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Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems *

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Abstract

By applying the Krasnoselskii's fixed point theorem on a suitable cone, several existence results for multiple positive solutions of a Fredholm integral equation are provided. Applications of these results to some nonlocal boundary-value problems are also given.

1 Introduction

We study the existence of multiple positive solutions of the abstract Fredholm integral equation

$$x(t)=\int_0^1 K(t,s)f(x(s))ds,\ \ t\in[0,1], \tag{1.1}$$

where the kernel $K(t,s)$ satisfies a continuity assumption in the L^1-sense and it is monotone and concave in t for a.a. s. Then we apply our results to the second order ordinary differential equation

$$(p(t)x'(t))'+\mu(t)f(x(t))=0,\ \ t\in[0,1], \tag{1.2}$$

associated with the nonlocal boundary conditions

$$x'(0)=\int_0^1 x'(s)dg(s) \quad \text{and} \quad x(1)=-\int_0^1 x'(s)dh(s) \tag{1.3}$$

or

$$x'(1)=\int_0^1 x'(s)dg_1(s) \quad \text{and} \quad x(0)=\int_0^1 x'(s)dh_1(s). \tag{1.4}$$

Notice that these problems are our motivation in investigating the abstract problem (1.1). Here $f:\mathbb{R}\to\mathbb{R}$, $p:[0,1]\to(0,\infty)$, $\mu:[0,1]\to[0,\infty)$ are

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continuous functions and $g, h, g_1, h_1 \colon [0,1] \to \mathbb{R}$ are nondecreasing functions. In these boundary conditions all integrals are meant in the sense of Riemann-Stieljes.

Boundary-value problems with integral boundary conditions constitute a very interesting class of problems, because they include as special cases two, three, multi-point and nonlocal boundary-value problems. For such problems and comments on their importance, we refer to the recent papers [7, 9, 15, 18]. Especially, the existence of positive solutions for such problems is the subject of several recent papers [5, 8, 9, 10, 11, 12, 13, 15, 16, 19]. Moreover, since a boundary-value problem may usually be reduced to an integral equation, the existence of positive solutions for a boundary-value problem is closely connected to an analogous problem for an integral equation. Indeed, the existence of positive solutions for integral equations, or, generally, for operator equations, is a problem which appeared early in the literature. For more details we refer to the books by Krasnoselskii [16] and Agarwal, O'Regan and Wong [1], as well as to the recent papers [2, 3, 17, 20, 21] and the references therein. For more information about the general theory of integral equations and their relation with boundary-value problems we refer to the book of Corduneanu [6] on finite intervals and the most recent book of Agarwal and O'Regan [4] on infinite intervals. We find it convenient to compare a little our setting with the ideas expressed in [2]. The subject studied in [2] is an integral equation of the form

$$y(t) = \theta(t) + \int_0^1 k(t,s)[g(y(s)) + h(y(s))]ds, \;\; t \in [0,1], \tag{1.4}$$

where h, g are continuous functions, g is sub-multiplicative on $(0, +\infty)$ and these two functions and h/g satisfy some monotonicity conditions. The procedure in [2] is to succeed existence via the solutions of an approximation of (1.4). In the present paper we do not assume any monotonicity condition on f and our existence results are obtained by a direct application of the Krasnoselskii's theorem. Notice that, as one can see, all conditions required for our integral problem (1.1) are inherited from the corresponding properties of the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4).

And indeed the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4) (though they are equivalent in a sense explained in the last section) are reduced to the abstract Fredholm integral equation (1.1). That is why we investigate when equation (1.1) admits multiple positive solutions. The conditions established on the kernel K and the force f are rather simple and such that the obtained results are original and may give as corollaries new results for the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4). Our paper is motivated mainly by the papers [13, 16, 18] and, among others, our results generalize and improve several recent results of the authors [14] and Ma and Castaneda [19].

The results here are obtained by applying the well known fixed point theorem due to Krasnoselskii [16], which states as follows:

Theorem 1.1 *Let $\mathcal{B}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume Ω_1,*

Ω_2 are open subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \mathrm{cl}\,\Omega_1 \subset \Omega_2$, and let

$$A\colon\ \mathbb{K} \cap (\mathrm{cl}\,\Omega_2 \setminus \Omega_1) \to \mathbb{K}$$

be a completely continuous operator such that either

$$\|Au\| \le \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_1 \quad \textit{and} \quad \|Au\| \ge \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_2,$$

or

$$\|Au\| \ge \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_1 \quad \textit{and} \quad \|Au\| \le \|u\|,\ \ u \in \mathbb{K} \cap \partial\Omega_2.$$

Then A has a fixed point in $\mathbb{K} \cap (\mathrm{cl}\,\Omega_2 \setminus \Omega_1)$.

