ELECTRONIC JOURNAL OF DIFFERENTIAL EQUATIONS, Vol. **2002**(2002), No. 29, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Regularity for solutions to the Navier-Stokes equations with one velocity component regular *

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Abstract

In this paper, we establish a regularity criterion for solutions to the Navier-stokes equations, which is only related to one component of the velocity field. Let (u,p) be a weak solution to the Navier-Stokes equations. We show that if any one component of the velocity field u, for example u_3, satisfies either $u_3 \in L^\infty(\mathbb{R}^3 \times (0,T))$ or $\nabla u_3 \in L^p(0,T;L^q(\mathbb{R}^3))$ with $1/p + 3/2q = 1/2$ and $q \geq 3$ for some $T > 0$, then u is regular on $[0,T]$.

1 Introduction

We consider the initial-value problem of the Navier-Stokes equations in $\mathbb{R}^3$ as follows:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u = -\nabla p, \\ &\operatorname{div} u = 0, \quad (x, t=0) = a(x) \end{aligned} \tag{1.1}$$

in which $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity field and the scalar pressure function of the fluid at the point $(x,t) \in \mathbb{R}^3 \times (0,+\infty)$, while a is a given initial velocity field, and ν is the viscosity coefficient of the fluid. We will assume that $\nu \equiv 1$ for simplicity.

For any given $a \in L^2(\mathbb{R}^3)$ and $\operatorname{div} a = 0$ in the sense of the distribution, a global weak solution $u \in L^\infty(0,\infty;L^2(\mathbb{R}^3))$ to (1.1) with $\nabla u \in L^2(\mathbb{R}^3 \times (0,\infty))$ was constructed by Leary [6] for a long time ago. But the regularity and the uniqueness of his weak solutions remain yet open, in the general case, till now, in spite of the great efforts made. Moreover, the uniqueness of the weak solutions follows if the regularity for weak solutions can be obtained. Cf. [10]. Therefore many regularity criteria have been obtained for weak solutions. In a space-time cylinder $\mathbb{R}^3 \times (0,T)$, the regularity class was showed for weak solutions which belong to the class $L^p(0,T;L^q(\mathbb{R}^3))$ by Serrin [8] in the case of $1/p + 3/2q < 1$ and $q > 3$; by Fabes, Jones and Riviere [4], Sohr and von Wahl [11] in the case of $1/p + 3/2q = 1$ and $q > 3$. Also see Giga [5] and Takahashi [9]. For the critical case as $p = \infty$ and $q = 3$, W.von Wahl showed that $C([0,T];L^3(\mathbb{R}^3))$ is

Mathematics Subject Classifications: 35Q30, 76D05.
Key words: Navier-Stokes equations, weak solutions, regularity.

Submitted November 7, 2001. Published March 17, 2002.

a regularity class for weak solutions. Also see [2]. Another kind of regularity class was obtained by da Veiga [1], in which he showed that u is regular if $\nabla u \in L^\alpha(0,T;L^\beta(\mathbb{R}^3))$ with $1/\alpha + 3/2\beta = 1/2$ and $1 < \alpha \leq 2$. It is obvious that, for the assumptions of all regularity criteria, it need that every components of the velocity field must satisfies the same assumptions, and it don't make any difference between the components of the velocity field. Thus, it don't exploit the relation between the regularity of velocity field and that of any one component of the velocity field, which is hard to be used in some applications. As pointed out by Neustupa and Penel [7], it is interesting to know how to effect the regularity of the velocity field by the regularity of only one component of the velocity field, when one try to construct an counter-example of solutions to (1.1), which develops blowup at finite time, i.e., construct an irregular solution. In this respect, the first result was obtained by Neustupa and Penel [7]. They showed that, for the suitable weak solutions which essentially differ from the usual weak solutions in that they should verify a generalized energy inequality, if u_3 is essentially bounded in one subdomain D, then u has no singular points in D. Their arguments depend heavily on the partial regularity of the suitable weak solutions developed by Caffarelli, Kohn and Nirenberg [3].

In this paper, we are also interested in establishing the regularity criteria which is only related to one component of the velocity field, and which is valid for any weak solutions satisfying the energy inequality. Actually, let u be any weak solution to the Navier-Stokes equations (1.1) in $L^\infty(0,\infty;L^2(\mathbb{R}^3)) \cap L^2_{loc}(0,\infty;H^1(\mathbb{R}^3))$ which verifies the energy inequality, if any one component, e.g., u_3 of the velocity field u satisfying either $u_3 \in L^\infty(\mathbb{R}^3 \times (0,T))$ or $\nabla u_3 \in L^p(0,T;L^q(\mathbb{R}^3))$ with $1/p + 3/2q = 1/2$ and $q \geq 3$, then u is regular. Our analysis is motivated by the argument of Neustupa and Penel [7], and here we improve their arguments.

This paper is organized as follows: in section 2, we state our main results after introducing some notations, in section 3, we given the proof of the main result.

2 Main Result

In this section, we first introduce some notations and the definition of weak solutions, then state our main result.

Let $L^p(\mathbb{R}^3), 1 \leq p \leq +\infty$, represent the usual Lesbegue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(\mathbb{R}^3)$ denote the set of all C^∞ real vector-valued functions $\phi = (\phi_1,\phi_2,\phi_3)$ with compact support in $\mathbb{R}^3$, such that $\operatorname{div}\phi = 0$. $L^p_\sigma(\mathbb{R}^3)$, $1 \leq p < \infty$, is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ with respect to $\|\cdot\|_p$. $H^m(\mathbb{R}^3)$ denotes the usual Sobolev Space. Finally, given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X), 1 \leq p \leq +\infty$, the set of function $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|^p_X dt < +\infty$. Let $Q_T = \mathbb{R}^3 \times (0,T)$. At last, by symbol C, we denote a generic constant whose value is unessential to our aims, and it may change from line to line.

