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Stabilization of heteregeneous Maxwell's equations by linear or nonlinear boundary feedbacks *

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Abstract

We examine the question of stabilization of the (nonstationary) heteregeneous Maxwell's equations in a bounded region with a Lipschitz boundary by means of linear or nonlinear Silver-Müller boundary condition. This requires the validity of some stability estimate in the linear case that may be checked in some particular situations. As a consequence we get an explicit decay rate of the energy, for instance exponential, polynomial or logarithmic decays are available for appropriate feedbacks. Based on the linear stability estimate, we further obtain certain exact controllability results for the Maxwell system.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary Γ. In that domain we consider (non-stationary) Maxwell's equations with a nonlinear boundary condition:

$$
\begin{gathered}
\varepsilon \frac{\partial E}{\partial t} - \operatorname{curl} H = 0 \text{ in } Q := \Omega \times]0, +\infty[, \\
\mu \frac{\partial H}{\partial t} + \operatorname{curl} E = 0 \text{ in } Q, \\
\operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q, \\
H \times \nu + g(E \times \nu) \times \nu = 0 \text{ on } \Sigma := \Gamma \times]0, +\infty[, \\
E(0) = E_0, H(0) = H_0 \text{ in } \Omega.
\end{gathered}
\tag{1.1}
$$

Above and below ε and μ are real, positive functions of class $L^\infty(\Omega)$, ν is the exterior unit normal vector to Γ. The function $g : \mathbb{R}^3 \to \mathbb{R}^3$ is assumed to be

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continuous and satisfies

$$(g(E) - g(F)) \cdot (E - F) \geq 0, \forall E, F \in \mathbb{R}^3 \text{ (monotonicity)}, \tag{1.2}$$

$$g(0) = 0, \tag{1.3}$$

$$g(E) \cdot E \geq m|E|^2, \forall E \in \mathbb{R}^3 : |E| \geq 1, \tag{1.4}$$

$$|g(E)| \leq M(1 + |E|), \forall E \in \mathbb{R}^3, \tag{1.5}$$

for some positive constants m, M.

The system (1.1) occurs in electromagnetic theory in which $E(x,t)$, $H(x,t)$ are the electric and magnetic fields at the point $x \in \Omega$ at time t, and $\varepsilon(x)$, $\mu(x)$ are the electric permittivity and magnetic permeability, respectively, at point x. When $g(\xi) \equiv \xi$, the boundary condition occuring in (1.1) is the classical Silver-Müller boundary condition and has its own interest. The classical Silver-Müller boundary condition is a first order approximation to the so-called "transparent" boundary condition. The latter boundary condition, which is non-local, corresponds to the complete transmission of electromagnetic waves through the boundary Γ while the Silver-Müller boundary condition, although dissipative, allows for reflexions back into the region Ω. This system with Silver-Müller boundary condition has attracted the attention of many authors and stability results were obtained under different geometrical conditions in [2, 12, 25] when $\varepsilon = \mu = 1$. For smooth coefficients ε, μ and g not necessarily linear, explicit decay rates for solutions of (1.1) were recently given in [8] when Ω is a connected domain with a smooth boundary Γ consisting of a single connected component.

The aims of our paper are multiple and may be summarized as follows: First we concentrate our attention to the linear case. We give a necessary and sufficient condition for exponential decay of the energy

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} (\varepsilon |E(x,t)|^2 + \mu |H(x,t)|^2)\, dx$$

of the solutions of (1.1) in the case of Silver-Müller boundary condition. This condition is actually the validity of a kind of stability estimate that we call the (ε, μ)-stability estimate. The second aim is to check this stability estimate in some particular cases, in other words find sufficient conditions ensuring the validity of the aforementioned stability estimate. The nonsmooth case will require special attention since standard regularity results fail [3, 6, 7, 5, 20]. In a third step we use the so-called Russell's principle "controllability via stability" to obtain new controllability results for Maxwell's equations extending former results from [26, 15, 16, 22, 12, 23, 24]. Finally, for the general system (1.1) using Liu's principle [19], based on the above Russell's principle and then on the (ε, μ)-stability estimate, and with the help of a new integral inequality [8], we give sufficient conditions on g which lead to an explicit decay rate of the energy. The main interest of this approach is that the controllability results in the linear case as well as the stabilization results with general feedbacks are only based on the (ε, μ)-stability estimate, an estimate which may be obtained by different techniques, like the multiplier method [12, 21], microlocal analysis

[25], combinations of them [8] or any method entering in a linear framework (like nonharmonic analysis for instance, see [14]). In other words any method proving the (ε, μ)-stability estimate directly yields the above mentioned controllability and stability results.

The schedule of the paper is the following one: Well-posedness of problem (1.1) is analyzed in section 2 under the above assumptions on Ω, ε and μ and appropriate conditions on g. Section 3 is devoted to the proof of the equivalence between the (ε, μ)-stability estimate and exponential stability of (1.1) in the case of Silver-Müller boundary condition, we further check this (ε, μ)-stability estimate in some particular situations. We give in section 4 some exact controllability results based on the (ε, μ)-stability estimate. Section 5 is devoted to the stability results of (1.1) for general nonlinear feedbacks g.

2 Well-posedness of the problem

The main goal of this section is to show the well-posedness of problem (1.1) under appropriate conditions on the mapping g using nonlinear semigroup theory. To this end we introduce the Hilbert spaces (see e.g. [16, 23])

$$J(\Omega, \varepsilon) = \{E \in L^2(\Omega)^3 |\, \operatorname{div}(\varepsilon E) = 0 \text{ in } \Omega\}, \tag{2.1}$$

$$\mathcal{H} = J(\Omega, \varepsilon) \times J(\Omega, \mu), \tag{2.2}$$

equipped with their natural norm induced by the inner product

$$(E, E')_\varepsilon = \int_\Omega \varepsilon(x) E(x) \cdot E'(x)\, dx, \forall E, E' \in J(\Omega, \varepsilon),$$

$$\left(\begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E' \\ H' \end{pmatrix} \right)_{\mathcal{H}} = (E, E')_\varepsilon + (H, H')_\mu, \forall \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E' \\ H' \end{pmatrix} \in \mathcal{H}.$$

Now define the (nonlinear) operator A from $\mathcal{H}$ into itself as follows:

$$D(A) = \{\begin{pmatrix} E \\ H \end{pmatrix} \in \mathcal{H} |\, \operatorname{curl} E, \operatorname{curl} H \in L^2(\Omega)^3; \tag{2.3}$$

$$E \times \nu, H \times \nu \in L^2(\Gamma)^3 \text{ satisfying}$$

$$H \times \nu + g(E \times \nu) \times \nu = 0 \text{ on } \Gamma\}, \tag{2.4}$$

$$A \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -\varepsilon^{-1} \operatorname{curl} H \\ \mu^{-1} \operatorname{curl} E \end{pmatrix}, \forall \begin{pmatrix} E \\ H \end{pmatrix} \in D(A). \tag{2.5}$$

Let us notice that the boundary conditions (2.4) is meaningful due to the assumption (1.5): Indeed the properties $E \times \nu, H \times \nu \in L^2(\Gamma)^3$ and (1.5) give a meaning to the boundary condition (2.4) as an equality in $L^2(\Gamma)^3$.

We then see that formally problem (1.1) is equivalent to

$$\begin{aligned} \frac{\partial U}{\partial t} + AU &= 0, \\ U(0) &= U_0, \end{aligned} \tag{2.6}$$

when $U = \begin{pmatrix} E \\ H \end{pmatrix}$ and $U_0 = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}$.

We shall prove that this problem (2.6) has a unique solution using nonlinear semigroup theory (see e.g. [27]) by showing that A is a maximal monotone operator adapting an argument from section 3 of [8]. But first we recall a density result from [4] allowing integration by parts.

Theorem 2.1 ([4]) *Introduce the Hilbert space*

$$W = \{E \in L^2(\Omega)^3 | \operatorname{curl} E \in L^2(\Omega)^3 \text{ and } E \times \nu \in L^2(\Gamma)^3\}, \tag{2.7}$$

with the norm

$$||E||_W^2 = \int_\Omega (|E|^2 + |\operatorname{curl} E|^2)dx + \int_\Gamma |E \times \nu|^2 d\sigma.$$

Then $H^1(\Omega)^3$ is dense in W.

Lemma 2.2 *For all $\begin{pmatrix} E \\ H \end{pmatrix} \in D(A)$, one has*

$$\int_\Omega (\operatorname{curl} E \cdot H - \operatorname{curl} H \cdot E)\, dx = \int_\Gamma H \times \nu \cdot E d\sigma. \tag{2.8}$$

Proof: We first remark that (2.8) holds for all $\begin{pmatrix} E \\ H \end{pmatrix}$ in $H^1(\Omega)^3 \times H^1(\Omega)^3$ owing to Green's formula. By density (see Theorem 2.1) it still holds in $W \times W$. We conclude since $D(A)$ is clearly continuously embedded into $W \times W$. $\square$

We next prove the following density result, which is closely related to Lemma 2.3 of [23].

