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Denseness of domains of differential operators in Sobolev spaces. *

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Abstract

Denseness of the domain of differential operators plays an essential role in many areas of differential equations and functional analysis. This, in turn, deals with dense sets in Soblev spaces. Denseness for functions of a single variable was formulated and proved, in a very general form, in the book by Yakubov and Yakubov [8, Theorem 3.4.2/1]. In the same book, denseness for functions of several variables was formulated. However, the proof of such result is complicated and needs a series of constructions which are presented in this paper. We also prove some independent and new results.

1 Introduction

We denote by $\mathbb{R}^r$ the r-dimensional real Euclidean space. For a bounded (open) domian G in $\mathbb{R}^r$ its boundary is denoted by ∂G: $\partial G = \overline{G}/G$.

A bounded domain $G \subset \mathbb{R}^r$ is said to be a C^ℓ, where $\ell = 1, 2, \dots$, if there exists a finite number of open balls G_i, $i = 1, \dots, N$, such that $\partial G \subset \cup_{i=1}^N G_i$, $G_i \cap \partial G \neq \emptyset$, $i = 1, \dots, N$, and if there exist ℓ-fold differentiable real vector-functions $f^{(i)}(x) = (f_1^{(i)}(x), \dots, f_r^{(i)}(x))$ defined in G_i such that $y = f^{(i)}(x)$ is a one-to-one mapping from G_i onto a bounded domain $\mathbb{R}^r$, where $G_i \cap \partial G$ is a part of the hyper-plane $\{y : y \in \mathbb{R}^r; y_r = 0\}$ and $G_i \cap G$ is a simply connected domain in the half-space $\mathbb{R}^r_+ = \{(y', y_r) : y' \in \mathbb{R}^{r-1}; y_r > 0\}$. On the Jacobian of f we assumed that

$$\frac{\partial(f_1^{(i)}, \dots, f_r^{(i)})}{\partial(x_1, \dots, x_r)} \neq 0, \quad x \in \overline{G}_i.$$

In this case we write $\partial G \in C^\ell$ and say that ∂G admits a *local rectification* by means of smooth non-degenerate transformations of coordinates. The coordinates $f^{(i)}$ will be called *local coordinates* in G_i.

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Let

$$L_\nu u = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')D^\alpha u(x') + \sum_{p=0}^{m_\nu-1} K_{\nu p}\frac{\partial^p u(x')}{\partial n^p}, \quad x'\in\partial G,\ \nu=1,\dots,m, \tag{1.1}$$

where $D^\alpha := D_1^{\alpha_1}\cdots D_r^{\alpha_r}$, $D_j := -i\frac{\partial}{\partial x_j}$, $j=1,\dots,r$, $\alpha := (\alpha_1,\dots,\alpha_r)$ is a multi-index, $|\alpha| := \sum_{j=1}^r \alpha_j$, $x := (x_1,\dots,x_r)$, $x' := (x'_1,\dots,x'_r)$, n is a normal vector to the boundary ∂G at the point $x'\in\partial G$. Then $L_\nu u$ is called *normal* if $m_j \neq m_k$ for $j\neq k$ and for any vector σ, normal to the boundary ∂G at the point $x'\in\partial G$,

$$L_{\nu 0}(x',\sigma) = \sum_{|\alpha|=m_\nu} b_{\nu\alpha}(x')\sigma^\alpha \neq 0, \quad \nu=1,\dots,m,$$

and the operator $K_{\nu p}$ from $W_q^{m_\nu-p}(\partial G)$ into $L_q(\partial G)$ is compact, where $\sigma^\alpha = \sigma_1^{\alpha_1}\cdots\sigma_r^{\alpha_r}$, $\partial G\in C^\ell$, $q\in(1,\infty)$.

Let E_0 and E_1 be two Banach spaces continuously embedded into a Banach space E: $E_0\subset E$, $E_1\subset E$. Such spaces are called an *interpolation couple* and is denoted by $\{E_0,E_1\}$. Consider the Banach space

$$\begin{gathered} E_0+E_1 := \big\{u=u_0+u_1 : u_j\in E_j,\ j=0,1\big\} \\ \|u\|_{E_0+E_1} := \inf_{u=u_0+u_1,\ u_j\in E_j} (\|u_0\|_{E_0}+\|u_1\|_{E_1}). \end{gathered}$$

Due to Triebel [7, 1.3.1], the functional

$$K(t,u) := \inf_{u=u_0+u_1,\ u_j\in E_j}\Big(\|u_0\|_{E_0}+t\|u_1\|_{E_1}\Big), \quad u\in E_0+E_1,$$

is continuous on $(0,\infty)$ in t, and the following estimate holds:

$$\min\{1,t\}\|u\|_{E_0+E_1} \le K(t,u) \le \max\{1,t\}\|u\|_{E_0+E_1}.$$

An *interpolation space* for $\{E_0,E_1\}$ by the K-method is defined as follows:

$$\begin{gathered} (E_0,E_1)_{\theta,p} := \big\{u\in E_0+E_1 : \|u\|_{(E_0,E_1)_{\theta,p}}<\infty,\ 0<\theta<1,\ 1\le p<\infty,\big\} \\ \|u\|_{(E_0,E_1)_{\theta,p}} := \Big(\int_0^\infty t^{-1-\theta p}K^p(t,u)\,dt\Big)^{1/p} \\ (E_0,E_1)_{\theta,\infty} := \big\{u\in E_0+E_1 : \|u\|_{(E_0,E_1)_{\theta,\infty}}<\infty,\ 0<\theta<1\big\} \\ \|u\|_{(E_0,E_1)_{\theta,\infty}} := \sup_{t\in(0,\infty)} t^{-\theta}K(t,u). \end{gathered}$$