Definition. A measurable vector u on Q_T is called a weak solution to the Navier-Stokes equations (1.1), if

1. $u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$;
2. u verifies the Navier-Stokes equations in the sense of the distribution, i.e.,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^3} \big(u(x,t)\cdot \frac{\partial \phi}{\partial t} - \nabla u(x,t)\cdot\nabla\phi + (u(x,t)\cdot\nabla)\phi\cdot u(x,t)\big)dxdt \\ + \int_{\mathbb{R}^3} a(x)\phi(x,0)dx &= 0\end{aligned}$$

for every $\phi \in C_0^\infty(\mathbb{R}^3\times\mathbb{R})$ with $\operatorname{div}\phi = 0$.

Our main result can be stated as follows.

Theorem 2.1 *Let the initial velocity $a \in L^2_\sigma(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$, and u is the weak solution to (1.1) which satisfies the energy inequality. If any one component of the velocity field u, e.g., u_3 satisfies either $u_3 \in L^\infty(\mathbb{R}^3\times(0,T))$ or $\nabla u_3 \in L^p(0,T;L^q(\mathbb{R}^3))$ with $1/p+3/2q = 1/2$ for $q \geq 3$, then u is regular on $[0,T]$, and for $t\in[0,T]$ the following two estimates are satisfied.*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \leq \|a\|_2^2, \tag{2.1}$$

$$\|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(s)\|_2^2 ds \leq A \tag{2.2}$$

In the various cases, A depends on the initial data as follows:

$$\begin{aligned} A_1 &= \|\nabla a\|_2^2 + CM_1^2\|a\|_2^2\big(1+\|a\|_2^4M_1^2\big) + C\|a\|_2^2\big(\|a\|_2M_1 + \|\nabla a\|_2\big) \\ &\quad \times\big[\|\nabla a\|_2^2 + \|a\|_2^2M_1^2\big(1+\|a\|_2^4M_1^2\big)\big]e^{C\|a\|_2\left(\|a\|_2M_1+\|\nabla a\|_2\right)} \end{aligned}$$

where $M_1 := \|\nabla u_3\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} < \infty$,

$$\begin{aligned} A_2 &= \|\nabla a\|_2^2 + CA_3\big\{\|a\|_2^{\frac{4q}{q-3}}M_2 + \|a\|_2\big(\|a\|_{H^1(\mathbb{R}^3)}(1+M_2^{1/2}e^{CM_2})\big) + M_2\big\} \\ &\quad + C\|a\|_2^2M_2^{\frac{q-3}{q}}, \end{aligned}$$

where $M_2 := \int_0^T \|\nabla u_3\|^p_{L^p(0,T;L^q(\mathbb{R}^3))}ds < \infty$,

$$\begin{aligned} A_3 &= \big\{\|\nabla a\|_2^2 + C\|a\|_2^2M_2^{\frac{q-3}{q}}\big\} \\ &\quad \times\exp\big\{M_2\|a\|_2^{\frac{4q}{q-3}} + \|a\|_2\|a\|_{H^1(\mathbb{R}^3)}\big(1+M^{1/2}e^{CM_2}\big) + M_2\big\}, \\ A_4 &= \|\nabla a\|_2^2 + C\big(M_0^2 + M_0^{8/3}\|a\|_2^{4/3}\big)\|a\|_2^2 \\ &\quad + A_5\|a\|_2\big(\|\nabla a\|_2 + M_0\|a\|_2 + M_0^{4/3}\|a\|_2^{5/3}\big), \end{aligned}$$

where $M_0 := \|u_3\|_{L^\infty(\mathbb{R}^3\times(0,T))} < \infty$, and

$$\begin{aligned} A_5 &= \big(\|\nabla a\|_2^2 + C(M_0^2 + \|a\|_2^{4/3} M_0^{8/3})\|a\|_2^2\big) \\ &\quad \times \exp\big\{C\|a\|_2\big(\|\nabla a\|_2 + M_0\|a\|_2 + M_0^{4/3}\|a\|_2^{5/3}\big)\big\}. \end{aligned}$$

The constant C, above, is independent of T. Thus T can approach $+\infty$.

Remarks. 1. The weak solution $u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ was constructed by Leray [6] as the initial velocity field $a \in L^2_\sigma(\mathbb{R}^3)$, such that u satisfies the energy inequality.
2. Neustupa and Penel [7] showed that, for the suitable weak solution u, if u_3 is essentially bounded in a subdomain D of a time-space cylinder $\Omega \times (0,T)$, then u has no singular points in D.
3. For solutions to the Navier-Stokes equations, either the class $L^p(0,T;L^q)$ for $1/p + 3/2q \le 1/2$ and $q > 3$, or $C([0,T];L^3(\Omega))$ is a regularity class, Cf. [4], [5], [8], [9],[10] and [2]. Further, if $\nabla u \in L^\alpha(0,T;L^\beta(\mathbb{R}^3))$ with $1/\alpha + 3/2\beta = 1$ for $1 < \alpha \le 2$, then u also is regular, see [1].

3 Proof of the main theorem

In this section, we present the proof of Theorem 2.1, preceded by the following Lemmas. The first lemma follows from direct calculations.