Lemma 2.3 *If $g(0) = 0$, the domain of the operator A is dense in $\mathcal{H}$.*

Proof: Let us denote by P_ε the projection on $J(\Omega, \varepsilon)$ in $L^2(\Omega)^3$ endowed with the inner product $(\cdot, \cdot)_\varepsilon$. As $\mathcal{D}(\Omega)^3$ is dense in $L^2(\Omega)^3$, $P_\varepsilon \mathcal{D}(\Omega)^3$ is dense in $J(\Omega, \varepsilon)$. Consequently $P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3$ is dense in $\mathcal{H} = J(\Omega, \varepsilon) \times J(\Omega, \mu)$.

Moreover as in Lemma 2.3 of [23] we can show that for any $\chi \in \mathcal{D}(\Omega)^3$, we have

$$\begin{gathered} \operatorname{curl}(P_\varepsilon \chi) = \operatorname{curl} \chi \text{ in } \Omega, \\ P_\varepsilon \chi \times \nu = 0 \text{ on } \Gamma. \end{gathered}$$

Since $g(0) = 0$, we conclude that

$$P_\varepsilon \mathcal{D}(\Omega)^3 \times P_\mu \mathcal{D}(\Omega)^3 \subset D(A).$$

Therefore the above density result implies that $D(A)$ is dense in $\mathcal{H}$. $\square$

Remark 2.4 We notice that the above arguments also show that for any $\chi \in C^\infty(\bar{\Omega})^3$, we have

$$\begin{aligned} \operatorname{curl}(P_\varepsilon \chi) &= \operatorname{curl} \chi \text{ in } \Omega, \\ P_\varepsilon \chi \times \nu &= \chi \times \nu \text{ on } \Gamma. \end{aligned}$$

Lemma 2.5 *Under the above assumptions on g, A is a maximal monotone operator.*

Proof: We start with the monotonicity: A is monotone if and only if

$$(AU - AV, U - V)_{\mathcal{H}} \geq 0, \forall U, V \in D(A).$$

From the definition of A and the inner product in $\mathcal{H}$, we see that this is equivalent to

$$\int_\Omega \{\operatorname{curl}(E - E') \cdot (H - H') - (E - E') \cdot \operatorname{curl}(H - H')\}\, dx \geq 0,$$

for all $\begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E' \\ H' \end{pmatrix} \in D(A)$. Lemma 2.2 yields equivalently

$$\int_\Gamma (H - H') \times \nu \cdot (E - E')\, d\sigma \geq 0, \forall \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E' \\ H' \end{pmatrix} \in D(A).$$

Using the boundary condition (2.4), we arrive at

$$\int_\Gamma (g(E \times \nu) - g(E' \times \nu)) \cdot (E \times \nu - E' \times \nu)\, d\sigma \geq 0, \forall \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E' \\ H' \end{pmatrix} \in D(A).$$

We then conclude using the monotonicity assumption on g (assumption (1.2)).

Let us now pass to the maximality. This means that for all $\begin{pmatrix} F \\ G \end{pmatrix}$ in $\mathcal{H}$, we are looking for $\begin{pmatrix} E \\ H \end{pmatrix}$ in $D(A)$ such that

$$(I + A) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}. \tag{2.9}$$

Formally, we then have

$$H = G - \mu^{-1} \operatorname{curl} E, \tag{2.10}$$

and

$$\varepsilon E + \operatorname{curl}(\mu^{-1} \operatorname{curl} E) = \varepsilon F + \operatorname{curl} G. \tag{2.11}$$

This last equation in E will have a unique solution by adding a boundary condition on E. Indeed using the identity (2.10), we see that (2.4) is formally equivalent to

$$-\mu^{-1} \operatorname{curl} E \times \nu + g(E \times \nu) \times \nu = -G \times \nu \text{ on } \Gamma. \tag{2.12}$$

We now remark that the variational formulation of problem (2.11)-(2.12) is the following one: Find $E \in W_\varepsilon$ such that

$$a(E,E') = \int_\Omega \{\varepsilon F \cdot E' + G \cdot \operatorname{curl} E'\}\, dx, \forall E' \in W_\varepsilon, \tag{2.13}$$

where the Hilbert space W_ε is defined by

$$W_\varepsilon = \{E \in L^2(\Omega)^3 |\operatorname{curl} E \in L^2(\Omega)^3, \operatorname{div}(\varepsilon E) \in L^2(\Omega) \text{ and } E \times \nu \in L^2(\Gamma)^3\},$$

with the norm

$$||E||^2_{W_\varepsilon} = \int_\Omega (|E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div}(\varepsilon E)|^2) dx + \int_\Gamma |E \times \nu|^2 d\sigma.$$

and the form a is defined by

$$\begin{aligned} a(E,E') &= \int_\Omega \{\mu^{-1} \operatorname{curl} E \cdot \operatorname{curl} E' + \varepsilon E \cdot E' + s \operatorname{div}(\varepsilon E) \operatorname{div}(\varepsilon E')\}\, dx \\ &\quad + \int_\Gamma g(E \times \nu) \cdot E' \times \nu \, d\sigma, \end{aligned}$$

$s > 0$ being a parameter appropriately chosen later on.

We now introduce the mapping

$$\mathcal{A} : W_\varepsilon \to W'_\varepsilon : u \to \mathcal{A}u,$$

where $\mathcal{A}u(v) = a(u,v)$. As the right-hand side of (2.13) defines an element of W'_ε, the solvability of (2.13) is equivalent to the surjectivity of $\mathcal{A}$. For that purpose we make use of Corollary 2.2 of [27] which proves that $\mathcal{A}$ is surjective if $\mathcal{A}$ is monotone, hemicontinuous, bounded and coercive. It then remains to check these properties.

From the monotonicity of g, we directly check that $\mathcal{A}$ is monotone. Moreover the continuity of g leads to the hemicontinuity of $\mathcal{A}$ while the properties (1.5) of g implies the boundedness of $\mathcal{A}$. It then remains to show that $\mathcal{A}$ is coercive, i.e.,

$$\frac{\mathcal{A}E(E)}{||E||_{W_\varepsilon}} \to +\infty \text{ as } ||E||_{W_\varepsilon} \to +\infty.$$

From the definition of $\mathcal{A}$ it is equivalent to

$$\frac{a(E,E)}{||E||_{W_\varepsilon}} \to +\infty \text{ as } ||E||_{W_\varepsilon} \to +\infty. \tag{2.14}$$

For a fixed $E \in W_\varepsilon$ we set

$$\Gamma^+ = \{x \in \Gamma : |(E \times \nu)(x)| > 1\}, \Gamma^- = \{x \in \Gamma : |(E \times \nu)(x)| \le 1\}.$$

The properties of g imply that

$$a(E,E) \ge \int_\Omega \{\mu^{-1} |\operatorname{curl} E|^2 + \varepsilon |E|^2 + s |\operatorname{div}(\varepsilon E)|^2\}\, dx + m \int_{\Gamma^+} |E \times \nu|^2\, d\sigma.$$

Moreover from the definition of Γ^-, we have

$$||E||_{W_\varepsilon}^2 \leq \int_\Omega (|E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div}(\varepsilon E)|^2)dx + \int_{\Gamma^+} |E \times \nu|^2 d\sigma + |\Gamma|.$$

These two inequalities show that there exists a positive constant β (independent on E) such that

$$a(E, E) \geq \beta(||E||_{W_\varepsilon}^2 - |\Gamma|).$$

Consequently we have

$$\frac{a(E, E)}{||E||_{W_\varepsilon}} \geq \beta(||E||_{W_\varepsilon} - \frac{|\Gamma|}{||E||_{W_\varepsilon}}),$$

which leads to (2.14).