$W_q^\ell((0,\infty);E)$, $1\le q<\infty$, with ℓ integer, denotes a Banach space of functions $u(x)$ with values from E which have generalized derivatives up to ℓ-th order, inclusive, on $(0,1)$ and the norm $\|u\|_{W_q^\ell((0,\infty);E)} := \sum_{k=0}^\ell \big(\int_0^1 \|u^{(k)}(x)\|_E^q\,dx\big)^{1/q}$ is finite.

Let the embedding $E_0 \subset E_1$ be continuous. Consider the Banach space $W_q^\ell((0,\infty);E_0,E_1) := L_q((0,\infty);E_0) \cap W_q^\ell((0,\infty);E_1)$ with the norm

$$\|u\|_{W_q^\ell((0,\infty);E_0,E_1)} := \|u\|_{L_q((0,\infty);E_0)} + \|u^{(\ell)}\|_{L_q((0,\infty);E_1)}\,.$$

Let G be an open set of $\mathbb{R}^r$, in particular, $G = \mathbb{R}^r$ and $G = \mathbb{R}^r_+$. Then, $W_q^m(G)$ is a Banach space of functions $u(x)$ that have generalized derivatives on G up to the m-th order inclusive, for which the following norm is finite:

$$\|u\|_{W_q^m(G)} := \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|^q_{L_q(G)}\Big)^{1/q}.$$

Let s_0 and s_1 be non-negative integers, $0<\theta<1$, $1<p<\infty$, $1\le q\le\infty$ and $s=(1-\theta)s_0+\theta s_1$. From Triebel [7, Theorem 4.3.2/1, formula 2.4.2/16] it follows that if $s=(1-\theta)s_0+\theta s_1=(1-\theta')s_0'+\theta' s_1'$, then,

$$(W_p^{s_0}(G), W_p^{s_1}(G))_{\theta,q} = (W_p^{s_0'}(G), W_p^{s_1'}(G))_{\theta',q}.$$

Consider the space

$$B^s_{p,q}(G) := (W_p^{s_0}(G), W_p^{s_1}(G))_{\theta,q},$$

where s_0, s_1 are non-negative integers, $0<\theta<1$, $1<p<\infty$, $1\le q\le\infty$ and $s=(1-\theta)s_0+\theta s_1$. For s positive and not an integer, set

$$W_p^s(G) := B^s_{p,p}(G) := (W_p^{s_0}(G), W_p^{s_1}(G))_{\theta,p}\,.$$

The closure of a set M of E by the norm of E is denoted by $\overline{M}|_E$ and sometimes by $\overline{M}$. The set M is called *dense* in E if $\overline{M}|_E = E$.

The following two equalities are application of Theorem 2.5 in this paper. For $(r-1)/q<2$, we have:

$$\overline{W_q^3\big(G; u|_{\partial G}=0, \frac{\partial^2 u}{\partial n^2}\Big|_{\partial G} + u(x_0')=0\big)}\Big|_{W_q^2(G)} = W_q^2(G; u|_{\partial G}=0),$$

$$\overline{W_q^4\big(G; u|_{\partial G}=0, \frac{\partial^2 u}{\partial n^2}\Big|_{\partial G} + u(x_0')=0\big)}\Big|_{W_q^3(G)}$$
$$= W_q^3\big(G; u|_{\partial G}=0, \frac{\partial^2 u}{\partial n^2}\Big|_{\partial G} + u(x_0')=0\big)\,,$$

where $x_0' \in \partial G$ is a fixed point of the boundary ∂G. These equalities are simple but are new. Such equalities without the term $u(x_0')$ follow from interpolation theorems of Grisvard-Seeley type (see, Grisvard [3], Seeley [6]). The following equality is also known:

$$\overline{C_0^\infty(G)}\Big|_{W_q^k(G)} = W_q^k\Big(G; u|_{\partial G} = \frac{\partial u}{\partial n}\Big|_{\partial G} = \dots = \frac{\partial^{k-1} u}{\partial n^{k-1}}\Big|_{\partial G} = 0\Big).$$

Moreover, one can add first derivatives of the function u at some fixed points of the boundary ∂G, and integral terms with $u(x')$ in addition to $u(x_0')$ in boundary conditions.

