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Differential equations with several deviating arguments: Sturmian comparison method in oscillation theory, II *

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Abstract

We study the oscillation of solutions to the differential equation

$$\dot{x}(t) + a_1(t)x[r(t)] + a_2(t)x[p(t)] = 0, \quad t \geq t_0$$

which has a retarded argument $r(t)$ and an advanced argument $p(t)$. We obtain oscillation and non-oscillation conditions which are closed to be necessary. We provide examples to show that our results are best possible and compare them with known results.

1 introduction

This paper is a continuation of the investigation on equations with two deviating arguments started in [4]. We are concerned with the following two basic problems in the oscillation theory:
1) Sufficient conditions for the existence of a non-oscillatory solution.
2) Sufficient conditions for all solutions to be oscillatory.

Methods of investigation differ significantly for these two problems. For the first problem, it is enough to prove the existence of a sign preserving solution. In this case, various fixed point methods are applied, or a monotone sequence defined which converges to a non-oscillatory solution.

The investigation of the second oscillation problem can not employ methods that characterize only some solutions of the equation. Thus the proof is usually done by contradiction, i.e. the assumption that there exists a non-oscillatory solution is inconsistent with the constraints on the equation parameters.

In the works [7, 8, 9, 15, 16, 18, 19], a constructive method for proving that all the solutions are oscillatory was proposed and was called Sturmian comparison method. It is based on the generalization of the classical Sturm comparison theorem to functional-differential equations and inequalities . This theorem

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was originally formulated for the second order ODE. The Sturmian comparison method was described in detail in [4]. We give here just a brief description of the method.

The main idea of the method is to obtain the widest possible set of functional differential inequalities (the "testing" equations) associated with the equation, such that at least one of the solutions is a so called "slowly oscillating". This together with the Sturmian Comparison Theorem (see Theorem 2.1 in this paper) yields that all solutions oscillate.

The proposed method is constructive. Indeed, it is enough to find *only one* solution which enjoys certain properties instead of checking the fact that *all* the solutions of the "test" equation oscillate. Note that the results usually called Sturmian comparison oscillation theorems [5, 6, 22, 29] are not a part of Sturmian comparison theorem.

The Sturmian comparison method has been applied to the first order delay differential equations [9, 10, 15, 16, 18, 19], neutral differential equations [7, 8, 18], the second order delay differential equations [7, 8, 9, 18], impulsive delay differential equation [1, 2, 3] and difference equations [11, 12, 13, 14, 17].

Note the search for the "slowly oscillating" solutions can be treated as the search for "big half-cycles" [35] of a given length. Thus this method incidentally solves a rather difficult problem of the evaluation of the length of sign preserving intervals [9, 16, 18, 19].

This method turned out to be efficient for the investigation of so called "critical cases" as well. This means that the oscillation of all solutions of functional DE with almost constant coefficients is studied in the case when the limit equation has a non-oscillatory solution (see [20, 21] for the application of Sturmian comparison method in such cases).

The present paper is a continuation of [4], where the basic ideas of Sturmian comparison method were described for a rather general functional DE and this method was applied to DE of the first order with two retarded arguments.

This paper develops the Sturmian comparison method for the mixed differential equation

$$\dot{x}(t) + a_1(t)x[r(t)] + a_2(t)x[p(t)] = 0, \quad t \geq t_0, \tag{1.1}$$

with nonnegative coefficients $a_i(t)$, one delayed argument ($r(t) \leq t$) and one advanced argument ($p(t) \geq t$). To the best of our knowledge, oscillation properties of such equations has not been studied before except special cases of the autonomous equations [27, 32, 33, 37, 38] and equations with constant delays [39].

Sufficient conditions for all solutions of (1.1) to be oscillating are obtained here. The examples presented here show that these conditions are rather sharp. Further we obtain sufficient conditions for the existence of a non-oscillatory solution of (1.1). Similar non-oscillation results are obtained for (1.1) with non-positive coefficients.