At this stage we need to show that the solution $E \in W_\varepsilon$ of (2.13) and H given by (2.10) are such that the pair $\begin{pmatrix} E \\ H \end{pmatrix}$ belongs to $D(A)$ and satisfies (2.9). We first show that εE is divergence free (compare with Theorem 7.1 of [6]) by taking test functions $E' = \nabla\phi$ with $\phi \in D(\Delta_\varepsilon^{Dir})$, where $D(\Delta_\varepsilon^{Dir})$ is the domain of the operator Δ_ε^{Dir} with Dirichlet boundary conditions defined by

$$D(\Delta_\varepsilon^{Dir}) = \{\phi \in \overset{\circ}{H}{}^1(\Omega) | \Delta_\varepsilon\phi := \operatorname{div}(\varepsilon\nabla\phi) \in L^2(\Omega)\},$$
$$\Delta_\varepsilon^{Dir}\phi = \Delta_\varepsilon\phi, \forall\phi \in D(\Delta_\varepsilon^{Dir}).$$

In that case (2.13) becomes (the boundary term disappears since ϕ is zero on Γ)

$$\int_\Omega \{\varepsilon E \cdot \nabla\phi + s \operatorname{div}(\varepsilon E)\Delta_\varepsilon\phi\}\, dx = \int_\Omega \varepsilon F \cdot \nabla\phi\, dx, \forall\phi \in D(\Delta_\varepsilon^{Dir}).$$

Since εE and εF have a divergence in $L^2(\Omega)$, by Green's formula in the above left-hand side and right-hand side (allowed since ϕ is in $H^1(\Omega)$, see for instance [1]), we obtain

$$\int_\Omega \operatorname{div}(\varepsilon E)\{\phi + s\Delta_\varepsilon\phi\}\, dx = 0, \forall\phi \in D(\Delta_\varepsilon^{Dir}),$$

since εF is divergence free. Taking $s > 0$ such that $-s^{-1}$ is not an eigenvalue of Δ_ε^{Dir} (always possible since Δ_ε^{Dir} is a negative selfadjoint operator with a discrete spectrum), we conclude that

$$\operatorname{div}(\varepsilon E) = 0 \text{ in } \Omega.$$

Using this fact and the identity (2.10), we see that (2.13) is equivalent to

$$\int_\Omega \{\varepsilon E \cdot E' - H \cdot \operatorname{curl} E'\}\, dx + \int_\Gamma g(E \times \nu) \cdot E' \times \nu\, d\sigma$$
$$= \int_\Omega \varepsilon F \cdot E'\, dx, \forall E' \in W_\varepsilon.$$

Taking first test functions $E' = P_\varepsilon \chi$ with $\chi \in \mathcal{D}(\Omega)^3$ by Lemma 2.3 we get

$$\varepsilon E - \operatorname{curl} H = \varepsilon F \text{ in } \mathcal{D}'(\Omega).$$

This means that the first identity in (2.9) holds since the above identity yields $\operatorname{curl} H \in L^2(\Omega)$.

Now taking test functions $E' = P_\varepsilon \chi$ with $\chi \in C^\infty(\bar{\Omega})^3$ by Remark 2.4 and Green's formula (see Lemma 2.2), we get (2.4).

Finally from (2.10) and the fact that μG is divergence free, μH is also divergence free. $\square$

Nonlinear semigroup theory [27] allows to conclude the following existence results:

Corollary 2.6 *If g satisfies the assumptions of Lemma 2.5, for all $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in \mathcal{H}$, the problem (1.1) admits a unique (weak) solution $\begin{pmatrix} E \\ H \end{pmatrix} \in C(\mathbb{R}_+, \mathcal{H})$. If moreover $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in D(A)$, the problem (1.1) admits a unique (strong) solution $\begin{pmatrix} E \\ H \end{pmatrix} \in W^{1,\infty}(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, D(A))$.*

We finish this section by showing the dissipativity of our system.

Lemma 2.7 *If g satisfies the assumptions of Lemma 2.5, then the energy*

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega \{\varepsilon |E(t,x)|^2 + \mu |H(t,x)|^2\}\, dx \tag{2.15}$$

is non-increasing and

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \int_\Gamma g(E(t) \times \nu) \cdot E(t) \times \nu \, d\sigma dt, \forall 0 \le S < T < \infty. \tag{2.16}$$

Moreover for $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in D(A)$, we have

$$\mathcal{E}'(t) = -\int_\Gamma g(E(t) \times \nu) \cdot E(t) \times \nu \, d\sigma. \tag{2.17}$$

Proof: Since $D(A)$ is dense in $\mathcal{H}$ it suffices to show (2.17). For $\begin{pmatrix} E_0 \\ H_0 \end{pmatrix} \in D(A)$, we have

$$\mathcal{E}'(t) = \int_\Omega \{\varepsilon E(t,x) \cdot E'(t,x) + \mu H(t,x) \cdot H'(t,x)\}\, dx.$$

By (1.1), we get

$$\begin{aligned}\mathcal{E}'(t) &= \int_\Omega \{E(t,x)\cdot \operatorname{curl} H(t,x) - H(t,x)\cdot \operatorname{curl} E(t,x)\}\, dx \\ &= -\left(A\begin{pmatrix} E(t) \\ H(t)\end{pmatrix}, \begin{pmatrix} E(t) \\ H(t)\end{pmatrix}\right)_{\mathcal{H}}.\end{aligned}$$

We conclude by Lemma 2.5. □

3 Exponential stability in the linear case

In this section we find a necessary and sufficient condition for exponential stability in the linear case (i.e. when $g(E) = E$). This condition is the validity of a stability estimate that will be checked in some particular cases.

We start with the following definition.

Definition 3.1 *We say that Ω satisfies the (ε, μ)-stability estimate if there exist $T > 0$ and two non negative constants C_1, C_2 (which may depend on T) with $C_1 < T$ such that*

$$\int_0^T \mathcal{E}(t)\, dt \le C_1\mathcal{E}(0) + C_2 \int_0^T \int_\Gamma |H(t)\times\nu|^2\, d\sigma dt, \tag{3.1}$$

for all solutions $\begin{pmatrix} E(t) \\ H(t)\end{pmatrix}$ *of (1.1) with $g(E) = E$.*

Let us give an equivalent formulation of that property:

Lemma 3.2 *Ω satisfies the (ε, μ)-stability estimate if and only if there exist $T > 0$ and a positive constant C (which may depend on T) such that*

$$\mathcal{E}(T) \le C \int_0^T \int_\Gamma |H(t)\times\nu|^2\, d\sigma dt, \tag{3.2}$$

for all solutions $\begin{pmatrix} E(t) \\ H(t)\end{pmatrix}$ *of (1.1) with $g(E) = E$.*

Proof: $\Rightarrow$: Since $\mathcal{E}(t)$ is non-increasing, the estimate (3.1) implies that

$$T\mathcal{E}(T) \le C_1\mathcal{E}(0) + C_2 \int_0^T \int_\Gamma |H(t)\times\nu|^2\, d\sigma dt.$$

By Lemma 2.7 we get

$$T\mathcal{E}(T) \le C_1\mathcal{E}(T) + (C_1 + C_2) \int_0^T \int_\Gamma |H(t)\times\nu|^2\, d\sigma dt.$$

This yields (3.2) with $C = \frac{C_1+C_2}{T-C_1}$.
$\Leftarrow$: From the monotonicity of $\mathcal{E}$ we may write

$$\int_0^T \mathcal{E}(t)\,dt \leq T\mathcal{E}(0).$$

Again Lemma 2.7 yields

$$\int_0^T \mathcal{E}(t)\,dt \leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}(\mathcal{E}(T) + \int_0^T \int_\Gamma |H(t)\times\nu|^2\,d\sigma dt).$$

Using the assumption (3.2) we obtain

$$\int_0^T \mathcal{E}(t)\,dt \leq \frac{T}{2}\mathcal{E}(0) + \frac{T}{2}(1+C)\int_0^T \int_\Gamma |H(t)\times\nu|^2\,d\sigma dt,$$

which is nothing else than (3.1). □

We now show that the (ε,μ)-stability estimate is equivalent to the exponential stability for a linear feedback.

Theorem 3.3 *Ω satisfies the (ε,μ)-stability estimate if and only if there exist two positive constants M and ω such that*

$$\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \tag{3.3}$$

for all solutions $\begin{pmatrix} E(t) \\ H(t) \end{pmatrix}$ *of (1.1) with $g(E) = E$.*

Proof: Let us start with the necessity of (3.3): By (3.2) and the identity (2.16) of Lemma 2.7 we have

$$\mathcal{E}(T) \leq C(\mathcal{E}(0) - \mathcal{E}(T)).$$

This estimate is equivalent to

$$\mathcal{E}(T) \leq \gamma\mathcal{E}(0),$$

with $\gamma = \frac{C}{1+C}$ which is < 1.