2 Functions of several real variables

Lemma 2.1 *Let* $\varphi_j \in W_q^{\ell-j-\frac{1}{q}}(\mathbb{R}^{r-1})$*, where* ℓ, j *are integer numbers,* $0 \le j \le \ell-1$*,* $q \in (1,\infty)$*. Then, there exist functions* $u_j(y_r,\lambda) = u_j(y', y_r, \lambda)$*,* $\lambda > 0$ *belonging to the Banach space* $W_q^\ell(\mathbb{R}_+^r) = W_q^\ell((0,\infty); W_q^\ell(\mathbb{R}^{r-1}), L_q(\mathbb{R}^{r-1}))$ *and satisfying*

$$\frac{\partial^j u_j(y',0,\lambda)}{\partial y_r^j} = \varphi_j(y'), \quad y' \in \mathbb{R}^{r-1}, \tag{2.1}$$

such that the following estimate holds

$$\sum_{k=0}^{\tilde{\ell}} \lambda^{\tilde{\ell}-k} \|u_j\|_{W_q^k(\mathbb{R}_+^r)} \le C\Big(\|\varphi_j\|_{W_q^{\tilde{\ell}-j-\frac{1}{q}}(\mathbb{R}^{r-1})} + \lambda^{\tilde{\ell}-j-\frac{1}{q}}\|\varphi_j\|_{L_q(\mathbb{R}^{r-1})}\Big), \tag{2.2}$$

where $0 \le j \le \tilde{\ell}-1$*,* $\tilde{\ell} \le \ell$*.*

Proof In the Banach space $E = L_q(\mathbb{R}^{r-1})$ consider the operator $A = (-\Lambda + I)^2$. By virtue of Lemma 3.1, for $k = 1, 2, \dots,$

$$D(A^k) = D(\Lambda^{2k}) = W_q^{2k}(\mathbb{R}^{r-1}), \quad D(A^{\frac{k}{2}}) = D(\Lambda^k) = W_q^k(\mathbb{R}^{r-1}).$$

Consider the functions

$$u_j(y_r,\lambda) = \mathrm{e}^{-y_r(A+\lambda^2 I)^{1/2}} g_j, \tag{2.3}$$

where $g_j \in E$. Since

$$u_{jy_r}^{(j)}(y_r,\lambda) = (-1)^j \mathrm{e}^{-y_r(A+\lambda^2 I)^{1/2}}(A+\lambda^2 I)^{j/2} g_j,$$

the functions in (2.3) satisfy (2.1) if $(-1)^j(A+\lambda^2 I)^{\frac{j}{2}} g_j = \varphi_j$. Consequently,

$$u_j(y_r,\lambda) = (-1)^j \mathrm{e}^{-y_r(A+\lambda^2 I)^{1/2}}(A+\lambda^2 I)^{-j/2}\varphi_j. \tag{2.4}$$

Since

$$\begin{aligned} A^{\frac{k}{2}} u_j(y_r,\lambda) &= (-1)^j A^{k/2}(A+\lambda^2 I)^{-\frac{j}{2}} \mathrm{e}^{-y_r(A+\lambda^2 I)^{1/2}}\varphi_j, \\ u_j^{(k)}(y_r,\lambda) &= (-1)^{k+j}(A+\lambda^2 I)^{\frac{k-j}{2}} \mathrm{e}^{-y_r(A+\lambda^2 I)^{1/2}}\varphi_j, \end{aligned}$$

for $k \le \ell$, we have

$$
\begin{aligned}
\lambda^{(\ell-k)q}\|u_j\|^q_{W^k_q(\mathbb{R}^r_+)} &= \lambda^{(\ell-k)q}\|u_j\|^q_{W^k_q((0,\infty);W^k_q(\mathbb{R}^{r-1}),L_q(\mathbb{R}^{r-1}))} \\
&= \lambda^{(\ell-k)q}\|u_j\|^q_{W^k_q((0,\infty);E(A^{\frac{k}{2}}),E)} \\
&\le C\lambda^{(\ell-k)q}\Big(\|A^{\frac{k}{2}}u_j\|^q_{L_q((0,\infty);E)} + \|u_j^{(k)}\|^q_{L_q((0,\infty);E)}\Big) \\
&\le C\lambda^{(\ell-k)q}\int_0^\infty \Big(\|A^{\frac{k}{2}}(A+\lambda^2 I)^{-\frac{j}{2}}\mathrm{e}^{-y_r(A+\lambda^2I)^{\frac12}}\varphi_j\|^q_E \\
&\quad + \|(A+\lambda^2I)^{\frac{k-j}{2}}\mathrm{e}^{-y_r(A+\lambda^2I)^{\frac12}}\varphi_j\|^q_E\Big)dy_r \\
&\le C\lambda^{(\ell-k)q}\|(A+\lambda^2I)^{-\frac{\ell-k}{2}}\|^q_{B(E)}(\|A^{\frac{k}{2}}(A+\lambda^2I)^{-\frac{k}{2}}\|^q_{B(E)}+1) \\
&\quad \times \int_0^\infty \|(A+\lambda^2I)^{\frac{\ell-j}{2}}\mathrm{e}^{-y_r(A+\lambda^2I)^{\frac12}}\varphi_j\|^q_E dy_r.
\end{aligned}
$$

By virtue of Lemma 3.2 and Theorem 3.3, we have

$$
\lambda^{(\ell-k)q}\|u_j\|^q_{W^k_q(\mathbb{R}^r_+)} \le C\Big(\|\varphi_j\|^q_{(E,E(A^\ell))_{\frac{\ell-j}{2\ell}-\frac{1}{2\ell q},q}} + \lambda^{(\ell-j)q-1}\|\varphi_j\|^q_E\Big). \tag{2.5}
$$