2 Sturmian comparison theorem

Suppose $-\infty < \alpha, \beta < \infty$, $k = 1, 2$, and the following assumptions hold:

(A1) The functions $a_k(t)$ are continuous on (α, β), $a_k(t) \geq 0$.

(A2) The functions $r(t), p(t)$ are monotone increasing on (α, β) with continuous derivatives and $r(t) \leq t, p(t) \geq t$.

We can extend functions $r(t), p(t)$, without loss of monotonicity and differentiability, in such a way that the range of these functions include $[\alpha, \beta]$. Then there exist continuously differentiable functions $g(t), s(t)$ such that $r[g(t)] = t$ and $p[s(t)] = t$ for $t \in (\alpha, \beta)$.

In this and the next sections, we assume that conditions (A1) and (A2) hold for (1.1). We define the following differential operators on the set of continuous functions on $[s(\alpha), q(\beta)]$ with continuous derivatives on (α, β):

$$(lx)(t) := x'(t) + a_1(t)x[r(t)] + a_2(t)x[p(t)], \quad t \in (\alpha, \beta), \tag{2.1}$$

$$(\tilde{l}y)(t) := -y'(t) + q'(t)\tilde{a}_1[q(t)]y[q(t)] + s'(t)\tilde{a}_2[s(t)]y[s(t)], \quad t \in (\alpha, \beta). \tag{2.2}$$

Here $\tilde{a}_k(t)$, $k = 1, 2$ are continuous functions on $(r(\alpha), p(\beta))$. Let us define the following differential inequalities and equations:

$$(lx)(t) \leq 0, \quad t \in (\alpha, \beta), \tag{2.3}$$

$$(lx)(t) = 0, \quad t \in (\alpha, \beta), \tag{2.4}$$

$$(\tilde{l}y)(t) \geq 0, \quad t \in (\alpha, \beta). \tag{2.5}$$

Definition (see [9]) An interval (α, β) is called *a regular half-cycle* for (2.5) if

$$r(\beta) > \alpha, \quad \beta > p(\alpha),$$

and there exist a solution $y(t)$ of (2.5) such that

$$\begin{gathered} y(\alpha) = y(\beta) = 0, \quad y(t) > 0, \quad t \in (\alpha, \beta), \\ y(t) \leq 0, \quad t \in (s(\alpha), \alpha) \cup (\beta, q(\beta)). \end{gathered} \tag{2.6}$$

The definition of regular half-cycle for (2.3) and (2.4) is similar.

Definition (see [9]) a) A solution $x(t)$, $t_0 \leq t < \infty$ of a differential equation or inequality is called *non-oscillatory* if there exists T such that $x(t) \neq 0$, $t \geq T$ and *oscillatory* otherwise.
b) An oscillatory solution x of a differential equation or inequality is called *regular oscillatory* if for every T there exists its regular half-cycle $(\alpha, \beta), \alpha > T$ and *quick oscillatory* otherwise.

The following theorem is a general Sturmian comparison theorem [4, Theorem 1] which is formulated for (1.1).

Theorem 2.1 *Suppose (α,β) is a regular half-cycle for (2.5) ,*

$$\tilde{a}_1(t) \ge 0, \quad t \in (\beta, q(\beta)); \quad \tilde{a}_2(t) \ge 0, \quad t \in (s(\alpha), \alpha)), \tag{2.7}$$

$$a_1(t) \ge \begin{cases} 0, & t \in (\alpha, q(\alpha)), \\ \tilde{a}_1(t), & t \in (q(\alpha), \beta), \end{cases} \qquad a_2(t) \ge \begin{cases} \tilde{a}_2(t), & t \in (\alpha, s(\beta)), \\ 0, & t \in (s(\beta), \beta), \end{cases} \tag{2.8}$$

and at least one of (2.7)-(2.8) is strict on some subinterval of (α,β). Then (2.3) has no positive solutions on $(s(\alpha), q(\beta))$.