Applying this argument on $[(m-1)T, mT]$, for $m = 1, 2, \cdots$ (which is valid since Maxwell's system is invariant by a translation in time), we will get

$$\mathcal{E}(mT) \leq \gamma\mathcal{E}((m-1)T) \leq \cdots \leq \gamma^m\mathcal{E}(0), m = 1, 2, \cdots$$

Therefore we have

$$\mathcal{E}(mT) \leq e^{-\omega mT}\mathcal{E}(0), m = 1, 2, \cdots$$

with $\omega = \frac{1}{T}\ln\frac{1}{\gamma} > 0$. For an arbitrary positive t, there exists one positive integer m such that $(m-1)T < t \leq mT$. By the nonincreasing property of $\mathcal{E}$, we conclude that

$$\mathcal{E}(t) \leq \mathcal{E}((m-1)T) \leq e^{-\omega(m-1)T}\mathcal{E}(0) \leq \frac{1}{\gamma}e^{-\omega t}\mathcal{E}(0).$$

Let us pass to the sufficiency: From the boundary condition (2.4) and Lemma 2.7, for any $T > 0$, we may write

$$\begin{aligned}\int_0^T \int_\Gamma |H(t)\times\nu|^2\, d\sigma dt &= \int_0^T \int_\Gamma |E(t)\times\nu|^2\, d\sigma dt \\ &= \mathcal{E}(0) - \mathcal{E}(T).\end{aligned}$$

With the help of (3.3), we get

$$\int_0^T \int_\Gamma |H(t)\times\nu|^2\, d\sigma dt \geq \mathcal{E}(0)(1 - Me^{-\omega T}). \tag{3.4}$$

The exponential decay (3.3) also implies

$$\int_0^T \mathcal{E}(t)dt \leq M\mathcal{E}(0)\frac{1-e^{-\omega T}}{\omega}.$$

Consequently for all $C_1 > 0$, we may write

$$\int_0^T \mathcal{E}(t)dt \leq C_1\mathcal{E}(0) + \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)\mathcal{E}(0). \tag{3.5}$$

Choosing T large enough so that $1 - Me^{-\omega T} > 0$ and $C_1 < \min\{\frac{M(1-e^{-\omega T})}{\omega}, T\}$, (3.4) and (3.5) yield (3.1) with

$$C_2 = \left(\frac{M(1-e^{-\omega T})}{\omega} - C_1\right)(1 - Me^{-\omega T})^{-1}.$$

□

Let us now give some examples of domains satisfying the (ε,μ)-stability estimate for some particular coefficients ε and μ. More precisely we assume that Ω is a Lipschitz polyhedron, in the sense that Ω is a bounded, simply connected Lipschitz domain with piecewise plane boundary. We further suppose that Ω is occupied by an electromagnetic medium of piecewise constant electric permittivity ε and piecewise constant magnetic permeability μ, i. e., we assume that there exists a partition $\mathcal{P}$ of Ω in a finite set of Lipschitz polyhedra $\Omega_1, \cdots, \Omega_J$ such that on each Ω_j, $\varepsilon = \varepsilon_j$ and $\mu = \mu_j$, where ε_j and μ_j are positive constants.

We first start with examples inspired from [11, 21]. For that purpose we make the following definition.

Definition 3.4 *We say that Ω is (ε,μ)-substarlike if there exists $\varphi \in W^{2,\infty}(\Omega)$ satisfying*

$$\Delta\varphi(x)|\xi|^2 - 2(D^2\varphi(x)\xi)\cdot\xi \geq \alpha|\xi|^2, \forall \xi \in \mathbb{R}^3, \forall \text{ a.e. } x \in \Omega, \tag{3.6}$$

$$\frac{\partial\varphi}{\partial\nu} > 0 \text{ on } \Gamma, \tag{3.7}$$

$$\frac{\partial\varphi}{\partial\nu_{i_F}}(\varepsilon_{i_F} - \varepsilon_{i'_F}) \leq 0 \text{ on } F, \forall F \in \mathcal{F}_{int}, \tag{3.8}$$

$$\frac{\partial\varphi}{\partial\nu_{i_F}}(\mu_{i_F} - \mu_{i'_F}) \leq 0 \text{ on } F, \forall F \in \mathcal{F}_{int}, \tag{3.9}$$

for some positive real number α. Hereabove and below, for any interior interface F (written in short $F \in \mathcal{F}_{int}$) the different indices i_F and i'_F are such that $F = \bar{\Omega}_{i_F} \cap \bar{\Omega}_{i'_F}$ and are fixed once and for all. For a fixed subdomain Ω_j, ν_j is the exterior unit normal vector along its boundary.

Before giving some examples of domains being (ε, μ)-substarlike, let us show that this (geometrical) property and a density result (which could be interpreted as a geometrical property, cf. [20]) guarantee that Ω satisfies the (ε, μ)-stability estimate. To formulate that result we recall that the space $PH^1(\Omega, \mathcal{P})$ of piecewise H^1 (scalar) function in Ω is defined by

$$PH^1(\Omega, \mathcal{P}) := \{u \in L^2(\Omega) | u_{|\Omega_j} \in H^1(\Omega_j), \forall j = 1, \cdots, J\}.$$

Theorem 3.5 *If Ω is (ε, μ)-substarlike, $PH^1(\Omega, \mathcal{P})^3 \cap W_\varepsilon$ is dense in W_ε and $PH^1(\Omega,\mathcal{P})^3 \cap W_\mu$ is dense in W_μ, then Ω satisfies the (ε, μ)-stability estimate.*

Proof: It suffices to show that the estimate (3.1) holds for any strong solution $\begin{pmatrix} E(t) \\ H(t) \end{pmatrix}$ of (1.1) with $g(E) = E$ and appropriate constants T, C_1, C_2.

Fixing a function $\varphi \in W^{2,\infty}(\Omega)$ from Definition 3.4 we define the multiplier $m = \nabla\varphi$. We now prove that the following estimate holds for all $t \geq 0$:

$$\mathcal{E}(t) \leq \frac{1}{\alpha} \int_\Omega \varepsilon\mu \frac{d}{dt} \{(E \times H) \cdot m\} \, dx + \frac{R_1}{2\alpha} I^0_{ext}, \tag{3.10}$$

where we set

$$\begin{aligned} I^0_{ext} &= \int_\Gamma (\mu|H \times \nu|^2 + \varepsilon|E \times \nu|^2) \, d\sigma, \\ R_1 &= \max_{x\in\Gamma} \frac{|m(x) \cdot \nu(x)|^2 + |m(x)|^2}{2m(x) \cdot \nu(x)}. \end{aligned}$$

The proof of that estimate follows the line of Lemma 8.20 of [13] but adapted to our setting. Indeed starting from

$$I := \int_\Omega \varepsilon\mu \frac{d}{dt} \{(E \times H) \cdot m\} \, dx$$

and using the first two identities of (1.1) we get

$$I = \int_\Omega \{\mu(\operatorname{curl} H \times H) \cdot m + \varepsilon(\operatorname{curl} E \times E) \cdot m\} \, dx.$$

Assume for a moment that E (resp. H) belongs to $PH^1(\Omega, \mathcal{P})^3 \cap W_\varepsilon$ (resp. $PH^1(\Omega, \mathcal{P})^3 \cap W_\mu$). Using the standard identity

$$\operatorname{curl} H \times H = (H \cdot \nabla)H - \frac{1}{2}\nabla|H|^2,$$

we obtain

$$I = -\frac{1}{2}\int_\Omega \left\{\mu\nabla|H|^2\cdot m + \varepsilon\nabla|E|^2\cdot m\right\}\,dx + I_\mu(H) + I_\varepsilon(E),$$

where for shortness we have set

$$I_\mu(H) = \int_\Omega \mu\,((H\cdot\nabla)H)\cdot m\,dx.$$

Using Green's formula we get

$$\begin{aligned} I &= \frac{1}{2}\int_\Omega \left\{\mu|H|^2 + \varepsilon|E|^2\right\}\operatorname{div} m\,dx \\ &- \frac{1}{2}\int_\Gamma (\varepsilon|E|^2 + \mu|H|^2)(m\cdot\nu)\,d\sigma \\ &- \frac{1}{2}\sum_{F\in\mathcal{F}_{int}}\int_F [\varepsilon|E|^2 + \mu|H|^2]_F(m\cdot\nu_{i_F})\,d\sigma \\ &\quad + I_\mu(H) + I_\varepsilon(E), \end{aligned} \tag{3.11}$$

where $[h]_F$ means the jump of the function h through the interface F, i.e., $[h]_F = h_{i_F} - h_{i'_F}$.