Since

$$
(E,E(A^\ell))_{\frac{\ell-j}{2\ell}-\frac{1}{2\ell q},q} = (L_q(\mathbb{R}^{r-1}),W^{2\ell}_q(\mathbb{R}^{r-1}))_{\frac{\ell-j}{2\ell}-\frac{1}{2\ell q},q} = W_q^{\ell-j-\frac1q}(\mathbb{R}^{r-1}), \tag{2.6}
$$

from (2.5) and (2.6) it follows that a function defined by (2.4) belongs to the space $W^\ell_q(\mathbb{R}^r_+) = W^\ell_q((0,\infty);W^\ell_q(\mathbb{R}^{r-1}),L_q(\mathbb{R}^{r-1}))$ and estimate (2.2) holds. □

Theorem 2.2 *Let $\varphi_j \in W_q^{\ell-j-\frac1q}(\mathbb{R}^{r-1})$, $0 \le j \le m-1$, $m \le \ell$. Then, there exist functions $u(y_r,\lambda) = u(y',y_r,\lambda)$, $\lambda > 0$, belonging to the Banach space $W^\ell_q(\mathbb{R}^r_+) = W^\ell_q((0,\infty);W^\ell_q(\mathbb{R}^{r-1}),L_q(\mathbb{R}^{r-1}))$ and satisfying*

$$
\frac{\partial^j u(y',0,\lambda)}{\partial y_r^j} = \varphi_j(y'), \quad y' \in \mathbb{R}^{r-1},\ j=0,\dots,m-1, \tag{2.7}
$$

such that the following estimate holds

$$
\sum_{k=0}^{\tilde\ell}\lambda^{\tilde\ell-k}\|u\|_{W^k_q(\mathbb{R}^r_+)} \le C\sum_{j=0}^{m-1}\Big(\|\varphi_j\|_{W_q^{\tilde\ell-j-\frac1q}(\mathbb{R}^{r-1})} + \lambda^{\tilde\ell-j-\frac1q}\|\varphi_j\|_{L_q(\mathbb{R}^{r-1})}\Big), \tag{2.8}
$$

where $m \le \tilde\ell \le \ell$.

Proof By virtue of Lions and Magenes [5, Theorem 1.3.2], if

$$
U_j(y_r,\lambda) = \sum_{p=1}^m c_{pj}u_j(py_r,\lambda), \tag{2.9}
$$

where

$$u_j(py_r,\lambda)=(-1)^j \mathrm{e}^{-py_r(A+\lambda^2 I)^{1/2}}(A+\lambda^2 I)^{-\frac{j}{2}}\varphi_j, \tag{2.10}$$

the operator A is defined in the proof of Lemma 2.1, and the complex numbers c_{pj} satisfy the systems

$$\sum_{p=1}^{m} p^k c_{pj}=\begin{cases}0, & k\neq j,\\ 1, & k=j,\end{cases} \quad k=0,\dots,m-1, \tag{2.11}$$

then

$$\frac{\partial^k U_j(0,\lambda)}{\partial y_r^k}=\begin{cases}0, & k\neq j,\\ \varphi_j, & k=j.\end{cases} \tag{2.12}$$

Consequently, the function

$$u(y_r,\lambda)=\sum_{j=0}^{m-1} U_j(y_r,\lambda)=\sum_{j=0}^{m-1}\sum_{p=1}^{m} c_{pj}u_j(py_r,\lambda) \tag{2.13}$$

belongs to the space $W_q^\ell(\mathbb{R}_+^r)=W_q^\ell((0,\infty);W_q^\ell(\mathbb{R}^{r-1}),L_q(\mathbb{R}^{r-1}))$ and satisfies (2.7). From (2.2) and (2.9)–(2.13) for the function $u(y_r,\lambda)$, it follows estimate (2.8). $\square$

Corollary 2.3 *For* $\lambda>0$, *there exists a continuous operator which is a continuation of* $\mathbb{R}(\lambda):\ (\varphi_0,\dots,\varphi_{m-1})\to\mathbb{R}(\lambda)(\varphi_0,\dots,\varphi_{m-1})$ *from* $\dot{+}_{j=0}^{m-1} W_q^{\ell-j-\frac{1}{q}}(\mathbb{R}^{r-1})$ *into* $W_q^\ell(\mathbb{R}_+^r)$, *such that*

$$\frac{\partial^j\mathbb{R}(\lambda)(\varphi_0,\dots,\varphi_{m-1})(y',0)}{\partial y_r^j}=\varphi_j(y'),\quad y'\in\mathbb{R}^{r-1},\ j=0,\dots,m-1$$

and

$$\sum_{k=0}^{\tilde\ell}\lambda^{\tilde\ell-k}\|\mathbb{R}(\lambda)(\varphi_0,\dots,\varphi_{m-1})\|_{W_q^k(\mathbb{R}_+^r)} \le C\sum_{j=0}^{m-1}\Big(\|\varphi_j\|_{W_q^{\tilde\ell-j-\frac{1}{q}}(\mathbb{R}^{r-1})}+\lambda^{\tilde\ell-j-\frac{1}{q}}\|\varphi_j\|_{L_q(\mathbb{R}^{r-1})}\Big).$$