This paper is organized as follows: In Section 2 we show how the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4) can be written as a Fredholm integral equation of the form (1.1). Section 3 contains the basic lemmas concerning equation (1.1). These lemmas imply several corollaries, which in Section 4 lead to our main existence results. In Section 5 we specify the results of Section 4 to the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4) and we close the paper with a discussion.

2 The boundary-value problems (1.2)-(1.3) and (1.2)-(1.4) reformulated

In what follows we shall denote by $\mathbb{R}$ the real line, by I the interval $[0, 1]$ and by $C(I)$ the space of all continuous functions $x : I \to \mathbb{R}$. The space $C(I)$ endowed with the sup-norm $\|\cdot\|$ is a Banach space. Also we denote by $L^1(I)$ the space of all functions $x : I \to \mathbb{R}$ which are Lebesgue integrable on I endowed with the usual norm $\|\cdot\|_{L^1}$.

In the sequel we shall use nondecreasing functions $g, h, g_1, h_1 : I \to [0, \infty)$, with $g(0) = h(0) = g_1(1) = h_1(1) = 0$ and the continuous function $p : I \to (0, \infty)$ such that

$$\int_0^1 \frac{1}{p(s)} dg(s) < \frac{1}{p(0)}, \tag{2.1}$$

if we have the conditions (1.3) and

$$\int_0^1 \frac{1}{p(s)} dg_1(s) < \frac{1}{p(1)}, \tag{2.2}$$

if we have the conditions (1.4).

As we pointed out in the introduction, to search for solutions to (1.2)-(1.3) and (1.2)-(1.4), we first reformulate them as operator equations of the form $x = Ax$, where A is an appropriate integral operator.

To find operator A consider first the boundary-value problem (1.2) – (1.3), where for simplicity we put

$$z(t) := \mu(t) f(x(t)),\ \ t \in I.$$

Integrate (1.2) from 0 to t and get

$$x'(t) = q(t)p(0)x'(0) - q(t)\int_0^t z(s)ds, \tag{2.3}$$

where we have set $q(t) := 1/p(t)$, $t \in I$. Taking into account the first condition in (1.3) we obtain

$$x'(0) = p(0)x'(0)\int_0^1 q(s)dg(s) - \int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s)$$

or

$$p(0)x'(0)\left[q(0) - \int_0^1 q(s)dg(s)\right] = -\int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s)$$

and so

$$p(0)x'(0) = -\alpha\int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s),$$

where α is a constant defined by

$$\alpha := \frac{1}{q(0) - \int_0^1 q(s)dg(s)}.$$

Then, from (2.3) we get

$$x'(t) = -\alpha q(t)\int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s) - q(t)\int_0^t z(\theta)d\theta.$$

Thus for $t \in I$, we have

$$x(t) = x(0) - \alpha\int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s)\int_0^t q(r)dr - \int_0^t q(s)\int_0^s z(\theta)d\theta ds,$$

and so

$$x(1) = x(0) - \alpha\int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s)\int_0^1 q(r)dr - \int_0^1 q(s)\int_0^s z(\theta)d\theta ds.$$

On the other hand taking into account the second condition in (1.3) we obtain

$$\begin{aligned} x(1) &= -\int_0^1 \Big(-\alpha q(s)\int_0^1 q(r)\int_0^r z(\theta)d\theta dg(r) - q(s)\int_0^s z(\theta)d\theta\Big)dh(s) = \\ &= \alpha\int_0^1 q(s)dh(s)\int_0^1 q(r)\int_0^r z(\theta)d\theta dg(r) + \int_0^1 q(s)\int_0^s z(\theta)d\theta dh(s). \end{aligned}$$

Therefore it follows that

$$\begin{aligned} x(0) = \alpha &\int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s)\int_0^1 q(r)dr + \int_0^1 q(s)\int_0^s z(\theta)d\theta ds + \\ &+ \alpha\int_0^1 q(s)dh(s)\int_0^1 q(r)\int_0^r z(\theta)d\theta dg(r) + \int_0^1 q(s)\int_0^s z(\theta)d\theta dh(s). \end{aligned}$$

Hence the solution x has the form

$$\begin{aligned} x(t) =&\alpha \int_0^1 q(s) \int_0^s z(\theta)d\theta dg(s) \int_0^1 q(r)dr+\\ &+ \alpha \int_0^1 q(s) \int_0^s z(\theta)d\theta dg(s) \int_0^1 q(r)dh(r)+\\ &+ \int_0^1 q(s) \int_0^s z(\theta)d\theta ds + \int_0^1 q(s) \int_0^s z(\theta)d\theta dh(s)-\\ &- \alpha \int_0^1 q(s) \int_0^s z(\theta)d\theta dg(s) \int_0^t q(r)dr - \int_0^t q(s) \int_0^s z(\theta)d\theta ds =\\ =&\alpha \int_0^1 q(s) \int_0^s z(\theta)d\theta dg(s) \int_t^1 q(r)dr+\\ &+ \alpha \int_0^1 q(s) \int_0^s z(\theta)d\theta dg(s) \int_0^1 q(r)dh(r) + \int_0^1 q(s) \int_0^s z(\theta)d\theta dh(s)+\\ &+ \int_t^1 q(s) \int_0^s z(\theta)d\theta ds, \quad t \in I. \end{aligned}$$