Using Green's formula we also transform $I_\mu(H)$ as follows:

$$\begin{aligned} I_\mu(H) &= -\sum_{i,j=1}^3 \int_\Omega \mu H_i\partial_j(m_iH_j)\,dx \\ &\quad + \int_\Gamma \mu(H\cdot\nu)(H\cdot m)\,d\sigma \\ &\quad + \sum_{F\in\mathcal{F}_{int}}\int_F [\mu(H\cdot\nu_{i_F})(H\cdot m)]_F\,d\sigma. \end{aligned}$$

Leibniz's rule yields

$$\begin{aligned} I_\mu(H) &= -\sum_{i,j=1}^3 \int_\Omega \mu H_iH_j\partial_jm_i\,dx - \int_\Omega H\cdot m\operatorname{div}(\mu H)\,dx \\ &\quad + \int_\Gamma \mu(H\cdot\nu)(H\cdot m)\,d\sigma \\ &\quad + \sum_{F\in\mathcal{F}_{int}}\int_F [\mu(H\cdot\nu_{i_F})(H\cdot m)]_F\,d\sigma. \end{aligned}$$

Inserting this identity into (3.11) and introducing the 3×3 matrix

$$M = \operatorname{div} mI - 2Dm,$$

we have obtained

$$2I = \int_\Omega (\mu MH\cdot H + \varepsilon ME\cdot E)\,dx - 2I_0 - I_{\varepsilon,int}(E) - I_{\mu,int}(H) - I_{ext}, \tag{3.12}$$

where we have set

$$
\begin{aligned}
I_{\mu,int}(H) &= \sum_{F\in\mathcal{F}_{int}} \int_F \left\{[\mu|H|^2]_F(m\cdot\nu_{i_F}) - 2[\mu(H\cdot\nu_{i_F})(H\cdot m)]_F\right\} d\sigma,\\
I_{ext} &= \int_\Gamma \{m\cdot\nu(\varepsilon|E|^2+\mu|H|^2) - 2\varepsilon(m\cdot E)(E\cdot\nu)\\
&\qquad\qquad -2\mu(m\cdot H)(H\cdot\nu)\}\, d\sigma,\\
I_0 &= \int_\Omega (E\cdot m\,\mathrm{div}(\varepsilon E) + H\cdot m\,\mathrm{div}(\mu H))\, dx.
\end{aligned}
$$

From the assumption (3.8) (resp. (3.9)) as well as the properties of E (resp. H) through the interior interfaces we readily show that $I_{\varepsilon,int}(E)\le 0$ (resp. $I_{\mu,int}(H)\le 0$). Combined with (3.12) we get

$$
2I \ge \int_\Omega (\mu MH\cdot H + \varepsilon ME\cdot E)\, dx - 2I_0 - I_{ext}.
$$

At this stage we remark that the assumption (3.7) guarantees that $m\cdot\nu>0$ on Γ, therefore by Lemma 8.21 of [13] we get

$$
I_{ext} \le R_1 I^0_{ext},
$$

Finally by the assumption (3.6) we arrive at

$$
2I \ge \alpha \int_\Omega (\mu|H|^2+\varepsilon|E|^2)\, dx - 2I_0 - R_1 I^0_{ext}.
$$

By the density assumptions this inequality remains valid in $W_\varepsilon\times W_\mu$ and, therefore, for any strong solution of (1.1) since $D(A)$ is continuously embedded into $W_\varepsilon\times W_\mu$. From the property $\mathrm{div}(\varepsilon E)=\mathrm{div}(\mu H)=0$ it actually reduces to (3.10).

We now take advantage of the estimate (3.10). From the boundary condition (2.4) (recalling that $g(E)=E$) we have

$$
I^0_{ext} \le M\int_\Gamma |H\times\nu|^2\, d\sigma, \tag{3.13}
$$

where $M=\max_{x\in\Gamma}\max\{\mu(x),\varepsilon(x)\}$. Integrating the estimate (3.10) from 0 to T and taking into account (3.13) we have

$$
\begin{aligned}
\int_0^T \mathcal{E}(t)\, dt &\le \frac{1}{\alpha}\int_\Omega \varepsilon\mu\{(E(T)\times H(T)) - (E(0)\times H(0))\}\cdot m\, dx \qquad (3.14)\\
&\quad + \frac{R_1M}{2\alpha}\int_{\Sigma_T} |H\times\nu|^2\, d\sigma.
\end{aligned}
$$

Since we readily check that

$$
\left|\int_\Omega \varepsilon\mu(E(t)\times H(t))\cdot m\, dx\right| \le C\mathcal{E}(t)\le C\mathcal{E}(0),
$$

with $C = (\max_j \mu_j\varepsilon_j) \max_{x\in\Omega} |m(x)|$ (which is independent of T), we arrive at

$$\int_0^T \mathcal{E}(t)\,dt \le \frac{2C}{\alpha}\mathcal{E}(0) + \frac{R_1 M}{2\alpha}\int_{\Sigma_T} |H\times\nu|^2\,d\sigma. \tag{3.15}$$

This proves the estimate (3.1) by choosing T large enough, i.e., $T > \frac{2C}{\alpha}$. $\square$

Remark 3.6 We gave in [20, Thm 5.1] sufficient conditions on $\Omega, \mathcal{P}$ and ε_j implying the density of $PH^1(\Omega,\mathcal{P})^3 \cap W_\varepsilon$ into W_ε, in particular this density always holds in the homogeneous case i.e. $\varepsilon_j = \varepsilon_1$, for all $j = 1,\cdots,J$. Examples of non-homogeneous structures satisfying these conditions are easily built.

We now give some examples of domains which are (ε,μ)-substarlike:

Example 3.7 Assume that the domain Ω is strictly star-shaped with respect to the origin 0 and that the subdomains are nested in the following sense: the origin belongs to Ω_1 and

$$\bar\Omega_j \cap \bar\Omega_{j+1} = \partial\Omega_j \setminus \partial\Omega_{j-1}, \forall j \ge 2.$$

Then $\varphi(x) = |x|^2/2$ directly satisfies (3.6) and (3.7), while (3.8) and (3.9) are equivalent to

$$\varepsilon_j \le \varepsilon_{j+1} \text{ and } \mu_j \le \mu_{j+1}\,\forall j = 1,\cdots,J.$$

Example 3.8 Assume that the domain Ω is strictly star-shaped with respect to the origin 0 and that the subdomains are around the origin in the sense that the origin belongs to the boundary of all subdomains Ω_j. Then $\varphi(x) = |x|^2/2$ directly satisfies (3.6) and (3.7), while (3.8) and (3.9) are trivially satisfied.

For these two above examples, particular partitions can be given for which in addition to the above assumptions, the density results hold for adequate choice of ε_j and μ_j.

An example of a non star-shaped domain which is (ε,μ)-substarlike is now built using arguments similar to those in [21]. Indeed we remark that a domain Ω is (ε,μ)-substarlike if there exists $\phi \in W^{2,\infty}(\Omega)$ satisfying (3.7), (3.8) and (3.9), as well as

$$\Delta\varphi = 1, \text{ in } \Omega, \tag{3.16}$$

$$\lambda_3(\varphi) < \frac{1}{2}, \tag{3.17}$$

where $\lambda_i(\varphi), i = 1,2,3$, denote the eigenvalues of the matrix $D^2\varphi$ enumerated in increasing order.

Indeed if such a function exists we see that

$$\Delta\varphi(x)|\xi|^2 - 2(D^2\varphi(x)\xi)\cdot\xi \ge (1 - 2\lambda_3(\varphi))|\xi|^2, \forall\xi\in\mathbb{R}^3, \forall \text{ a.e. } x\in\Omega,$$

which directly yields (3.6).

In [21] the assumption (3.17) is replaced by the stronger one

$$\lambda_1(\varphi) > 1/4. \tag{3.18}$$

Indeed this assumption yields (3.17) since

$$\lambda_3(\varphi) = 1 - \lambda_1(\varphi) - \lambda_2(\varphi) \le 1 - 2\lambda_1(\varphi).$$

The advantage of (3.18) is that it is easier to check.

Example 3.9 We first take the 2D domain $\tilde{\Omega}$ with vertices $A = (-1-\delta/2, y_1)$, $B = (1-\delta/2, y_1)$, $C = (1-\delta/2, -\delta)$, $D = (1+\delta/2, -\delta)$, $E = -A$, $F = -B$, $G = -C$, $H = -D$ with $\delta > 0$ such that $\delta < 2$. We define the function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(x) = \frac{1}{4}|x|^2 + P(x),$$

where $P = \Re f$ when f is the holomorphic function

$$f(z) = \frac{1}{2}\eta i \frac{z^2}{z_0},$$

with $z_0 = -1 + i\delta$. Then one readily checks that for δ small enough $\tilde{\varphi}$ satisfies (3.7), (3.16) as well as (3.18). Moreover taking an interface I on the line $x = -1$ one checks that

$$\frac{\partial\tilde{\varphi}}{\partial x_1} < 0 \text{ on } I.$$

Defining $\tilde{\Omega}_1 = \tilde{\Omega} \cap \{x : x_1 < -1\}$ and $\tilde{\Omega}_2 = \tilde{\Omega} \cap \{x : x_1 > -1\}$, we see that (3.8) and (3.9) hold if

$$\varepsilon_1 \le \varepsilon_2 \text{ and } \mu_1 \le \mu_2.$$

Extending this domain into a prism $\Omega = \tilde{\Omega} \times]-1,1[$ divided by the two subdomains $\Omega_j = \tilde{\Omega}_j \times]-1,1[$, we check that

$$\varphi(x) = \tilde{\varphi}(x_1, x_2) + x_3^2/4,$$

satisfies (3.7), (3.16) (3.18) and (3.8) and (3.9) under the above assumptions on ε and μ.