Proof Define a continuation operator as

$$\mathbb{R}(\lambda)(\varphi_0,\dots,\varphi_{m-1}):=\sum_{j=0}^{m-1}\sum_{p=1}^{m} c_{pj}u_j(py_r,\lambda)$$

and apply Theorem 2.2. $\square$

Theorem 2.4 *Let the following conditions be satisfied:*

1. *$b_{\nu\alpha} \in C^{\ell-m_\nu}(\overline{G})$, operators $K_{\nu p}$ from $W_q^{m_\nu-p}(\partial G)$ into $L_q(\partial G)$ and from $W_q^{\ell-p}(\partial G)$ into $W_q^{\ell-m_\nu}(\partial G)$ are compact, where $\ell \geq \max\{m_\nu\}+1$, $q \in (1,\infty)$, $\partial G \in C^\ell$.*

2. *System (1.1) is normal.*

3. *$f_\nu \in W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)$, $\nu = 1,\dots,m$.*

Then, there exist functions $u(x,\lambda)$, $\lambda > 0$, belonging to the Sobolev space $W_q^\ell(G)$ and satisfying

$$L_\nu u(x',\lambda) = f_\nu(x'), \quad x' \in \partial G,\ \nu = 1,\dots,m, \tag{2.14}$$

where L_ν are defined in (1.1), such that the following estimate holds

$$\sum_{k=0}^{\tilde\ell} \lambda^{\tilde\ell-k}\|u\|_{W_q^k(G)} \leq C\sum_{\nu=1}^m\Big(\|f_\nu\|_{W_q^{\tilde\ell-m_\nu-\frac{1}{q}}(\partial G)} + \lambda^{\tilde\ell-m_\nu-\frac{1}{q}}\|f_\nu\|_{L_q(\partial G)}\Big), \tag{2.15}$$

where $\max\{m_\nu\}+1 \leq \tilde\ell \leq \ell$.

Proof Consider the balls G_i, $i = 1,\dots,N$, from $\mathbb{R}^r$, which cover the ∂G, i.e.,

$$\partial G \subset \cup_{i=1}^N G_i; \quad G_i \cap \partial G \neq 0, \quad i = 1,\dots,N.$$

Let $\{\theta_i(x)\}$ be a partition of unity subordinate to a cover of ∂G by $\{G_i\}$ (see, e. g., Lions and Magenes [5, 2.5.1]). The functions $\theta_i(x)$ have the following properties:

1. The support of the function $\theta_i(x)$ belongs to the set G_i, i.e., $\theta_i(x) = 0$ outside of G_i;

2. Functions $\theta_i(x)$ are infinitely differentiable on $\mathbb{R}^r$;

3. $0 \leq \theta_i(x) \leq 1$; $\sum_{i=1}^N \theta_i(x) \equiv 1$, $x \in \partial G$.

In G_i we introduce a system of curvilinear coordinates $y_1(x'),\dots,\ y_r(x')$, where $x' \in \partial G$. Assume that $y_r(x') = n(x')$ is the normal vector, while $y_1(x'),\dots,y_{r-1}(x')$ are tangential vectors on ∂G. The operators L_ν may be expressed in these curvilinear coordinates as

$$\begin{aligned}\tilde L_\nu \tilde u := & c_\nu(y',0)\frac{\partial^{m_\nu}\tilde u(y',0)}{\partial y_r^{m_\nu}} + \sum_{|\alpha|\leq m_\nu, |\alpha_r|<m_\nu} c_{\nu\alpha}(y',0)\frac{\partial^\alpha \tilde u(y',0)}{\partial y_1^{\alpha_1}\cdots\partial y_{r-1}^{\alpha_{r-1}}\partial y_r^{\alpha_r}} \\ & + \sum_{p=0}^{m_\nu-1}\tilde K_{\nu p}\frac{\partial^p \tilde u(y',0)}{\partial y_r^p}, \quad \nu = 1,\dots,m,\ (y',0) \in f^{(i)}(G_i\cap\partial G),\end{aligned} \tag{2.16}$$

where

$$\tilde{u}(y) := u((f^{(i)})^{-1}(y)), \quad y \in f^{(i)}(G_i \cap G),$$
$$\tilde{K}_{\nu p}\frac{\partial^p \tilde{u}(y',0)}{\partial y_r^p} := \Big(K_{\nu p}\frac{\partial^p u(x')}{\partial n^p}\Big)((f^{(i)})^{-1}(y',0)), \quad (y',0) = f^{(i)}(x'),$$

where $c_\nu,\ c_{\nu\alpha} \in C^{\ell-m_\nu}(f^{(i)}(G_i \cap G))$, and $f^{(i)}$ is defined in a similar way as in the beginning of the paper. It holds that $c_\nu(y',0) \neq 0$ for $(y',0) \in f^{(i)}(G_i \cap \partial G)$.