Lemma 3.1 *Let $a \in L^2_\sigma(\mathbb{R}^3)$. Then for any $t \ge 0$,*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \le \|a\|_2^2. \tag{3.1}$$

Let $\omega = \operatorname{curl} u(x,t)$. Then ω satisfies weakly the equations

$$\frac{\partial \omega}{\partial t} - \Delta\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u. \tag{3.2}$$

For the third component ω_3, of ω, one obtain the following estimate.

Lemma 3.2 *Let $a \in L^2_\sigma(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. If $u_3 \in L^\infty(\mathbb{R}^3 \times (0,T))$, then*

$$\|\omega_3(t)\|_2^2 + \int_0^t \|\nabla\omega_3(s)\|_2^2 ds \le \|\omega_3(0)\|_2^2 + M_0^2\|a\|_2^2 \tag{3.3}$$

for any $0 \le t \le T$ with $M_0 = \|u_3\|_{L^\infty(\mathbb{R}^3\times(0,T))}$.

Proof. Multiplying both sides of the third equation of (3.2) by ω_3 and integrating for $x \in \mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega_3\|_2^2 + \|\nabla\omega_3\|_2^2 = \int_{\mathbb{R}^3} (\omega\cdot\nabla)u_3\cdot\omega_3 dx. \tag{3.4}$$

Using integration by parts and the Cauchy inequality, the right term of (3.4) can be estimated as

$$\begin{aligned}\int_{\mathbb{R}^3}(\omega\cdot\nabla)u_3\cdot\omega_3dx &= -\int_{\mathbb{R}^3}(\omega\cdot\nabla)\omega_3\cdot u_3dx\\ &\leq \|\nabla\omega_3\|_2\|u_3\|_\infty\|\omega\|_2\\ &\leq \frac{1}{2}\|\nabla\omega_3\|_2^2+\frac{1}{2}\|u_3\|_\infty^2\|\nabla u\|_2^2,\end{aligned}$$

where we used the fact that $\|\omega\|_2\leq\|\nabla u\|_2$. Thus

$$\frac{d}{dt}\|\omega_3\|_2^2+\|\nabla\omega_3\|_2^2\leq M_0^2\|\nabla u\|_2^2. \tag{3.5}$$

Integrating (3.5) over $(0,t)$ gives (3.3). □

Lemma 3.3 *Let $a\in L^2_\sigma(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$. If $\nabla u_3\in L^p(0,T;L^q(\mathbb{R}^3))$ with $1/p+3/2q=1/2$ for $q\geq 3$, then*

$$\|\omega_3(t)\|_2^2+\int_0^t\|\nabla\omega_3(s)\|_2^2ds\leq\begin{cases}C\big(\|a\|^2_{H^1(\mathbb{R}^3)}+\|a\|_2^2M_1^2\big) & \text{for } q=3,\\ C\|a\|_{H^1(\mathbb{R}^3)}\big(1+M_2e^{CM_2}\big) & \text{for } q>3\end{cases} \tag{3.6}$$

for $0<t\leq T$. Here $M_1:=\|\nabla u_3\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}$ and $M_2:=\int_0^T\|\nabla u_3(s)\|_q^pds$ for $q>3$.

Proof. If $q=3$, by the Hölder inequality and the Sobolev inequality, the right term of (3.4) can be estimated as

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}(\omega\cdot\nabla)u_3\cdot\omega_3dx\Big| &\leq \|\omega\|_2\|\nabla u_3\|_3\|\omega_3\|_6\\ &\leq C\|\omega\|_2\|\nabla u_3\|_3\|\nabla\omega_3\|_2\\ &\leq \frac{1}{2}\|\nabla\omega_3\|_2^2+C\|\nabla u_3\|_3^2\|\nabla u\|_2^2.\end{aligned}$$

Then

$$\frac{d}{dt}\|\omega_3\|_2^2+\|\nabla\omega_3\|_2^2\leq C\|\nabla u_3\|_2^2\|\nabla u\|_2^2$$

which implies (3.6) as $q=3$.

If $q>3$, by the Hölder inequality and the Gagliardo-Nirenberg inequality, we estimate the right term of (3.4) as

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}(\omega\cdot\nabla)u_3\cdot\omega_3dx\Big| &\leq \|\omega\|_2\|\nabla u_3\|_q\|\omega_3\|_{2q/(q-2)}\\ &\leq C\|\omega\|_2\|\nabla u_3\|_q\|\omega_3\|_2^{1-3/q}\|\nabla\omega_3\|_2^{3/q}\\ &\leq \frac{1}{2}\|\nabla\omega_3\|_2^2+C\|\nabla u_3\|_q^p\|\omega_3\|_2^2+C\|\nabla u\|_2^2.\end{aligned}$$

Then

$$\frac{d}{dt}\|\omega_3\|_2^2 + \|\nabla\omega_3\|_2^2 \le C\|\nabla u_3\|_q^p\|\omega_3\|_2^2 + C\|\nabla u\|_2^2 \tag{3.7}$$

which implies that

$$\frac{d}{dt}\big(e^{-C\int_0^t \|\nabla u_3\|_q^p ds}\|\omega_3\|_2^2\big) \le Ce^{-C\int_0^t \|\nabla u_3\|_q^p ds}\|\nabla u\|_2^2.$$

Thus using Lemma 3.1,

$$\|\omega_3\|_2^2 \le Ce^{CM_2}\big(\|\omega_3(0)\|_2^2 + \|a\|_2^2\big). \tag{3.8}$$