Now for simplicity we set $b := \int_0^1 q(s)dh(s)$ and get

$$\begin{aligned} x(t) &= \alpha \int_0^1 q(s) \int_0^s z(\theta)d\theta dg(s)\Big(b + \int_t^1 q(s)ds\Big)+\\ &\quad + \int_0^1 q(s) \int_0^s z(\theta)d\theta dh(s) + \int_0^1 q(s)\chi_{[t,1]}(s) \int_0^s z(\theta)d\theta ds =\\ &= \int_0^1 q(s) \int_0^s z(\theta)d\theta d_s\rho(t,s), \quad t \in I, \end{aligned}$$

where

$$\rho(t,s) := \alpha g(s)\Big(b + \int_t^1 q(r)dr\Big) + h(s) + s\chi_{[t,1]}(s).$$

Finally by using Fubini' s theorem we obtain

$$x(t) = \int_0^1 z(\theta) \int_\theta^1 q(s)d_s\rho(t,s)d\theta = \int_0^1 f(x(\theta))\Big(\mu(\theta) \int_\theta^1 q(s)d_s\rho(t,s)\Big)d\theta.$$

This is an equation of the form (1.1), where the kernel $K(t,\theta)$ is defined by

$$K(t,\theta) := \mu(\theta) \int_\theta^1 q(s)d_s\rho(t,s). \tag{2.4}$$

Here we have the following:

Theorem 2.1 *A function $x \in C^1(I)$ is a solution of the boundary-value problem* $(1.2) - (1.3)$ *if and only if x is a solution of equation* (1.1), *whose the kernel K is given by* (2.4).

Proof. The *only if* part is proved above. For the *if* part, first we assume that $x : I \to \mathbb{R}$ is a continuous function satisfying (1.1). Then x is differentiable with derivative

$$x'(t) = \int_0^1 \frac{\partial K(t,\theta)}{\partial t} f(x(\theta))d\theta,$$

where

$$\frac{\partial K(t,\theta)}{\partial t} = \begin{cases} -(\alpha\phi(\theta)+1)\mu(\theta)q(t), & \text{if } \theta < t \\ -\alpha\phi(\theta)\mu(\theta)q(t), & \text{if } t < \theta \end{cases} \tag{2.5}$$

and $\phi(\theta) := \int_\theta^1 q(r)dg(r)$. Thus we get

$$x'(t) = q(t)\Big[-\alpha \int_0^1 z(\theta)\phi(\theta)d\theta - \int_0^t z(\theta)d\theta\Big], \tag{2.6}$$

where, recall that, $z(t) = \mu(t)f(x(t)),\ \ t \in I$.

Now, from (2.6) it follows that equation (1.2) is satisfied. Moreover the first condition in (1.3) is satisfied. Indeed, from the obvious relation

$$-\alpha q(0) + 1 = -\alpha \int_0^1 q(s)dg(s)$$

we get

$$\begin{aligned} \int_0^1 x'(s)dg(s) &= -\alpha \int_0^1 z(\theta)\phi(\theta)d\theta \int_0^1 q(s)dg(s) - \int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s) = \\ &= -\alpha q(0)\int_0^1 z(\theta)\phi(\theta)d\theta + \int_0^1 z(\theta)\phi(\theta)d\theta - \\ &\quad - \int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s) = \\ &= x'(0) + \int_0^1 z(\theta)\int_\theta^1 q(s)dg(s)d\theta - \int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s) = \\ &= x'(0), \end{aligned}$$

since from Fubini's theorem we have

$$\int_0^1 z(\theta)\int_\theta^1 q(s)dg(s)d\theta = \int_0^1 q(s)\int_0^s z(\theta)d\theta dg(s). \tag{2.7}$$

Finally, we can use (2.7) with h in place of g to show that the second condition in (1.3) is also satisfied. □

Next following the same procedure as above it is not hard to see that a function $x \in C^1(I)$ is a solution of the boundary-value problem (1.2) – (1.4) if and only if x is a solution of equation (1.1), where now the kernel K is given by

$$K(t,\theta) := \mu(\theta)\int_0^\theta q(s)d_s\rho_1(t,s), \tag{2.8}$$

the measure ρ_1 is defined by

$$\rho_1(t,s) := \alpha_1 g_1(s)\Big(b_1 + \int_0^t q(r)dr\Big) + h_1(s) + s\chi_{[0,t]}(s)$$

and the constants α_1, b_1 are given by

$$\alpha_1 := \frac{1}{q(1) - \int_0^1 q(s)dg_1(s)} \quad \text{and} \quad b_1 := \int_0^1 q(s)dh_1(s).$$

3 On the Fredholm integral equation (1.1)

In this section we present some auxiliary facts needed to show the existence of solutions of the general integral equation (1.1). Then we will return to the specific problems (1.2)-(1.3) and (1.2)-(1.4).