Note further that for this example the density of $PH^1(\Omega, \mathcal{P})^3 \cap W_\varepsilon$ (resp. $PH^1(\Omega, \mathcal{P})^3 \cap W_\mu$) into W_ε (resp. into W_μ) always holds.

All these examples are based on the multiplier technique, we finish this section by giving one coming from microlocal analysis which, up to now, is only applicable for smooth coefficients ε and μ.

Example 3.10 Suppose that Ω is a connected domain with a smooth boundary Γ consisting of a single connected component and assume that $\varepsilon = \mu = 1$. Then Ω satisfies the $(1,1)$-stability estimate. Indeed thanks to Lemma 4.1 of [25] the estimate (3.1) holds with $C_1 = 0$ provided the set $\ker \mathcal{A}$ is reduced to $\{0\}$, this last property being proved using Proposition 3.18 of [1].

Note that this example could be deduced from Proposition 4.1 of [8] but there the estimate (3.1) is obtained using the multiplier technique as well as microlocal analysis.

4 Exact controllability results

Using the results of the previous section we deduce the exact controllability of the heterogeneous Maxwell's system, thereby extending earlier results from [16, 12, 21, 23]. More precisely for all $(E_0, H_0) \in \mathcal{H}$, we are looking for a time $T > 0$ and a control $J \in L^2(\Gamma\times]0,T[)^3$ such that the solution (E, H) of

$$\begin{gathered} \varepsilon \frac{\partial E}{\partial t} - \operatorname{curl} H = 0 \text{ in } Q_T := \Omega\times]0,T[, \\ \mu \frac{\partial H}{\partial t} + \operatorname{curl} E = 0 \text{ in } Q_T, \\ \operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \text{ in } Q_T, \\ H \times \nu = J \text{ on } \Sigma_T := \Gamma\times]0,T[, \\ E(0) = E_0, H(0) = H_0 \text{ in } \Omega, \end{gathered} \tag{4.1}$$

satisfies

$$E(T) = H(T) = 0. \tag{4.2}$$

Theorem 4.1 *If Ω satisfies the (ε, μ)-stability estimate, then for $T > 0$ sufficiently large, for all $(E_0, H_0) \in \mathcal{H}$ there exists a control $J \in L^2(\Sigma_T)^3$ satisfying*

$$J \cdot \nu = 0 \text{ on } \Sigma_T, \tag{4.3}$$

such that the solution $(E, H) \in C([0, T], \mathcal{H})$ of (4.1) is at rest at time T, i.e., satisfies (4.2).

Proof: In fact, it follows as in Theorem 1.1 of [17] that the estimate (3.2) of Lemma 3.2 implies the observability estimate

$$\|(\Phi_0, \Psi_0)\|^2_{\mathcal{H}} \le C \int_0^T \int_\Gamma |\Phi(t) \times \nu|^2 d\sigma dt, \quad \forall (\Phi_0, \Psi_0) \in \mathcal{F}, \tag{4.4}$$

with the same T and C as in (3.2), where (Φ, Ψ) is the solution of

$$\begin{gathered} \varepsilon \frac{\partial \Phi}{\partial t} - \operatorname{curl} \Psi = 0 \text{ in } Q_T, \\ \mu \frac{\partial \Psi}{\partial t} + \operatorname{curl} \Phi = 0 \text{ in } Q_T, \\ \operatorname{div}(\varepsilon \Phi) = \operatorname{div}(\mu \Psi) = 0 \text{ in } Q_T, \\ \Psi \times \nu = 0 \text{ on } \Sigma_T, \\ \Phi(T) = \Phi_0, \Psi(T) = \Psi_0 \text{ in } \Omega, \end{gathered}$$

and where

$$\mathcal{F} = \{(\Phi_0, \Psi_0) \in \mathcal{H} :\, \Phi \times \nu|_{\Sigma_T} \in L^2(\Sigma_T)^3\}.$$

As is well-known, the observability estimate (4.4) implies exact controllability of (4.1) in time T. The control J of minimum norm in $L^2(\Sigma_T)^3$ that steers the solution of (4.1) from (E_0, H_0) to $(0,0)$ at time T is constructed according to J.-L. Lions' Hilbert Uniqueness Method. However, for purposes of deriving our decay rates for the nonlinear system (1.1), we shall need to work with the control constructed according to Russell's principle that steers the solution of (4.1) from (E_0, H_0) to $(0,0)$. The control time T for this control will in general be greater than the T appearing in the observability estimate (4.4), however.

Although the construction of a feasible control via Russell's principle is quite standard, we include it for the sake of completeness. Moreover for further purposes we prefer to solve the inverse problem: Given $(P_0, Q_0) \in \mathcal{H}$, we are looking for $K \in L^2(\Sigma_T)^3$ satisfying (4.3) such that the solution $(P, Q) \in C([0,T], \mathcal{H})$ of

$$\begin{gathered}
\varepsilon \frac{\partial P}{\partial t} - \operatorname{curl} Q = 0 \text{ in } Q_T, \\
\mu \frac{\partial Q}{\partial t} + \operatorname{curl} P = 0 \text{ in } Q_T, \\
\operatorname{div}(\varepsilon P) = \operatorname{div}(\mu Q) = 0 \text{ in } Q_T, \\
Q \times \nu = K \text{ on } \Sigma_T, \\
P(T) = P_0, Q(T) = Q_0 \text{ in } \Omega,
\end{gathered} \tag{4.5}$$

satisfies

$$P(0) = Q(0) = 0. \tag{4.6}$$

Indeed if the above problem has a solution the conclusion follows by setting

$$E(t) = -P(T-t), H(t) = Q(T-t).$$

We solve problem (4.5) and (4.6), using a backward and an inward Maxwell system with Silver-Müller boundary conditions: First given (F_0, I_0) in $\mathcal{H}$, we consider $(F, I) \in C([0,T], \mathcal{H})$ the unique solution of

$$\begin{gathered}
\varepsilon \frac{\partial F}{\partial t} - \operatorname{curl} I = 0 \text{ in } Q_T, \\
\mu \frac{\partial I}{\partial t} + \operatorname{curl} F = 0 \text{ in } Q_T, \\
\operatorname{div}(\varepsilon F) = \operatorname{div}(\mu I) = 0 \text{ in } Q_T, \\
I \times \nu - (F \times \nu) \times \nu = 0 \text{ on } \Sigma_T, \\
F(T) = F_0, I(T) = I_0 \text{ in } \Omega.
\end{gathered} \tag{4.7}$$

Its existence follows from Corollary 2.6 by setting $\tilde{E}(t) = -F(T-t)$ and $\tilde{H}(t) = I(T-t)$. Moreover applying Theorem 3.3 to $(\tilde{E}(t), \tilde{H}(t))$ we get

$$\mathcal{E}(F(t), I(t)) \leq M e^{-\omega(T-t)} \mathcal{E}(F_0, I_0). \tag{4.8}$$

Second we consider $(G, J) \in C([0,T], \mathcal{H})$ the unique solution of (whose existence and uniqueness still follow from Corollary 2.6)

$$\begin{aligned}
&\varepsilon \frac{\partial G}{\partial t} - \operatorname{curl} J = 0 \text{ in } Q_T, \\
&\mu \frac{\partial J}{\partial t} + \operatorname{curl} G = 0 \text{ in } Q_T, \\
&\operatorname{div}(\varepsilon G) = \operatorname{div}(\mu J) = 0 \text{ in } Q_T, \\
&J \times \nu + (G \times \nu) \times \nu = 0 \text{ on } \Sigma_T, \\
&G(0) = F(0), J(0) = I(0) \text{ in } \Omega.
\end{aligned} \tag{4.9}$$

We now take $P = G - F$ and $Q = J - I$. From (4.7) and (4.9), the pair (P, Q) satisfies (4.5) with

$$K = -(G \times \nu) \times \nu - (F \times \nu) \times \nu. \tag{4.10}$$

Let us further consider the mapping Λ from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\Lambda((F_0, I_0)) = (G(T), J(T)).$$

We show that for $T > 0$ such that $d := Me^{-\omega T} < 1$, the mapping $\Lambda - I$ is invertible by proving that $||\Lambda||_{\mathcal{L}(\mathcal{H},\mathcal{H})} < 1$. Indeed using successively the definition of Λ, Lemma 2.7, the initial conditions of problem (4.9) and the estimate (4.8) we have

$$\begin{aligned}
||\Lambda((F_0, I_0))||^2_{\mathcal{H}} &= 2\mathcal{E}((G(T), J(T))) \leq 2\mathcal{E}((G(0), J(0))) \\
&\leq 2\mathcal{E}((F(0), I(0))) \leq 2Me^{-\omega T}\mathcal{E}(F_0, I_0) = d||(F_0, I_0)||^2_{\mathcal{H}}.
\end{aligned}$$

Consequently $||\Lambda||_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \sqrt{d} < 1$.