Substituting (3.8) into (3.7), (3.6) follows for $q > 3$. □

Lemma 3.4 *Let $a \in L^2_\sigma(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. If $\nabla u_3 \in L^p(0,T;L^q(\mathbb{R}^3))$ with $1/p + 3/2q = 1/2$ for $q \ge 3$, then for any $t \ge 0$,*

$$\|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(s)\|_2^2 ds \le A_1\,, \tag{3.9}$$

where, for $q = 3$,

$$\begin{aligned} A_1 &= \|\nabla a\|_2^2 + CM_1^2\|a\|_2^2\big(1 + \|a\|_2^4 M_1^2\big) + C\|a\|_2^2\big(\|a\|_2 M_1 + \|\nabla a\|_2\big) \\ &\quad \times\big[\|\nabla a\|_2^2 + \|a\|_2^2 M_1^2\big(1 + \|a\|_2^4 M_1^2\big)\big]e^{C\|a\|_2\big(\|a\|_2 M_1 + \|\nabla a\|_2\big)}. \end{aligned}$$

When $q > 3$, then for any $t \ge 0$,

$$\|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(s)\|_2^2 ds \le A_2 \tag{3.10}$$

with

$$\begin{aligned} A_2 &= \|\nabla a\|_2^2 + CA_3\big\{\|a\|_2^{\frac{4q}{q-3}} M_2 + \|a\|_2\big(\|a\|_{H^1(\mathbb{R}^3)}(1 + M_2^{1/2}e^{CM_2})\big) \\ &\quad + M_2\big\} + C\|a\|_2^2 M_2^{\frac{q-3}{q}} \\ A_3 &= \big\{\|\nabla a\|_2^2 + C\|a\|_2^2 M_2^{\frac{q-3}{q}}\big\}\exp\big\{M_2\|a\|_2^{\frac{4q}{q-3}} \\ &\quad + \|a\|_2\|a\|_{H^1(\mathbb{R}^3)}\big(1 + M^{1/2}e^{CM_2}\big) + M_2\big\}. \end{aligned}$$

The constant C, above, is independent of T. Thus T can approach $+\infty$.

Proof. The Navier-Stokes equations (1.1) can be rewritten as

$$\frac{\partial u}{\partial t} - \Delta u + \omega \times u = -\nabla(p + \frac{1}{2}|u|^2). \tag{3.11}$$

Multiplying both sides of (3.11) by Δu, it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|\Delta u\|_2^2 &= \int_{\mathbb{R}^3} \omega \times u \cdot \Delta u dx \\ &= \int_{\mathbb{R}^3} (\omega_2 u_3 - \omega_3 u_2)\Delta u_1 dx + \int_{\mathbb{R}^3} (\omega_3 u_1 - \omega_1 u_3)\Delta u_2 dx \\ &\quad + \int_{\mathbb{R}^3} (\omega_1 u_2 - \omega_2 u_1)\Delta u_3 dx. \end{aligned} \tag{3.12}$$

In the following, we estimate the each terms at the right hand side of (3.12) by the Hölder inequality, Young inequality and Gagliardo- Nirenberg inequality. Let δ be one small parameter determined later. One has that

$$\begin{aligned} I_1 &= \Big|\int_{\mathbb{R}^3} \omega_2 u_3 \Delta u_1 dx\Big| \le \|\Delta u_1\|_2 \|\omega_2\|_4 \|u_3\|_4 \\ &\le C\|\Delta u\|_2 \|u_3\|_2^{1/2} \|\nabla u_3\|_3^{1/2} \|\omega_2\|_2^{1/4} \|\nabla \omega_2\|_2^{3/4} \\ &\le C\|\Delta u\|_2^{\frac{7}{4}} \|u\|_2^{1/2} \|\nabla u_3\|_3^{1/2} \|\nabla u\|_2^{1/4} \\ &\le \delta \|\Delta u\|_2^2 + C(\delta)\|a\|_2^4 \|\nabla u_3\|_3^4 \|\nabla u\|_2^2, \end{aligned}$$

for $q = 3$, and

$$\begin{aligned} I_1 &= |\int_{\mathbb{R}^3} \omega_2 u_3 \Delta u_1 dx| \le \|\Delta u_1\|_2 \|\omega_2\|_{2r/(r-2)} \|u_3\|_r \\ &\le C\|\Delta u\|_2 \|u_3\|_2^{\frac{2rq+6q-6r}{r(5q-6)}} \|\nabla u_3\|_q^{\frac{3q(r-2)}{r(5q-6)}} \|\omega_2\|_2^{\frac{r-3}{r}} \|\nabla \omega_2\|_2^{\frac{3}{r}} \\ &\le C\|\Delta u\|_2^{\frac{r+3}{r}} \|u\|_2^{\frac{2rq+6q-6r}{r(5q-6)}} \|\nabla u_3\|_q^{\frac{3q(r-2)}{r(5q-6)}} \|\nabla u\|_2^{\frac{r-3}{r}} \\ &\le \delta \|\Delta u\|_2^2 + C(\delta)\|a\|_2^{\frac{4q}{q-3}} \|\nabla u_3\|_q^p \|\nabla u\|_2^2, \end{aligned}$$

for $q > 3$ with $r = 9q/(2q+3) > 3$.