To unify our results in the cases appeared later, we consider a fixed number $\delta \in \{-1, +1\}$ and assume that the following conditions hold:

(H_f) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(x) > 0$, when $x > 0$.

(H_K) $K(\cdot,\cdot)$ maps the square $I \times I$ into $(0, +\infty)$ and it is such that:

a) For a.a. $s \in I$ the function $K(\cdot, s)$ is concave and the function $\delta K(\cdot, s)$ is non-increasing.

b) For all $t \in I$ we have $K(t,\cdot) \in L^1(I)$ and the function $t \to K(t,\cdot)$ is uniformly L^1-continuous.

Next define the cone

$$\mathbb{K}_\delta := \{x \in C(I) : x \geq 0,\ \ x \text{ is concave and } \delta x \text{ is nonincreasing}\}$$

as well as the operator

$$(Ax)(t) := \int_0^1 K(t,s)f(x(s))ds,\ \ x \in C(I).$$

Lemma 3.1 *Under the assumptions (H_f) and (H_K) the operator A is completely continuous and it maps $\mathbb{K}_\delta$ into $\mathbb{K}_\delta$.*

Proof. For all $x, y \in C(I)$ we have

$$\big|(Ax)(t) - (Ay)(t)\big| = \Big|\int_0^t K(t,s)(f(x(s)) - f(y(s)))ds\Big| \leq k\,\|f(x(\cdot)) - f(y(\cdot))\|,$$

where

$$k := \sup_{t\in I}\int_0^1 K(t,s)ds \quad (< +\infty).$$

Thus A is a continuous operator.

Let B be a bounded subset of $C(I)$. Then there is a certain $c > 0$ such that $|x(t)| \leq c$, for all $x \in B$ and $t \in I$. So we have

$$|(Ax)(t)| \leq k \sup_{0 \leq \xi \leq c} |f(\xi)|$$

for all $x \in B$ and $t \in I$.

Also, let $x \in B$. Then, for all $t_1, t_2 \in I$, it holds

$$|(Ax)(t_1) - (Ax)(t_2)| \leq \|K(t_1, \cdot) - K(t_2, \cdot)\|_{L^1} \sup_{0 \leq \xi \leq c} |f(\xi)|.$$

Therefore A maps bounded sets into compact sets. The previous facts show the complete continuity of A. Let $x \in \mathbb{K}_\delta$. Then $f(x(s)) \geq 0$, $s \in I$ and therefore $Ax(t) \geq 0$, $t \in I$.

Also, let $E \subset I$ be a set with Lebesgue measure zero such that for all $s \in I \setminus E$ it holds

$$\delta K(t_1, s) \leq \delta K(t_2, s), \text{ whenever } t_2 \leq t_1,$$

as well as

$$K(\lambda t_1 + (1-\lambda)t_2, s) \geq \lambda K(t_1, s) + (1-\lambda)K(t_2, s), \text{ for all } t_1, t_2 \in I, \ \lambda \in [0,1].$$

These facts are guaranteed by (H_K). The first inequality implies that δAx is a non-increasing function and the second one that Ax is concave. The proof is complete. □

Next we let $i := \frac{1}{2}(1-\delta)$ and observe that, for each $x \in \mathbb{K}_\delta$, it holds

$$\|x\| = x(i).$$

Define the continuous function

$$\Phi_i(\theta) := \Big| \int_i^\theta K(i, s) ds \Big|, \ \theta \in I.$$

and make the following assumptions:

(H1) There is a $v > 0$ such that $\Phi_i(1-i) \sup_{\xi \in [0,v]} f(\xi) < v$.

(H2) There are $\eta \in (0,1)$ and $u > 0$ such that

$$\Phi_i(\eta) \inf_{\xi \in [\zeta_i u, u]} f(\xi) > u,$$

where $\zeta_i := 1 - i + \eta(2i - 1)$.