Since $\Lambda - I$ is invertible for any $(P_0, Q_0) \in \mathcal{H}$, there exists a unique $(F_0, I_0) \in \mathcal{H}$ such that

$$(P_0, Q_0) = (P(T), Q(T)) = (G(T), J(T)) - (F(T), I(T)) = (\Lambda - I)(F_0, I_0). \tag{4.11}$$

The proof will be complete if we can show that $K \in L^2(\Sigma_T)^3$. For that purpose, we remark that Lemma 2.7 (identity (2.16) applied to $(\tilde{E}, \tilde{H})$ and (G, J)) yields

$$\begin{aligned}
\mathcal{E}((F(T), I(T)) - \mathcal{E}((F(0), I(0)) &= \int_{\Sigma_T} |F(t) \times \nu|^2 \, d\sigma dt, \\
\mathcal{E}((G(0), J(0)) - \mathcal{E}((G(T), J(T)) &= \int_{\Sigma_T} |G(t) \times \nu|^2 \, d\sigma dt.
\end{aligned}$$

Summing these two identities and using the initial conditions of problem (4.9), the final conditions of (4.7) and the definition of Λ, we obtain

$$\begin{aligned}
\int_{\Sigma_T} (|F \times \nu|^2 + |G \times \nu|^2) \, d\sigma dt &= \mathcal{E}((F(T), I(T)) - \mathcal{E}((G(T), J(T)) \\
&\leq \frac{1}{2}||(F_0, I_0)||^2_{\mathcal{H}}
\end{aligned}$$

Using the identity (4.11) and the boundedness of $(I-\Lambda)^{-1}$ we finally arrive at the estimate

$$\begin{aligned}\int_{\Sigma_T}(|F\times\nu|^2+|G\times\nu|^2)\,d\sigma dt &\leq \frac{1}{2}||(I-\Lambda)^{-1}(P_0,Q_0)||^2_{\mathcal{H}} \qquad (4.12)\\ &\leq \frac{1}{2(1-\sqrt{d})^2}||(P_0,Q_0)||^2_{\mathcal{H}}.\end{aligned}$$

This proves that K given by (4.10) belongs to $L^2(\Sigma_T)^3$. □

Remark 4.2 The sufficient conditions on Ω,ε,μ given in section 3 which guarantee that Ω satisfies the (ε,μ)-stability estimate are weaker than those found in [23, §6.2] to insure the validity of some observation estimates (leading to exact controllability results). Consequently the conjunction of Theorem 4.1 with the results from section 3 improves some exact controllability results from [23].

5 Stability in the nonlinear case

Liu's principle [19] consists in estimating the energy of the direct system by boundary terms using a retrograde system with final data equal to the final data of the direct system. These boundary terms are then estimated using Russell's principle and the properties of g. For our system the application of this principle is essentially based on the validity of the (ε,μ)-stability estimate. In a second step we use a new integral inequality to deduce decay rates of the energy using appropriate nonlinear feedbacks g.

We start with the new integral inequality obtained similarly to Theorem 9.1 of [13] and proved in detail in [8].

Theorem 5.1 *Let $\mathcal{E}:[0,+\infty)\to[0,+\infty)$ be a non-increasing mapping satisfying*

$$\int_S^\infty \phi(\mathcal{E}(t))\,dt\leq T\mathcal{E}(S),\forall S\geq 0, \qquad (5.1)$$

for some $T>0$ and some strictly increasing convex mapping ϕ from $[0,+\infty)$ to $[0,+\infty)$ such that $\phi(0)=0$. Then there exist $t_1>0$ and c_1 depending on T and $\mathcal{E}(0)$ such that

$$\mathcal{E}(t)\leq\phi^{-1}\left(\frac{\psi^{-1}(c_1t)}{c_1Tt}\right),\forall t\geq t_1, \qquad (5.2)$$

where ψ is defined by

$$\psi(t)=\int_t^1\frac{1}{\phi(s)}\,ds,\forall t>0. \qquad (5.3)$$

Remark 5.2 Theorem 5.1 yields exactly the same decay rate as in Theorem 9.1 of [13] when $\phi(t)=t^{1+\alpha}$ for some $\alpha>0$ (case leading to polynomial decay).

We now give the consequence of this result to our Maxwell system.

Theorem 5.3 *Assume that g satisfies the assumptions of Lemma 2.5 as well as*

$$|E|^2 + |g(E)|^2 \leq G(g(E) \cdot E), \forall |E| \leq 1, \tag{5.4}$$

for some concave strictly increasing function $G : [0, \infty) \to [0, \infty)$ such that $G(0) = 0$. If Ω satisfies the (ε, μ)-stability estimate, then there exist $c_2, c_3 > 0$ and $T_1 > 0$ (depending on T, $\mathcal{E}(0)$ and $|\Gamma|$) such that

$$\mathcal{E}(t) \leq c_3 G\left(\frac{\psi^{-1}(c_2 t)}{c_2 T^2 |\Gamma| t}\right), \forall t \geq T_1, \tag{5.5}$$

for all solution $\begin{pmatrix} E(t) \\ H(t) \end{pmatrix}$ of (1.1), where ψ is given by (5.3) for ϕ defined by

$$\phi(s) = T|\Gamma| G^{-1}(\frac{s}{c_3}). \tag{5.6}$$

Proof: Thanks to Lemma 2.3 it suffices to prove (5.5) for data in $D(A)$. In that case let $\begin{pmatrix} E(t) \\ H(t) \end{pmatrix}$ be the solution of (1.1) and consider (P, Q) the solution of problem (4.5) and (4.6) with $P_0 = E(T)$ and $Q_0 = H(T)$ with $T > 0$ sufficiently large (whose existence was established in Theorem 4.1). By (1.1) and (4.5) we may write

$$\begin{aligned} 0 &= \int_{Q_T} \{\varepsilon P \cdot (E' - \varepsilon^{-1} \operatorname{curl} H) + \mu Q \cdot (H' + \mu^{-1} \operatorname{curl} E) \\ &\quad + \varepsilon E \cdot (P' - \varepsilon^{-1} \operatorname{curl} Q) + \mu H \cdot (Q' + \mu^{-1} \operatorname{curl} P)\} \, dx dt. \end{aligned}$$

By integration by parts in t and Lemma 2.2 this identity becomes

$$\begin{aligned} 0 &= \left[\int_\Omega (\varepsilon P(t) \cdot E(t) + \mu Q(t) \cdot H(t)) \, dx \right]_0^T \\ &\quad + \int_{\Sigma_T} \{(H \times \nu) \cdot P + (Q \times \nu) \cdot E\} \, d\sigma dt. \end{aligned}$$

Using the boundary and initial/final conditions in (1.1) and (4.5), we obtain

$$\mathcal{E}(T) = -\frac{1}{2} \int_{\Sigma_T} \{g(E \times \nu) \cdot P \times \nu + E \cdot K\} \, d\sigma dt.$$

Since K is a tangential vector, we may write $E \cdot K = E_\tau \cdot K$, where the subscript τ denotes the tangential component. Since $|E_\tau| = |E \times \nu|$, Cauchy-Schwarz's inequality in $\mathbb{R}^3$ allows us to transform the last estimate into

$$\mathcal{E}(T) \leq \frac{1}{2} \int_{\Sigma_T} \{|g(E \times \nu)||P \times \nu| + |E \times \nu||K|\} \, d\sigma dt. \tag{5.7}$$

Let us remark that the estimate (4.12) and the final conditions on (P, Q) yield

$$\int_{\Sigma_T} (|F \times \nu|^2 + |G \times \nu|^2)\, d\sigma dt \leq \frac{1}{(1-\sqrt{d})^2}\mathcal{E}(T).$$

This estimate, the definition (4.10) of K, of P ($P = G - F$) and Cauchy-Schwarz's inequality lead to

$$\begin{aligned} \int_{\Sigma_T} |K|^2\, d\sigma dt &\leq \frac{2}{(1-\sqrt{d})^2}\mathcal{E}(T) &(5.8)\\ \int_{\Sigma_T} |P \times \nu|^2\, d\sigma dt &\leq \frac{2}{(1-\sqrt{d})^2}\mathcal{E}(T). &(5.9)\end{aligned}$$

We now estimate each term of the right-hand side of (5.7) as follows: Introduce

$$\begin{aligned} \Sigma_T^+ &= \{(x,t) \in \Sigma_T | |E(x,t) \times \nu(x)| > 1\},\\ \Sigma_T^- &= \{(x,t) \in \Sigma_T | |E(x,t) \times \nu(x)| \leq 1\}.\end{aligned}$$

By Cauchy-Schwarz's inequality we may write

$$\int_{\Sigma_T^+} |g(E\times\nu)||P\times\nu|\, d\sigma dt \leq (\int_{\Sigma_T^+} |P\times\nu|^2\, d\sigma dt)^{\frac{1}{2}} (\int_{\Sigma_T^+} |g(E\times\nu)|^2\, d\sigma dt)^{\frac{1}{2}}.$$