$$\begin{aligned} I_2 &= |\int_{\mathbb{R}^3} \omega_3 u_2 \Delta u_1 dx| \le \|\Delta u_1\|_2 \|\omega_3\|_3 \|u_2\|_6 \\ &\le \delta \|\Delta u\|_2^2 + C(\delta)\|\nabla u_2\|_2^2 \|\omega_3\|_2 \|\nabla \omega_3\|_2 \\ &\le \delta \|\Delta u\|_2^2 + C(\delta)\|\nabla \omega_3\|_2 \|\nabla u\|_2^3. \end{aligned}$$

Similar to the estimates as I_2,

$$I_3 = |\int_{\mathbb{R}^3} \omega_3 u_1 \Delta u_2 dx| \le \delta \|\Delta u\|_2^2 + C(\delta)\|\nabla \omega_3\|_2 \|\nabla u\|_2^3.$$

Similar to the estimates as I_1, one can obtain that

$$\begin{aligned} I_4 &= |\int_{\mathbb{R}^3} \omega_1 u_3 \Delta u_2 dx| \\ &\le \begin{cases} \delta \|\Delta u\|_2^2 + C(\delta)\|a\|_2^4 \|\nabla u_3\|_3^4 \|\nabla u\|_2^2, & \text{if } q = 3, \\ \delta \|\Delta u\|_2^2 + C(\delta)\|a\|_2^{\frac{4q}{q-3}} \|\nabla u_3\|_q^p \|\nabla u\|_2^2, & \text{if } q > 3. \end{cases} \end{aligned}$$

Let

$$I_5 = |\int_{\mathbb{R}^3} \omega_1 u_2 \Delta u_3 dx| \le |\int_{\mathbb{R}^3} \partial_2 u_3 \cdot u_2 \cdot \Delta u_3 dx| + |\int_{\mathbb{R}^3} \partial_3 u_2 \cdot u_2 \cdot \Delta u_3 dx|.$$

Then

$$\begin{aligned} I_{51} &= |\int_{\mathbb{R}^3} \partial_2 u_3 \cdot u_2 \cdot \Delta u_3 dx| \le \|\Delta u_3\|_2 \|\nabla u_3\|_q \|u_2\|_{2q/(q-2)} \\ &\le C\|\Delta u\|_2 \|\nabla u_3\|_q \|u\|_2^{\frac{q-3}{q}} \|\nabla u\|_2^{\frac{3}{q}} \\ &\le \delta \|\Delta u\|_2^2 + C(\delta)\|a\|_2^{\frac{2(q-3)}{q}} \|\nabla u_3\|_q^2 \|\nabla u\|_q^{\frac{6}{q}} \end{aligned}$$

and

$$\begin{aligned} I_{52} &= |\int_{\mathbb{R}^3} \partial_2 u_2 \cdot u_2 \cdot \Delta u_3 dx| = |\frac{1}{2}\int_{\mathbb{R}^3} |u_2|^2 \Delta\partial u_3 dx| \\ &\le |\int_{\mathbb{R}^3} u_2 \Delta u_2 \cdot \partial u_3 dx| + |\int_{\mathbb{R}^3} |\nabla u_2|^2 \partial_3 u_3 dx| \\ &\le \frac{\delta}{2}\|\Delta u_2\|_2^2 + C(\delta)\|\nabla u_3\|_q^2 \|u_2\|_{2q/(q-2)}^2 + \|\nabla u_3\|_q \|\nabla u_2\|_{2q/(q-2)}^2 \\ &\le \frac{\delta}{2}\|\Delta u\|_2^2 + C(\delta)\|\nabla u_3\|_q^2 \|u\|_2^{\frac{2(q-3)}{q}} \|\nabla u\|_2^{\frac{6}{q}} + C\|\nabla u_3\|_q \|\nabla u\|_2^{\frac{2q-3}{q}} \|\Delta u\|_2^{\frac{3}{q}} \\ &\le \delta\|\Delta u\|_2^2 + C(\delta)\|\nabla u_3\|_q^2 \|a\|_2^{\frac{2(q-3)}{q}} \|\nabla u\|_2^{\frac{6}{q}} + C(\delta)\|\nabla u_3\|_q^p \|\nabla u\|_2^2. \end{aligned}$$

Similarly,

$$\begin{aligned} I_6 &= |\int_{\mathbb{R}^3} \omega_2 u_1 \Delta u_3 dx| \le |\int_{\mathbb{R}^3} \partial_3 u_1 \cdot u_1 \cdot \Delta u_3 dx| + |\int_{\mathbb{R}^3} \partial_1 u_3 \cdot u_1 \cdot \Delta u_3 dx| \\ I_{61} &= |\int_{\mathbb{R}^3} \partial_3 u_1 \cdot u_1 \cdot \Delta u_3 dx| \\ &\le \delta\|\Delta u\|_2^2 + C(\delta)\|\nabla u_3\|_q^2 \|a\|_2^{\frac{2(q-3)}{q}} \|\nabla u\|_2^{\frac{6}{q}} + C(\delta)\|\nabla u_3\|_q^p \|\nabla u\|_2^2 \\ I_{62} &= |\int_{\mathbb{R}^3} \partial_1 u_3 \cdot u_1 \cdot \Delta u_3 dx| \\ &\le \delta\|\Delta u\|_2^2 + C(\delta)\|a\|_2^{\frac{2(q-3)}{q}} \|\nabla u_3\|_q^2 \|\nabla u\|_q^{\frac{6}{q}}. \end{aligned}$$