Lemma 3.2 *For each $x \in \mathbb{K}_\delta$ it holds*

$$\zeta_i \|x\| \leq x(t), \ t \in [i\eta, i + (1-i)\eta].$$

Proof. Concavity of x on I implies that

$$\frac{x(1)-x(0)}{1} \geq \frac{x(1)-x(\eta)}{1-\eta},$$

which gives

$$x(\eta) \geq (1-\eta)x(0) + \eta x(1).$$

Since x is nonnegative, we get

$$x(\eta) \geq (1-\eta)x(0) \text{ and } x(\eta) \geq \eta x(1),$$

which is written as

$$x(\eta) \geq \zeta_i x(i). \tag{3.1}$$

Now, if $\delta = +1$, we have $i = 0$, $\zeta_i = 1-\eta$, x is non-increasing and $\|x\| = x(0)$. If $\delta = -1$, then $i = 1$, $\zeta_i = \eta$, x is nondecreasing and $\|x\| = x(1)$. It is clear that these facts together with (3.1) complete the proof. □

Lemma 3.3 *If the assumptions (H_f), (H_K) and (H1) hold, then for all $x \in \mathbb{K}_\delta$, with $\|x\| = v$, we have $\|Ax\| < \|x\|$.*

Proof. If $\|x\| = v$, then $0 \leq x(s) \leq v$, for every $s \in I$ and, by (H1) and Lemma 3.1, we have

$$\begin{aligned}\|Ax\| = (Ax)(i) = \int_0^1 K(i,\theta)f(x(\theta))d\theta \leq \\ \leq sup_{\xi\in[0,v]}f(\xi)\int_0^1 K(i,\theta)d\theta = \Phi_i(1-i)\ sup_{\xi\in[0,v]}f(\xi) < v = \|x\|.\end{aligned}$$

Lemma 3.4 *If the assumptions (H_f), (H_K) and (H2) hold, then for all $x \in \mathbb{K}_\delta$, with $\|x\| = u$, we have $\|Ax\| > \|x\|$.*

Proof. Take a point $x \in \mathbb{K}_\delta$, with $\|x\| = u$. Then by Lemma 3.2 we have $\zeta_i\|x\| \leq x(s) \leq \|x\| = x(i)$, for every $s \in [i\eta, i+(1-i)\eta]$. Therefore from Lemma 3.1 it follows that

$$\begin{aligned}\|Ax\| = (Ax)(i) = \int_0^1 K(i,\theta)f(x(\theta))d\theta \geq \\ \geq inf_{\xi\in[\zeta_i u,u]}f(\xi)\Big|\int_i^\eta K(i,\theta)d\theta\Big| = \Phi_i(\eta)inf_{\xi\in[\zeta_i u,u]}f(\xi) > \\ > u = \|x\|.\end{aligned}$$

□

Now we assume that the quantities

$$T_0 := \lim_{u\to 0}\frac{f(u)}{u}, \quad \text{and} \quad T_\infty := \lim_{u\to\infty}\frac{f(u)}{u}$$

exist. The previous results imply the following corollaries.

Corollary 3.5 *Let the assumptions* (H_f), (H_K) *be satisfied. If* $T_0 = 0$, *there is* $m_0 > 0$ *such that for every* $m \in (0, m_0]$ *and for every* $x \in \mathbb{K}_\delta$, *with* $\|x\| = m$, *we have* $\|Ax\| < \|x\|$.

Proof. Let $\epsilon \in (0, \frac{1}{\Phi_i(1-i)}]$. Since $T_0 = 0$, there is a point $m_0 > 0$ such that for every $y \in (0, m_0]$ we have $f(y) < \epsilon y$. Let $m \in (0, m_0]$ be fixed. For every $y \in (0, m]$ we have $f(y) < \epsilon y \leq \epsilon m$. Thus assumption (H1) is valid with $v := m$. So, Lemma 3.3 applies.

Corollary 3.6 *Let the assumptions* (H_f), (H_K) *be satisfied. If* $T_\infty = +\infty$, *then there is* $M_0 > 0$ *such that for every* $M \geq M_0$ *and for every* $x \in \mathbb{K}_\delta$, *with* $\|x\| = M$, *we have* $\|Ax\| > \|x\|$.

Proof. Since $T_\infty = +\infty$, there is a $M_1 > 0$ such that for every $y \geq M_1$ we have

$$f(y) > \frac{1}{\Phi_i(\eta)\zeta_i} y.$$

Set $M_0 := \frac{1}{\zeta_i} M_1$. Then for any $M \geq M_0$ we have $\zeta_i M \geq M_1$. So, if $y \in [\zeta_i M, M]$, then $y \geq \zeta_i M \geq M_1$ and so

$$\Phi_i(\eta) f(y) > \frac{1}{\zeta_i} y \geq \frac{1}{\zeta_i} \zeta_i M = M.$$

Therefore assumption (H2) is valid with $u := M$ and Lemma 3.4 applies.

