\infty, 0]$ we need uniform boundedness and uniform decay of the family of functions as $|x'| \to \infty$. These properties are provided by the estimate (5.5) that gives a uniform exponential decay. Then it is easy to construct a finite ε-net by choosing a finite interval beyond which the functions of the family are smaller than ε and extending the elements of the ε-net from this interval by zero. □

The properties of the evolution operator $T(t)$ described in the above propositions allow us to apply the abstract general result (see, for example, [15] Chap. 1) that in our situation can be stated as follows:

Theorem 5.3 *The continuous semigroup $T(t)$, $T(t) = T_1(t) + T_2(t)$ with $T_1(t)$ uniformly compact and $T_2(t)$ uniformly contracting has the following properties: the ω-limit set A of the absorbing set B_a is a compact attractor for the metric space X; A is the maximal attractor in X and it is connected.*

6 Concluding remarks

Compactness of the attractor and ultimately its finite Hausdorff dimension (see [6]) for the free boundary problem modeling nonequilibrium solidification and SHS is a rather remarkable fact, especially in view of the surprising wealth of possible dynamical scenarios. The situation should be compared, perhaps, to the similar facts known for the Kuramoto-Sivashinsky equation or Navier-Stokes equations. In both cases the compactness is shown for finite intervals whose length enters also into the estimate on the Hausdorff dimension. In our case, however, the domain of the field variable is an infinite interval.

The compactness result was proved here in the presence of heat losses for any nonzero heat loss. Although we chose to operate in spaces of continuous uniformly bounded functions on the infinite interval, we believe that compactness can be established in spaces with weaker topology, specifically in the space of continuous functions bounded on each finite interval. In this case we would not need the heat loss term, but we would have less control over the behavior of solutions at infinity.

Results of this paper are proved for the kinetic function satisfying the bounds in (2.4). These bounds are quite physical and cover a wide range of important applications. Nonetheless, our numerical experimentation with different types of kinetic functions, including unbounded ones demonstrate that the asymptotic dynamics are insensitive to the behavior of the kinetic function for large temperatures. On the other hand, our results from [4] provide global existence for a wider class of kinetic functions, than in (2.4), namely for sublinear kinetics. Therefore it seems plausible that a compact attractor should exists for this case as well.

Finally, we should remark that the one-phase problem is to a degree a particular case of a more general two-phase problem (2.4). There are technical difficulties in implementation of the construction of this paper for the two-phase problem, as the field extends behind the interface where it is not necessarily decaying. At the same time numerical experiments show a great similarity in dynamical behavior of both problems. It would be, therefore, interesting to extend results of the present paper to the two-phase problem.

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