Let $\delta = 1/16$. Substituting above estimates into (3.12), it follows that

$$\frac{d}{dt}\|\nabla u\|_2^2 + \|\Delta u\|_2^2 \le CM_1^2\big(1 + \|a\|_2^4 M_1^2\big)\|\nabla u\|_2^2 + C\|\nabla\omega_3\|_2 \|\nabla u\|_2^3 \tag{3.13}$$

for $q = 3$, and

$$\begin{aligned} \frac{d}{dt}\|\nabla u\|_2^2 + \|\Delta u\|_2^2 \quad &\le \quad C\big(\|a\|_2^{\frac{4q}{q-3}}\|\nabla u_3\|_q^p + \|\nabla\omega_3\|_2\|\nabla u\|_2 + \|\nabla u_3\|_q^p\big)\|\nabla u\|_2^2 \\ &\quad + C\|a\|_2^{\frac{2(q-3)}{q}} \|\nabla u_3\|_q^2 \|\nabla u\|_2^{6/q} \end{aligned} \tag{3.14}$$

for $q > 3$. (3.13) implies that

$$\begin{aligned} &\frac{d}{dt}\big(\exp\big\{-C\int_0^t \|\nabla\omega_3\|_2\|\nabla u\|_2 ds\big\}\|\nabla u\|_2^2\big) \\ &\quad \le \quad CM_1^2\big(1 + \|a\|_2^4 M_1^2\big)\exp\big\{-C\int_0^t \|\nabla\omega_3\|_2\|\nabla u\|_2 ds\big\}\|\nabla u\|_2^2, \end{aligned}$$

which implies

$$\begin{aligned} &\|\nabla u(t)\|_2^2 \\ &\quad \le \quad \big[\|\nabla a\|_2^2 + CM_1^2\|a\|_2^2\big(1 + \|a\|_2^4 M_1^2\big)\big]\exp\big\{C\int_0^t \|\nabla\omega_3\|_2\|\nabla u\|_2 ds\big\} \\ &\quad \le \quad \big[\|\nabla a\|_2^2 + CM_1^2\|a\|_2^2\big(1 + \|a\|_2^4 M_1^2\big)\big]\exp\big\{C\|a\|_2\big(\|a\|_2 M_1 + \|\nabla u\|_2\big)\big\}, \end{aligned}$$

This inequality and (3.13) imply (3.9). From (3.14), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\big(\exp\big\{C\int_0^t\big[\|a\|_2^{\frac{4q}{q-3}}\|\nabla u_3\|_q^p+\|\nabla\omega_3\|_2\|\nabla u\|_2+\|\nabla u_3\|_q^p\big]ds\big\}\|\nabla u\|_2^2\big)\\
&\le\ C\|a\|_2^{\frac{2(q-3)}{q}}\|\nabla u_3\|_q^2\|\nabla u\|_2^{\frac{6}{q}}\\
&\quad\times\exp\big\{C\int_0^t\big[\|a\|_2^{\frac{4q}{q-3}}\|\nabla u_3\|_q^p+\|\nabla\omega_3\|_2\|\nabla u\|_2+\|\nabla u_3\|_q^p\big]ds\big\},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\nabla u(t)\|_2^2 &\le \big(\exp\big\{C\int_0^t\big[\|a\|_2^{\frac{4q}{q-3}}\|\nabla u_3\|_q^p+\|\nabla\omega_3\|_2\|\nabla u\|_2+\|\nabla u_3\|_q^p\big]ds\big\}\big)\\
&\quad\times\big\{\|\nabla a\|_2^2+C\|a\|_2^{\frac{2(q-3)}{q}}\big(\int_0^T\|\nabla u_3\|_q^p dt\big)^{\frac{q-3}{q}}\big(\int_0^T\|\nabla u\|_2 dt\big)^{\frac{3}{q}}\big\}\\
&\le \big\{\|\nabla a\|_2^2+C\|a\|_2^2 M_2^{\frac{q-3}{q}}\big\}\exp\big\{M_2\|a\|_2^{\frac{4q}{q-3}}\\
&\quad+\|a\|_2\|a\|_{H^1(\mathbb{R}^3)}\big(1+M_2^{1/2}e^{CM_2}\big)\big\}\triangleq A_3,
\end{aligned}
$$

This inequality and (3.14) imply (3.10). □

Lemma 3.5 *Let $a\in L^2_\sigma(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$. If $u_3\in L^\infty(\mathbb{R}^3\times(0,T))$, then for any $t\ge 0$,*

$$
\|\nabla u(t)\|_2^2+\int_0^t\|\Delta u(s)\|_2^2 ds\le A_4 \tag{3.15}
$$

with

$$
\begin{aligned}
A_4 &= \|\nabla a\|_2^2+C(M_0^2+M_0^{8/3}\|a\|_2^{\frac{4}{3}})\|a\|_2^2\\
&\quad+A_5\|a\|_2\big(\|\nabla a\|_2+M_0\|a\|_2+M_0^{4/3}\|a\|_2^{5/3}\big),\\
A_5 &= \big(\|\nabla a\|_2^2+C(M_0^2+\|a\|_2^{4/3}M_0^{8/3})\|a\|_2^2\big)\\
&\quad\times\exp\big\{C\|a\|_2\big(\|\nabla a\|_2+M_0\|a\|_2+M_0^{4/3}\|a\|_2^{5/3}\big)\big\}.
\end{aligned}
$$

In above, constant C is a absolute constant and independent of T. Thus T can be taken as $+\infty$.