Corollary 3.7 *Let the assumptions* (H_f), (H_K) *be satisfied. If* $T_0 = +\infty$, *then there is* $n_0 > 0$ *such that for every* $n \in (0, n_0)$ *and for every* $x \in \mathbb{K}_\delta$, *with* $\|x\| = n$, *we have* $\|Ax\| > \|x\|$.

Proof. Since $T_0 = +\infty$, there is a $n_1 > 0$ such that for every $y \in (0, n_1]$ it holds $\Phi_i(\eta) f(y) > \frac{1}{\zeta_i} y$. Set $n_0 := \zeta_i n_1$. Then for any $n \in (0, n_0]$ and $y \in [\zeta_i n, n]$ we have $0 < y \leq n \leq n_0 = \zeta_i n_1 < n_1$ and thus

$$\Phi_i(\eta) f(y) > \frac{1}{\zeta_i} y \geq \frac{1}{\zeta_i} \zeta_i n = n.$$

Therefore assumption (H2) is valid with $u := n$ and Lemma 3.4 applies.

Corollary 3.8 *Let the assumptions* (H_f), (H_K) *be satisfied. If* $T_\infty = 0$, *then there is* $N_0 > 0$ *large as we want such that for every* $x \in \mathbb{K}_\delta$, *with* $\|x\| = N_0$, *we have* $\|Ax\| < \|x\|$.

Proof. Let $\epsilon \in (0, \frac{1}{\Phi_i(1-i)})$. We distinguish two cases:
1) Assume, first, that f is bounded. Then there is a $k > 0$ such that $f(y) \leq k$, for all $y \geq 0$. We set $N_0 := \frac{k}{\epsilon}$. Then, for all $y \leq N_0$, it holds

$$f(y) \leq k = \epsilon N_0 < \frac{1}{\Phi_i(1-i)} N_0.$$

2) If f is not bounded, then there is a N_0 so large as we want such that

$$\sup_{[0,N_0]} f(y) = f(N_0) \leq \epsilon N_0 < \frac{1}{\Phi_i(1-i)} N_0.$$

In any case, assumption (H1) holds with $v := N_0$ and Lemma 3.3 applies. □

4 Existence results for equation (1.1)

In this section we state and prove our main results.

Theorem 4.1 *Assume that the functions f, K satisfy assumptions (H_f), (H_K) and, moreover, one of the following statements:*

(i) (H1) and (H2).

(ii) (H1) and $T_0 = +\infty$.

(iii) (H1) and $T_\infty = +\infty$.

(iv) (H2) and $T_0 = 0$.

(v) (H2) and $T_\infty = 0$.

(vi) $T_0 = 0$ and $T_\infty = +\infty$.

(vii) $T_0 = +\infty$ and $T_\infty = 0$.

Then equation (1.1) *admits at least one positive solution.*

Proof. The result of the theorem is obtained if we apply Theorem 1.1 to the completely continuous operator A on the cone $\mathbb{K}_\delta$ and use Lemmas 3.3 and 3.4 if (*i*) holds, Lemma 3.3 and Corollary 3.7 if (*ii*) holds, Lemma 3.3 and Corollary 3.6 if (*iii*) holds, Lemma 3.4 and Corollary 3.5 if (*iv*) holds, Lemma 3.4 and Corollary 3.8 if (*v*) holds, Corollaries 3.5 and 3.6, if (*vi*) holds, and Corollaries 3.7 and 3.8 if (*vii*) holds. In all cases we keep in mind Lemma 3.1. □

Remark. In any case the application of Theorem 1.1 provides information on the norm of the fixed points, namely information on the maximum value of the corresponding solution of the problem. For instance, in case (i) the norm of the solution lies in the interval $(min\{u,v\}, max\{u,v\})$. The way of getting exact information about the bounds of the norms of the solutions is exhibited in [8], where the meaning of the index of convergence is used.

Theorem 4.2 *Assume that the functions f, k satisfy the assumptions (H_f), (H_K), (H1) and (H2). Moreover, let one of the following statements hold:*

(i) $u < v$ *and* $T_0 = 0$.

(ii) $u < v$ *and* $T_\infty = +\infty$.

(iii) $v < u$ *and* $T_0 = +\infty$.

(iv) $v < u$ *and* $T_\infty = 0$.

Then equation (1.4) *admits at least two positive solutions. (In any case the remark of Theorem 4.1 keeps in force.)*

Proof. As in the proof of Theorem 4.1, we apply (twice) Theorem 1.1 on the completely continuous operator A defined on the cone $\mathbb{K}_\delta$ and use Lemmas 3.3, 3.4 in connection with Corollary 3.5 if (i) holds, Corollary 3.6 if (ii) holds, Corollary 3.7 if (iii) holds and Corollary 3.8 if (iv) holds. Again, keep in mind Lemma 3.1. □

Theorem 4.3 *Assume that the functions f, k satisfy the assumptions (H_f), (H_K), (H1) and (H2). Moreover, assume that one of the following conditions holds:*

(i) $u < v$, $T_0 = 0$ *and* $T_\infty = +\infty$.