We look for functions $u(x,\lambda) \in W_q^\ell(G)$, $\lambda > 0$, satisfying the relations

$$\tilde{L}_\nu \tilde{u} = \tilde{f}_\nu(y',0), \quad (y',0) \in f^{(i)}(G_i \cap \partial G), \quad \nu = 1,\dots,m, \tag{2.17}$$

where $\tilde{L}_\nu$ are operators defined in (2.16), and for which additionally

$$\frac{\partial^j \tilde{u}(y',0,\lambda)}{\partial y_r^j} = 0, \quad j \neq m_\nu, \quad j = 0,\dots,\max\{m_\nu\}-1, \quad (y',0) \in f^{(i)}(G_i \cap \partial G). \tag{2.18}$$

Consider for $(y',0) \in f^{(i)}(G_i \cap \partial G)$ the functions

$$\begin{aligned}
\varphi_j(y') :=& 0, \quad j \neq m_\nu, \quad j = 0,\dots,\max\{m_\nu\}-1,\\
\varphi_{m_\nu}(y') :=& \Big(c_\nu(y',0)\Big)^{-1}\Big[\tilde{f}_\nu(y',0)\\
&- \sum_{|\alpha|\le m_\nu, |\alpha_r|<m_\nu} c_{\nu\alpha}(y',0)\frac{\partial^\alpha \varphi_{\alpha_r}(y')}{\partial y_1^{\alpha_1}\cdots \partial y_{r-1}^{\alpha_{r-1}}} - \sum_{p=0}^{m_\nu-1} \tilde{K}_{\nu p}\varphi_p(y')\Big].
\end{aligned} \tag{2.19}$$

Since $W_q^s = (W_q^{s_0}, W_q^{s_1})_{\theta,q}$, $s > 0$ is not an integer, s_0, s_1 are integers, $0 < \theta < 1$, and $s = (1-\theta)s_0 + \theta s_1$, by virtue of the interpolation theorem [4] (see, e.g., [7, 1.16.4]) and condition 1 of Theorem 2.4, the operators $K_{\nu p}$ from $W_q^{k-p-\frac{1}{q}}(\partial G)$ into $W_q^{k-m_\nu-\frac{1}{q}}(\partial G)$, for $m_\nu + 1 \le k \le \ell$, are compact. Then, from conditions 1 and 3 of Theorem 2.4 and (2.19) it follows that $\varphi_j \in W_q^{\ell-j-\frac{1}{q}}(f^{(i)}(G_i \cap \partial G)$.

Let $\eta_i(x) \in C^\infty(\mathbb{R}^r)$, $i = 1,\dots,N$, and $\operatorname{supp}\eta_i \subset G_i$, $\eta_i(x) = 1$, $x \in \operatorname{supp}\theta_i$. Consider the function

$$\begin{aligned}
u(x,\lambda) :=& \overline{\mathbb{R}}(\lambda)(\varphi_0,\dots,\varphi_{\tilde{m}-1})(x,\lambda)\\
:=& \sum_{i=1}^N \eta_i(x)\mathbb{R}(\lambda)(\theta_i((f^{(i)})^{-1}(y',0))\varphi_0,\\
&\dots,\theta_i((f^{(i)})^{-1}(y',0))\varphi_{\tilde{m}-1})(f^{(i)}(x),\lambda),
\end{aligned} \tag{2.20}$$

where $\tilde{m} = \max\{m_\nu\} + 1$, for $x \in G$ (where $\eta_i(x)\mathbb{R}\{\quad\}(f^{(i)}(x), \lambda) = 0$ outside of G_i). By virtue of Corollary 2.3, for function (2.20) we have

$$\begin{aligned} \frac{\partial^j \tilde{u}(y', 0, \lambda)}{\partial y_r^j} &= \frac{\partial^j u((f^{(i)})^{-1}(y', 0), \lambda)}{\partial y_r^j} \\ &= \sum_{i=1}^{N} \theta_i((f^{(i)})^{-1}(y', 0))\varphi_j = \varphi_j(y'),\ y' \in \mathbb{R}^{r-1},\ j = 0, \dots, \tilde{m} - 1, \end{aligned} \tag{2.21}$$

where $\varphi_j(y')$ is defined in (2.19). From (2.19) and (2.21) we get (2.17) and (2.14). Since the mapping $f^{(i)}$ is a diffeomorphism of class C^ℓ then, by virtue of Corollary 2.3, function (2.20) satisfies estimate (2.15). □

Theorem 2.5 *Let the following conditions be satisfied:*

1. $b_{\nu\alpha} \in C^{\ell - m_\nu}(\overline{G})$, *operators* $K_{\nu p}$ *from* $W_q^{m_\nu - p}(\partial G)$ *into* $L_q(\partial G)$ *and from* $W_q^{\ell - p}(\partial G)$ *into* $W_q^{\ell - m_\nu}(\partial G)$ *are compact, where* $\ell \geq \max\{m_\nu\} + 1$, $q \in (1, \infty)$, $\partial G \in C^\ell$.
2. *System (1.1) is normal.*

Then, for integer $k \in [0, \ell]$,

$$\overline{W_q^\ell(G; L_\nu u = 0, \nu = 1, \dots, m)}\Big|_{W_q^k(G)} = W_q^k(G; L_\nu u = 0, m_\nu \leq k - 1). \tag{2.22}$$

Proof For $k = 0$, (2.22) follows from the known embedding $\overline{C_0^\infty(G)}\Big|_{L_q(G)} = L_q(G)$. Let $k \geq 1$. Obviously,

$$\overline{W_q^\ell(G; L_\nu u = 0, \nu = 1, \dots, m)}\Big|_{W_q^k(G)} \subset W_q^k(G; L_\nu u = 0, m_\nu \leq k - 1). \tag{2.23}$$