Proof. We need to re-estimate the terms at the right hand side of (3.12) by the assumption that $u_3\in L^\infty(Q_T)$ to replace the assumption that $\nabla u_3\in L^p(0,T;L^q(\mathbb{R}^3))$. In view of the above procedure, we don't use the assumption as estimating the I_2 and I_3, except the estimates about $\|\nabla\omega_3\|_2$ obtained in Lemmas 3.2 and 3.3. So there are same as $u_3\in L^\infty(Q_T)$, i.e.,

$$
\begin{aligned}
I_2 &= \big|\int_{\mathbb{R}^3}\omega_3u_2\Delta u_1dx\big|\le\delta\|\Delta u\|_2^2+C(\delta)\|\nabla\omega_3\|_2\|\nabla u\|_2^3\\
I_3 &= \big|\int_{\mathbb{R}^3}\omega_3u_1\Delta u_2dx\big|\le\delta\|\Delta u\|_2^2+C(\delta)\|\nabla\omega_3\|_2\|\nabla u\|_2^3.
\end{aligned}
$$

Now we estimate other terms. By Hölder inequality, the estimate of I_1 is easy. In fact,

$$\begin{aligned} I_1 = |\int_{\mathbb{R}^3} \omega_2 u_3 \Delta u_1 dx| &\leq \|\Delta u\|_2 \|u_3\|_\infty \|\omega_2\|_2 \\ &\leq \delta\|\Delta u\|_2^2 + C(\delta)\|u_3\|_\infty^2 \|\nabla u\|_2^2. \end{aligned}$$

Similarly,

$$I_4 = |\int_{\mathbb{R}^3} \omega_1 u_3 \Delta u_2 dx| \leq \delta\|\Delta u\|_2^2 + C(\delta)\|u_3\|_\infty^2 \|\nabla u\|_2^2.$$

By Hölder inequality again, one has

$$I_{51} = |\int_{\mathbb{R}^3} \partial_2 u_3 u_2 \Delta u_3 dx| \leq \delta\|\Delta u\|_2^2 + C_0 \int_{\mathbb{R}^3} |u_2|^2 |\partial_2 u_3|^2 dx.$$

Integrating by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |u_2|^2 |\partial_2 u_3|^2 dx \\ &\leq |2\int_{\mathbb{R}^3} u_2 \partial_2 u_2 \partial_2 u_3 u_3 dx| + |\int_{\mathbb{R}^3} |u_2|^2 |\partial_{22}^2 u_3 u_3 dx| \\ &\leq |\int_{\mathbb{R}^3} |\partial_2 u_2|^2 |u_3|^2 dx| + |\int_{\mathbb{R}^3} u_2 \partial_{22}^2 u_2 |u_3|^2 dx| \\ &\quad + \|\Delta u_3\|_2 \|u_2\|_6^2 \|u_3\|_2^{1/3} \|u_3\|_\infty^{2/3} \\ &\leq \|u_3\|_\infty^2 \|\nabla u_2\|_2^2 + \|\Delta u_2\| \|u_2\|_6 \|u_3\|_2^{2/3} \|u_3\|_\infty^{4/3} \\ &\quad + \|\Delta u_3\|_2 \|u_2\|_6^2 \|u_3\|_2^{1/3} \|u_3\|_\infty^{2/3} \\ &\leq \frac{\delta}{C_0}\|\Delta u\|_2^2 + C(\delta)\|a\|_2^{2/3} \|u_3\|_\infty^{4/3} \|\nabla u\|_2^4 \\ &\quad + C\big(\|u_3\|_\infty^2 + \|a\|_2^{4/3} \|u_3\|_\infty^{8/3}\big) \|\nabla u\|_2^2. \end{aligned}$$

For I_{52}, by integration by part, we can treat as

$$\begin{aligned} I_{52} &= |\int_{\mathbb{R}^3} \partial_3 u_2 u_2 \Delta u_3 dx| = \frac{1}{2}|\int_{\mathbb{R}^3} |u_2|^2 \Delta \partial_3 u_3 dx| \\ &\leq |\int_{\mathbb{R}^3} (u_2 \Delta u_2) \partial_3 u_3 dx| + |\int_{\mathbb{R}^3} |\nabla u_2|^2 \partial_3 u_3 dx| \\ &\leq |\int_{\mathbb{R}^3} (u_2 \Delta u_2) \partial_3 u_3 dx| + |2\int_{\mathbb{R}^3} (\nabla u_2 \cdot \nabla \partial_3 u_2) u_3 dx| \\ &\leq \delta\|\Delta u\|_2^2 + C_1(\delta) \int_{\mathbb{R}^3} |u_2|^2 |\partial_3 u_3|^2 dx + C(\delta)\|u_3\|_\infty^2 \|\nabla u_2\|_2^2. \end{aligned}$$