(ii) $v < u$, $T_0 = +\infty$ *and* $T_\infty = 0$.

Then equation (1.4) *admits at least three positive solutions. (In any case the remark of Theorem 4.1 keeps in force.)*

Proof. The result follows as in the previous theorems, where, now, we use Lemmas 3.3, 3.4 in connection with Corollaries 3.5, 3.6, if (i) holds and Corollaries 3.7, 3.8, if (ii) holds, q.e.d.

5 Back to the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4)

In this section we apply the previous results to get existence results for the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4). We need the following lemma.

Lemma 5.1 *If the function p is non-increasing, then the function $K(\cdot,\theta):I\to\mathbb{R}$ defined by (2.4) is concave for a.a. $\theta\in I$. Also if the function p is nondecreasing, then the function $K(\cdot,\theta):I\to\mathbb{R}$ defined by (2.8) is concave for a.a. $\theta\in I$.*

Proof. First we consider the function K defined by (2.4). It takes the form

$$\begin{aligned}K(t,\theta)&=\mu(\theta)\int_\theta^1 q(s)d_s\Big[\alpha\big(b+\int_t^1 q(r)dr\big)g(s)+h(s)+s\chi_{[t,1]}(s)\Big]=\\&=\mu(\theta)\Big[\alpha b\int_\theta^1 q(s)dg(s)+\alpha\int_t^1 q(r)dr\int_\theta^1 q(s)dg(s)+\\&\quad+\int_\theta^1 q(s)dh(s)+\int_{max\{\theta,t\}}^1 q(s)ds\Big]=\\&=\mu(\theta)\Big[\alpha b\phi(\theta)+\alpha\phi(\theta)\int_t^1 q(s)ds+\int_\theta^1 q(s)dh(s)+\\&\quad+\int_{max\{\theta,t\}}^1 q(s)ds\Big].\end{aligned}$$

Now fix $t,r,\theta\in I$ and consider the quantity

$$\begin{aligned}U:&=K(t,\theta)-K(r,\theta)-\frac{\partial K(r,\theta)}{\partial r}(t-r)=\\&=\mu(\theta)\Big(\alpha\phi(\theta)\int_t^1 q(s)ds+\int_{max\{\theta,t\}}^1 q(s)ds-\\&\quad-\alpha\phi(\theta)\int_r^1 q(s)ds-\int_{max\{\theta,r\}}^1 q(s)ds\Big)-\frac{\partial K(r,\theta)}{\partial r}(t-r).\end{aligned}$$

We distinguish the following six cases, where we take into account that ϕ,q are nonnegative functions and q is nondecreasing.
i) $t>r>\theta$. Then

$$\begin{aligned}U&=\mu(\theta)\Big(-\alpha\phi(\theta)\int_r^t q(s)ds-\int_r^t q(s)ds+(\alpha\phi(\theta)+1)q(r)\int_r^t ds\Big)=\\&=\mu(\theta)(\alpha\phi(\theta)+1)\int_r^t[q(r)-q(s)]ds\le 0.\end{aligned}$$

ii) $t>\theta>r$. Then

$$\begin{aligned}U&=\mu(\theta)\Big(\alpha\phi(\theta)\int_t^1 q(s)ds-\int_t^1 q(s)ds-\alpha\phi(\theta)\int_r^1 q(s)ds-\int_\theta^1 q(s)ds\Big)=\\&=\mu(\theta)\Big(-\alpha\phi(\theta)\int_r^t q(s)ds-\int_\theta^t q(s)ds+\alpha\phi(\theta)q(r)\int_r^t ds\Big)\le\\&\le\mu(\theta)\alpha\phi(\theta)\int_r^t[q(r)-q(s)]ds\le 0.\end{aligned}$$

iii) $\theta > t > r$. Then

$$U = \mu(\theta)\Big(-\alpha\phi(\theta)\int_r^t q(s)ds + \alpha\phi(\theta)\int_r^t q(r)ds\Big) =$$
$$= \mu(\theta)\alpha\phi(\theta)\int_r^t (q(r) - q(s))ds \le 0.$$

iv) $t < r < \theta$. Then

$$U = \mu(\theta)\Big(\alpha\phi(\theta)\int_t^r q(s)ds + \alpha\phi(\theta)q(r)(t-r)\Big) =$$
$$= \mu(\theta)\alpha\phi(\theta)\int_t^r (q(s) - q(r))ds \le 0.$$

v) $\theta < t < r$. Then

$$U = -\mu(\theta)(\alpha\phi(\theta) + 1)\int_t^r [q(r) - q(s)]ds \le 0.$$

vi) $t < \theta < r$. Then

$$U = \mu(\theta)\Big(\alpha\phi(\theta)\int_t^r q(s)ds + \int_\theta^r q(s)ds + (\alpha\phi(\theta)+1)q(r)\int_r^t ds\Big) \le$$
$$\le \mu(\theta)\Big(\alpha\phi(\theta)\int_t^r q(s)ds + \int_t^r q(s)ds + (\alpha\phi(\theta)+)\int_r^t q(r)ds\Big) =$$
$$= \mu(\theta)(\alpha\phi(\theta)+1)\int_t^r [q(s) - q(r)]ds \le 0.$$