Indeed, let $u_n \in W_q^\ell(G; L_\nu u = 0, \nu = 1, \dots, m)$ and let

$$\lim_{n \to \infty} \|u_n - u\|_{W_q^k(G)} = 0.$$

It is proved in Theorem 2.4 that from condition 1 of Theorem 2.5 it follows that operators $K_{\nu p}$ from $W_q^{k-p-\frac{1}{q}}(\partial G)$ into $W_q^{k-m_\nu-\frac{1}{q}}(\partial G)$, for $m_\nu + 1 \leq k \leq \ell$, are compact. Then, by virtue of [7, Theorem 4.7.1], for $m_\nu \leq k - 1$ we have

$$\begin{aligned} \lim_{n \to \infty} \|L_\nu u_n - L_\nu u\|_{W_q^{k-m_\nu-\frac{1}{q}}(\partial G)} &\leq C \lim_{n \to \infty} \|u_n - u\|_{W_q^{k-\frac{1}{q}}(\partial G)} \\ &\leq C \lim_{n \to \infty} \|u_n - u\|_{W_q^k(G)} = 0, \quad m_\nu \leq k - 1. \end{aligned} \tag{2.24}$$

Thus, $L_\nu u = 0$, $m_\nu \leq k - 1$, since $L_\nu u_n = 0$.

Now, we show the inverse inclusion

$$W_q^k(G; L_\nu u = 0, m_\nu \le k-1) \subset \overline{W_q^\ell(G; L_\nu u = 0, \nu = 1, \dots, m)}\Big|_{W_q^k(G)}. \quad (2.25)$$

Let $u \in W_q^k(G; L_\nu u = 0, m_\nu \le k-1)$. Then, there exists a sequence of functions $v_n(x) \in C^\infty(G)$, $n = 1, \dots, \infty$, such that

$$\lim_{n\to\infty} \|v_n - u\|_{W_q^k(G)} = 0. \quad (2.26)$$

From (2.26) and (2.24) it follows that

$$\lim_{n\to\infty} \|L_\nu v_n\|_{W_q^{k-m_\nu-\frac{1}{q}}(\partial G)} = \|L_\nu u\|_{W_q^{k-m_\nu-\frac{1}{q}}(\partial G)} = 0, \quad m_\nu \le k-1. \quad (2.27)$$

By virtue of Theorem 2.4 (for $\tilde{\ell} = k$, $\lambda = \lambda_0$), there exists a solution $w_n \in W_q^\ell(G)$ of the system

$$L_\nu w_n = -L_\nu v_n, \quad m_\nu \le k-1, \quad (2.28)$$

and

$$\|w_n\|_{W_q^k(G)} \le C \sum_{m_\nu \le k-1} \|L_\nu v_n\|_{W_q^{k-m_\nu-\frac{1}{q}}(\partial G)}. \quad (2.29)$$

Then, from (2.27) and (2.29) it follows that

$$\lim_{n\to\infty} \|w_n\|_{W_q^k(G)} = 0. \quad (2.30)$$

Let λ_n be a sequence tending to ∞, if with respect to n

$$\|L_\nu(v_n + w_n)\|_{W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)} \le C, \ m_\nu \ge k; \quad (2.31)$$

and

$$\lambda_n = \max_{m_\nu \ge k} \|L_\nu(v_n + w_n)\|^\delta_{W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)}, \quad (2.32)$$

where $\delta > q$, if $\|L_\nu(v_n + w_n)\|_{W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)}$ is not a bounded sequence at least for one $m_\nu \ge k$.

Apply Theorem 2.4 (for $\tilde{\ell} = \ell$, $\lambda = \lambda_n$) to the system

$$\begin{aligned} L_\nu g_n &= 0, \quad m_\nu \le k-1, \\ L_\nu g_n &= -L_\nu(v_n + w_n) \quad m_\nu \ge k. \end{aligned} \quad (2.33)$$

Then there exists a solution $g_n(x)$ of (2.33) on $W_q^\ell(G)$ and for this solution, as $n \to \infty$,

$$\begin{aligned} \lambda_n^{\ell-k} \|g_n\|_{W_q^k(G)} \le & C \sum_{m_\nu \ge k} \Big(\lambda_n^{\ell-m_\nu-\frac{1}{q}} \|L_\nu(v_n + w_n)\|_{L_q(\partial G)} \\ & + \|L_\nu(v_n + w_n)\|_{W_q^{\ell-m_\nu-\frac{1}{q}}(\partial G)} \Big). \end{aligned}$$

From this and (2.31),(2.32) we have

$$\begin{aligned} \|g_n\|_{W_q^k(G)} &\le C \sum_{m_\nu \ge k} (\lambda_n^{\ell - m_\nu - \frac{1}{q} + \frac{1}{\delta}} + \lambda_n^{\frac{1}{\delta}}) \lambda_n^{-\ell + k} \\ &\le C \sum_{m_\nu \ge k} (\lambda_n^{k - m_\nu - \frac{1}{q} + \frac{1}{\delta}} + \lambda_n^{-\ell + k + \frac{1}{\delta}}). \end{aligned} \tag{2.34}$$