The assumptions (1.4), (1.5) and the estimate (5.9) lead to

$$\int_{\Sigma_T^+} |g(E\times\nu)||P\times\nu|\, d\sigma dt \leq c_4 \mathcal{E}(T)^{\frac{1}{2}} (\int_{\Sigma_T^+} (E\times\nu)\cdot g(E\times\nu)\, d\sigma dt)^{\frac{1}{2}},$$

for some positive constant c_4. By (2.16) we arrive at

$$\int_{\Sigma_T^+} |g(E\times\nu)||P\times\nu|\, d\sigma dt \leq c_4 \mathcal{E}(T)^{\frac{1}{2}} (\mathcal{E}(0) - \mathcal{E}(T))^{\frac{1}{2}}. \tag{5.10}$$

Similarly by Cauchy-Schwarz's inequality, the estimate (5.9) and the assumption (5.4) we have

$$\begin{aligned} \int_{\Sigma_T^-} |g(E\times\nu)||P\times\nu|\, d\sigma dt &\leq (\int_{\Sigma_T} |P\times\nu|^2\, d\sigma dt)^{\frac{1}{2}} (\int_{\Sigma_T^-} |g(E\times\nu)|^2\, d\sigma dt)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{2}}{1-\sqrt{d}}\mathcal{E}(T)^{\frac{1}{2}} (\int_{\Sigma_T^-} G((E\times\nu)\cdot g(E\times\nu))\, d\sigma dt)^{\frac{1}{2}}.\end{aligned}$$

Jensen's inequality then yields

$$\begin{aligned} \int_{\Sigma_T^-} |g(E\times\nu)||P\times\nu|\, d\sigma dt &\leq \frac{\sqrt{2}}{1-\sqrt{d}} |\Sigma_T|^{\frac{1}{2}} \mathcal{E}(T)^{\frac{1}{2}} \cdot\\ &\quad \left(G\left(\frac{1}{|\Sigma_T|}\int_{\Sigma_T^-} (E\times\nu)\cdot g(E\times\nu)\, d\sigma dt\right)\right)^{\frac{1}{2}}.\end{aligned}$$

By (2.16), we arrive at

$$\int_{\Sigma_T^-} |g(E \times \nu)||P \times \nu| \, d\sigma dt \leq \frac{\sqrt{2}|\Sigma_T|^{\frac{1}{2}} \mathcal{E}(T)^{\frac{1}{2}}}{1-\sqrt{d}} \left(G \left(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{|\Sigma_T|} \right) \right)^{\frac{1}{2}} . \quad (5.11)$$

We proceed similarly for the second term of the right-hand side of (5.7). First Cauchy-Schwarz's inequality yields

$$\int_{\Sigma_T^-} |E \times \nu||K| \, d\sigma dt \leq (\int_{\Sigma_T^-} |K|^2 \, d\sigma dt)^{\frac{1}{2}} (\int_{\Sigma_T^-} |E \times \nu|^2 \, d\sigma dt)^{\frac{1}{2}}.$$

The assumption (5.4) and the estimate (5.8) directly lead to

$$\int_{\Sigma_T^-} |E \times \nu||K| \, d\sigma dt \leq \frac{\sqrt{2}}{1-\sqrt{d}} \mathcal{E}(T)^{\frac{1}{2}} (\int_{\Sigma_T} G\left((E \times \nu) \cdot g(E \times \nu)\right) \, d\sigma dt)^{\frac{1}{2}}.$$

Again Jensen's inequality and (2.16) yield

$$\int_{\Sigma_T^-} |E \times \nu||K| \, d\sigma dt \leq \frac{\sqrt{2}}{1-\sqrt{d}} |\Sigma_T|^{\frac{1}{2}} \mathcal{E}(T)^{\frac{1}{2}} \left(G \left(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{|\Sigma_T|} \right) \right)^{\frac{1}{2}} . \quad (5.12)$$

The assumption (1.4), the estimate (5.8) and (2.16) lead similarly to

$$\int_{\Sigma_T^+} |E \times \nu||K| \, d\sigma dt \leq c_5 \mathcal{E}(T)^{\frac{1}{2}} (\mathcal{E}(0) - \mathcal{E}(T))^{\frac{1}{2}}, \quad (5.13)$$

for some constant c_5.

The estimates (5.10), (5.11), (5.12) and (5.13) into the estimate (5.7) give

$$\mathcal{E}(T)^{\frac{1}{2}} \leq c_6 \{ (\mathcal{E}(0) - \mathcal{E}(T))^{\frac{1}{2}} + \left(G \left(|\Gamma|^{-1} T^{-1} (\mathcal{E}(0) - \mathcal{E}(T)) \right) \right)^{\frac{1}{2}} \},$$

for some positive constant c_6 (depending on T and $|\Gamma|$). This finally leads to

$$\begin{aligned} \mathcal{E}(0) &= \mathcal{E}(0) - \mathcal{E}(T) + \mathcal{E}(T) \\ &\leq \max\{1, c_6\} \{ (\mathcal{E}(0) - \mathcal{E}(T)) + G\left(|\Gamma|^{-1} T^{-1} (\mathcal{E}(0) - \mathcal{E}(T)) \right) \}. \end{aligned}$$

As $\frac{\mathcal{E}(0)-\mathcal{E}(T)}{T|\Gamma|} \leq \frac{\mathcal{E}(0)}{T|\Gamma|}$, the concavity of G yields a constant c_7 (depending continuously on T, $\mathcal{E}(0)$ and $|\Gamma|$) such that

$$\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T|\Gamma|} \leq c_7 G(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T|\Gamma|}).$$

These two estimates lead to

$$\mathcal{E}(0) \leq c_3 G(\frac{\mathcal{E}(0) - \mathcal{E}(T)}{T|\Gamma|}),$$

for some $c_3 > 0$ (depending on T, $\mathcal{E}(0)$ and $|\Gamma|$).

Using this argument in $[t, t+T]$ instead of $[0, T]$ we have shown that

$$\mathcal{E}(t) \leq \phi^{-1}(\mathcal{E}(t) - \mathcal{E}(t+T)), \forall t \geq 0, \tag{5.14}$$

when we recall that ϕ was defined by (5.6).

We conclude by Theorem 5.1 since Lemma 5.1 of [8] shows that the estimate (5.14) guarantees that $\mathcal{E}$ actually satisfies (5.1). $\square$

Corollary 5.4 *Assume that g satisfies the assumptions of Lemma 2.5 as well as*

$$|E|^{q+1} + |g(E)|^{q+1} \leq c_8 g(E) \cdot E, \forall |E| \leq 1, \tag{5.15}$$

for some positive constant c_8 and $q \geq 1$. If Ω satisfies the (ε, μ)-stability estimate and $q = 1$, then there exist two positive constants K and ω which may depend on $T, |\Gamma|$ and $\mathcal{E}(0)$ such that

$$\mathcal{E}(t) \leq K e^{-\omega t} \mathcal{E}(0), \tag{5.16}$$

for all solution $\begin{pmatrix} E(t) \\ H(t) \end{pmatrix}$ *of (1.1).*
If Ω satisfies the (ε, μ)-stability estimate and $q > 1$, then there exist a positive constant K_1 which may depend on $T, |\Gamma|$ and $\mathcal{E}(0)$ such that

$$\mathcal{E}(t) \leq K_1 (1+t)^{-\frac{2}{q-1}}, \tag{5.17}$$

for all solution $\begin{pmatrix} E(t) \\ H(t) \end{pmatrix}$ *of (1.1).*

Examples of functions g leading to an explicit decay rate (5.5) are given in [8]. Let us give the following illustration (compare with Examples 2.1 and 2.2 of [8])

Example 5.5 Suppose that g satisfies (1.5) and (1.2) to (1.4) as well as

$$E \cdot g(E) \geq c_0 |E|^{p+1},\ |g(E)| \leq C_0 |E|^{\alpha}, \forall |E| \leq 1, \tag{5.18}$$

for some positive constants c_0, C_0, $\alpha \in (0, 1]$ and $p \geq \alpha$. Then g satisfies (5.15) with $q = \frac{p+1}{\alpha} - 1$ (which is ≥ 1). By Corollary 5.4 we get an exponential decay if $p = \alpha = 1$ and the decay $t^{-\frac{2\alpha}{p+1-2\alpha}}$ if $p + 1 > 2\alpha$. A function g satisfying all these assumptions is given by

$$g(E) = \begin{cases} |E|^{\alpha-1} E & \text{if } |E| \leq 1, \\ E & \text{if } |E| \geq 1, \end{cases}$$

for some $\alpha \in (0, 1]$. In that case (5.18) holds for $p = \alpha$.

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