Therefore in any case we have $U \le 0$, which proves the result in case of (2.4). The other case can be proved by the same way. □

It is not hard to see that the kernel K defined by either (2.4), or (2.8) is a continuous, nonnegative function. Also, since obviously for $\delta = 1$ we have $\frac{\partial K(t,\cdot)}{\partial t} \le 0$ for a.a. $t \in I$, if K is defined by (2.4) and for $\delta = -1$ we have $\frac{\partial K(t,\cdot)}{\partial t} \ge 0$ for a.a. $t \in I$, if K is defined by (2.8), it follows that the function $t \to \delta K(t,\cdot)$ is non-increasing. Moreover, in both cases the function $t \to K(t,\cdot)$ is L_1-uniformly continuous. To check this fact, we observe that in case (2.4) for $t_1 < t_2$ it holds

$$\begin{aligned}
&\Big|\int_0^1 [K(t_1,\theta) - K(t_2,\theta)]d\theta\Big| = \\
&= \Big|\int_0^1 \mu(\theta)\int_\theta^1 q(s)d_s\Big[g(s)\int_{t_1}^{t_2} q(r)dr + s\chi_{[t_1,t_2]}(s)\Big]d\theta\Big| \\
&= \int_{t_1}^{t_2} q(r)dr\int_0^1 \mu(\theta)\int_\theta^1 q(s)dg(s)d\theta + \int_0^1 \mu(\theta)\int_{[\theta,1]\cap[t_1,t_2]} q(s)dsd\theta \\
&\le \int_{t_1}^{t_2} q(r)dr\Big[\int_0^1 \mu(\theta)d\theta\int_0^1 q(s)dg(s) + \int_0^1 \mu(\theta)d\theta\Big].
\end{aligned}$$

But $\int_{t_1}^{t_2} q(r)dr \to 0$, when $|t_1 - t_2| \to 0$.

Similarly for the case (2.8). Now, taking into account Lemma 5.1, we conclude that assumption (H_K) is satisfied. Therefore, in case assumptions (H1) and/or (H2) hold, Theorems 4.1, 4.2 and 4.3 apply also to the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4) and the corresponding results follow.

6 Concluding remarks

The multi-point boundary conditions for the problem considered in [19] are special cases of the conditions

$$x'(0) = -\sum_{i=1}^{k} b_i x'(\xi_i) + \sum_{i=1}^{k} c_i x'(\xi_i)$$

and

$$x(1) = \sum_{i=1}^{\lambda} d_i x'(\zeta_i) - \sum_{i=1}^{\lambda} r_i x'(\zeta_i),$$

which, obviously, are the discrete version of the conditions

$$x'(0) = -\int_0^1 x(s)dB(s) + \int_0^1 x'(s)dC(s) \tag{6.1}$$

and

$$x(1) = \int_0^1 x(s)dD(s) - \int_0^1 x'(s)dR(s), \tag{6.2}$$

respectively, with B, C, D, R nondecreasing functions and (without loss of generality) $B(0) = D(0) = 0$. But it is easy to see that in case $D(1) < 1$, (6.1), (6.2) can be written in the form (1.3) where

$$g(s) := \frac{B(1)}{1 - D(1)}\Big(R(s) + \int_0^s D(\theta)d\theta\Big) + C(s) + \int_0^s B(\theta)d\theta$$

and

$$h(s) := \frac{1}{1 - D(1)}\left(R(s) + \int_0^s D(\theta)d\theta\right).$$

Hence [19, Theorem1] is a special case of our Theorem 4.1.

Remark. We proved above that the boundary-value problems (1.2)-(1.3) and (1.2)-(1.4) admit one, or two, or three solutions according to the conditions which they satisfy. And it was their behavior which motivated us to distinguish the two cases for the kernel K of the integral equation (1.1). However, as the two problems is concerned, one can observe that existence of solutions of one of them guarantees existence of solutions of the other. Indeed, it is easy to see that they are equivalent and their equivalence follows by applying the transformation $x(t) \to x(1 - t)$, $p(t) \to p(1 - t)$, $\mu(t) \to \mu(1 - t)$ and $g_1(t) = -g(1 - t)$, $h_1(t) = -h(1 - t)$.

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