Since $\delta > q$, (2.34) and (2.31),(2.32) imply

$$\lim_{n\to\infty} \|g_n\|_{W_q^k(G)} = 0. \tag{2.35}$$

Now, it is easy to see that for the sequence of functions $u_n = v_n + w_n + g_n \in W_q^\ell(G)$ the relations

$$L_\nu u_n = 0, \quad \nu = 1, \dots, m, \tag{2.36}$$

$$\lim_{n\to\infty} \|u_n - u\|_{W_q^k(G)} = 0 \tag{2.37}$$

hold. Namely, (2.36) follows from (2.28) and (2.33), and (2.37) follows from (2.26), (2.30), and (2.35). So, the inverse inclusion (2.25) has also been proved. From (2.23) and (2.25) it follows (2.22). □

3 Appendix

Lemma 3.1 ([7, Lemma 2.5.3 and formula 2.5.3/11]) *Let* $1 < q < \infty$, $c_{r-1}\|(1+|x|^2)^{-\frac{r}{2}}\|_{L_1(\mathbb{R}^{r-1})} = 1$ *and*

$$\Big(P(t)u\Big)(x) = c_{r-1} \int_{\mathbb{R}^{r-1}} \frac{t}{(|x-y|^2 + t^2)^{r/2}} u(y)dy, \quad 0 < t < \infty,\ u \in L_q(\mathbb{R}^{r-1}).$$

If additionally $P(0) = I$, *then* $P(t)$ *is a holomorphic semigroup in the space* $L_q(\mathbb{R}^{r-1})$. *If* Λ *is a corresponding infinitesimal operator for* $P(t)$, *then*

$$\Lambda^{2m} u = (-1)^m \Delta^m u, \quad D(\Lambda^m) = W_q^m(\mathbb{R}^{r-1}), \quad m = 1, 2, \dots .$$

The estimate of the semigroup is a central point in the theory of differential-operator equations.

For $\varphi \in (0, \pi)$ we denote by Σ_φ the closed sector $\{\lambda \in \mathbb{C} : |\arg\lambda| \le \varphi\} \cup \{0\}$.

Let A be a closed, densely defined operator in a complex Banach space E. We say that A is of type φ with bound L if for every $\lambda \in \Sigma_\varphi$ the operator $A + \lambda I$ is invertible with bounded inverse and

$$\|(A + \lambda I)^{-1}\| \le \frac{L}{|\lambda| + 1}.$$

We note that, in particular, an operator of type φ is invertible. Moreover, an operator of type φ with $\varphi > \pi/2$ is the inverse of the generator of a holomorphic semigroup whose norm decays exponentially at infinity (see Fattorini

[2, Theorem 4.2.2]). If A is of type φ with bound L, then for every $\lambda \in \Sigma_\varphi$,

$$\|A(A+\lambda I)^{-1}\| = \|I - \lambda(A+\lambda I)^{-1}\| \le 1 + \frac{L|\lambda|}{|\lambda|+1} \le 1 + L\,.$$

We remark that, in view of the properties of the resolvent operator, if $A+\lambda I$ is invertible for $\lambda \in [0,\infty)$ and $\sup_{\lambda\in[0,\infty)} \|(\lambda+1)(A+\lambda I)^{-1}\| < \infty$, then there exists $\varphi > 0$ such that A is of type φ.

Lemma 3.2 ([8, Lemma 5.4.2/6]) *Let A be a closed, densely defined operator in a Banach space E and*

$$\|R(\lambda, A)\| \le C(1+|\lambda|)^{-1}, \quad |\arg\lambda| \ge \pi - \varphi,$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ is the resolvent of the operator A and $0 \le \varphi < \pi$. Then,

a) For $|\arg\lambda| \le \varphi$, $\alpha \in \mathbb{R}$ there exist fractional powers A^α and $(A+\lambda I)^\alpha$ for $0 \le \alpha \le \beta$ with

$$\|A^\alpha(A+\lambda I)^{-\beta}\| \le C(1+|\lambda|)^{\alpha-\beta}, \quad |\arg\lambda| \le \varphi;$$

b) For $|\arg\lambda| \le \varphi$ there exists the semigroup $\mathrm{e}^{-x(A+\lambda I)^{1/2}}$, which is holomorphic for $x > 0$ and strongly continuous for $x \ge 0$; moreover, for $\alpha \in \mathbb{R}$ and for some $\omega > 0$,

$$\|(A+\lambda I)^\alpha e^{-x(A+\lambda I)^{1/2}}\| \le C\mathrm{e}^{-\omega x|\lambda|^{1/2}}, \qquad x \ge x_0 > 0,\ |\arg\lambda| \le \varphi.$$

Theorem 3.3 ([8, Theorem 5.4.2/1]) *Let E be a complex Banach space, A be a closed operator in E of type φ with bound L. Moreover, let m be a positive integer, $p \in (1,\infty)$ and $\alpha \in (\frac{1}{2p}, m + \frac{1}{2p})$. Then, there exists C (depending only on L, φ, m, α and p) such that for every $u \in (E, E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}$ and $\lambda \in \Sigma_\varphi$,*

$$\int_0^\infty \Big\|(A+\lambda I)^\alpha \mathrm{e}^{-x(A+\lambda I)^{\frac12}} u\Big\|^p dx \le C\Big(\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}} + |\lambda|^{p\alpha-\frac12}\|u\|^p\Big)\,.$$

The proof of this theorem can be found in [1].

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