Integrating by parts again,

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}|u_2|^2|\partial_3 u_3|^2dx| \\
&\leq\ 2|\int_{\mathbb{R}^3}u_2\cdot\partial_3 u_2\cdot\partial_3 u_3\cdot u_3dx|+|\int_{\mathbb{R}^3}|u_2|^2\partial_{33}^2 u_3\cdot u_3dx| \\
&\leq\ |\int_{\mathbb{R}^3}|\partial_3 u_2|^2|u_3|^2dx|+|\int_{\mathbb{R}^3}u_2\partial_{33}^2 u_2\cdot|u_3|^2dx| \\
&\quad +\|\Delta u_3\|_2\|u_2\|_6^2\|u_3\|_2^{1/3}\|u_3\|_\infty^{2/3} \\
&\leq\ \|u_3\|_\infty^2\|\nabla u\|_2^2+\|\Delta u_2\|_2\|u_2\|_6\|u_3\|_2^{\frac{2}{3}}\|u_3\|_\infty^{4/3} \\
&\quad +\|\Delta u_3\|_2\|u_2\|_6^2\|u_3\|_2^{1/3}\|u_3\|_\infty^{2/3} \\
&\leq\ \frac{\delta}{C_1}\|\Delta u\|_2^2+C\|a\|_2^{2/3}\|u_3\|_\infty^{4/3}\|\nabla u\|_2^4 \\
&\quad +C\big(\|u_3\|_\infty^2+\|a\|_2^{4/3}\|u_3\|_\infty^{8/3}\big)\|\nabla u\|_2^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_5 &= |\int_{\mathbb{R}^3}\omega_1 u_2\Delta u_3dx| \\
&\leq 4\delta\|\Delta u\|_2^2+C\|a\|_2^{2/3}\|u_3\|_\infty^{4/3}\|\nabla u\|_2^4+C\big(\|u_3\|_\infty^2+\|a\|_2^{4/3}\|u_3\|_\infty^{8/3}\big)\|\nabla u\|_2^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_6 &= |\int_{\mathbb{R}^3}\omega_2 u_1\Delta u_3dx| \\
&\leq 4\delta\|\Delta u\|_2^2+C\|a\|_2^{2/3}\|u_3\|_\infty^{4/3}\|\nabla u\|_2^4+C\big(\|u_3\|_\infty^2+\|a\|_2^{4/3}\|u_3\|_\infty^{8/3}\big)\|\nabla u\|_2^2.
\end{aligned}
$$

Substituting above estimates into (3.12) and taking $\delta = 1/16$, one obtain that

$$
\begin{aligned}
\frac{d}{dt}\|\nabla u\|_2^2+\|\Delta u\|_2^2 &\leq C\big(\|\nabla\omega_3\|_2\|\nabla u\|_2+\|a\|_2^{2/3}\|u_3\|_\infty^{4/3}\|\nabla u\|_2^2\big)\|\nabla u\|_2^2 \\
&\quad +C\big(\|u_3\|_\infty^2+\|a\|_2^{4/3}\|u_3\|_\infty^{8/3}\big)\|\nabla u\|_2^2, \qquad (3.16)
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\nabla u(t)\|_2^2 &\leq \big\{\|\nabla a\|_2^2+C\big(M_0^2+\|a\|_2^{4/3}M_0^{8/3}\big)\|a\|_2^2\big\} \\
&\quad\times\exp\big\{C\|a\|_2\big(\|\nabla a\|_2+M_0\|a\|_2+\|a\|_2^{5/3}M_0^{4/3}\big)\big\},
\end{aligned}
$$

here we used Lemmas 3.1 and 3.2. The above inequality and (3.16) implies (3.15). □

Proof of Theorem 2.1 Since the initial velocity field a is in $L^2_\sigma(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)$, it is well-known that there is a $T_0 > 0$, such that there is a unique strong solution $u \in L^\infty(0, T_0; H^1(\mathbb{R}^3))\cap L^2(0, T_0; H^2(\mathbb{R}^3))$ to the Navier-Stokes equations (1.1).

See [11]. According the result about uniqueness [11], our weak solution identity with the strong solution in $(0, T_0)$. If $u_3 \in L^\infty(\mathbb{R}^3 \times (0,T))(t_0 < T \leq +\infty)$ or $\nabla u_3 \in L^s(0,T; L^q(\mathbb{R}^3))$ for $1/s + 3/2q = 1/2$ with $q \geq 3$, then Lemmas 3.1-3.5 imply

$$\sup_{0\leq t\leq T} \|u(t)\|_{H^1(\mathbb{R}^3)} + \int_0^T \big(\|\nabla u(s)\|_2^2 + \|\Delta u(s)\|_2^2\big) ds \leq C(\|a\|_{H^1(\mathbb{R}^3)}).$$

Thus the local strong solution u can be extended to time T, and also identity with the weak solution. Moreover the classical regular criteria [11] implies that u is a regular solution on $[0, T]$. □

Remark. The referee has kindly pointed out the recent works [12] -[14]. In [12], the authors consider the interior regularity of the suitable weak solutions under the assumption that one component of the velocity is assumed to belong to the anisotropic Lebesgue space $L^{p,q}$ (p in time and q in space) with $2/p + 3/q \leq 1/2$ for $p \geq 4$ and $q > 6$; In [13], the authors obtained the regular criteria under the different assumptions on three components of velocity; While in [14], the authors defined many regularity classes by means of derivatives of some components of velocity. For example, they obtained the regularity of the weak solutions which satisfy the energy inequality and $\partial_3 u$ belong to $L^p(0,T; L^q(\mathbb{R}^3))$ with $1/p + 3/2q \leq 3/4$ for $q \geq 2$, or $\partial_3 u_3 \in L^\infty(0,T; L^\infty(\mathbb{R}))$, etc. But our assumptions are different from theirs. Through our arguments also based on some estimates of vorticity as do in [12]-[14], the particular technique is different.

Acknowledgement. The author would like to express his sincere gratitude to researchers: Professor Tetsuro Miyakawa for his heuristic discussion about this problem, the anonymous referee for his helpful comments and for pointing out the existence of related works [12]-[14], Professor M. Pokorny for sending me his preprint [14], and for pointing out that the weak solution here should satisfy the energy inequality. The present work was completed while the author was visiting the Department of Mathematics at Kobe University as a reseach fellow of the Japan Society for the Promotion of Sciences (JSPS). The financial support and the warm hospitality at these institutions are gratefully acknowledged here.

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