Corollary 2.2 *Suppose (2.5) has a regular oscillatory solution on (t_0,∞) and*

$$a_k(t) \ge \tilde{a}_k(t) \ge 0, \quad k = 1, 2, \quad t \ge t_0.$$

Then (2.3) has no positive solution on (t_0,∞).

Remark Theorem 2.1 is concerned with the behavior of solutions of mixed differential equations and inequalities on a *finite interval* and not on a half-line. Therefore one can obtain from Theorem 2.1 not only explicit conditions of oscillation but also estimations of the length of the sign-preserving intervals of solutions. In this paper we will not consider this problem.

3 Construction of the testing equations

Lemma 3.1 *Let $r(\beta) > \alpha$, $p(\alpha) < \beta$ and for continuous functions $\varphi(t)$, $m(t)$ assume the following conditions hold:*

$$0 < \int_\alpha^t \varphi(s)ds < \pi, \ t \in (\alpha,\beta); \quad 0 < \int_t^\alpha \varphi(s)ds < \pi, \ t \in (s(\alpha), \alpha); \tag{3.1}$$

$$\int_\alpha^\beta \varphi(s)ds = \pi; \quad 0 < \int_{s(t)}^{q(t)} \varphi(s)ds < \frac{\pi}{2}, \quad t \in (r(\alpha), \beta); \tag{3.2}$$

$$m(t) \ge \begin{cases} -\varphi(t) \cot \int_{s(t)}^t \varphi(s)ds, & t \in (r(\beta), \beta), \\ \varphi(t) \cot \int_t^{q(t)} \varphi(s)ds, & t \in (\alpha, p(\alpha)). \end{cases} \tag{3.3}$$

Then the interval (α,β) is regular half-cycle for (2.5) where $\tilde{a}_k(t)$ defined by

$$\begin{aligned} q'(t)\tilde{a}_1[q(t)] := \csc \int_{s(t)}^{q(t)} \varphi(s)ds \exp\big(-\int_t^{q(t)} m(s)ds\big) \\ \times \Big\{ m(t) \sin \int_{s(t)}^t \varphi(s)ds + \varphi(t) \cos \int_{s(t)}^t \varphi(s)ds \Big\}, \end{aligned}$$

$$\begin{aligned} s'(t)\tilde{a}_2[s(t)] := \csc \int_{s(t)}^{q(t)} \varphi(s)ds \exp\big(\int_{s(t)}^t m(s)ds\big) \\ \times \Big\{ m(t) \sin \int_t^{q(t)} \varphi(s)ds - \varphi(t) \cos \int_t^{q(t)} \varphi(s)ds \Big\}. \end{aligned} \tag{3.4}$$

Proof. By direct calculations one can check that the function

$$y(t) := \sin \int_{\alpha}^{t} \varphi(s)ds \times \exp \int_{\alpha}^{t} m(s)ds$$

is a solution of the equation $(\tilde{l}y)(t) = 0$. Conditions (3.1)-(3.3) and $p(\beta) > \alpha$ make (α, β) an regular half-cycle for (2.5).

From Theorem 2.1 and Lemma 3.1 we obtain the following theorem.

Theorem 3.2 *Suppose for some functions $\varphi(t), m(t)$ conditions of Lemma 3.1 and (2.7)-(2.8) hold, where $\tilde{a}_i(t)$ are given by (3.4) and at least one of (2.7)-(2.8) is strict on some subinterval. Then (2.3) has no positive solution on $(r(\alpha), p(\beta))$.*

Corollary 3.3 *Suppose the conditions of Theorem 3.2 hold for a sequence of intervals (α_j, β_j), $\alpha_j \to \infty$. Then all solutions of (1.1) are oscillatory.*

Remark. No limitations are imposed on the coefficients $a_k(t)$ of (1.1) outside the set $\cup_{j=1}^{\infty}(\alpha_j, \beta_j)$ in Corollary 3.3.

Corollary 3.4 *Let $r(t) \to \infty$, $\int_{t_0}^{\infty} \varphi(s)ds = \infty$, and*

$$0 < \int_{s(t)}^{q(t)} \varphi(s)ds < \frac{\pi}{2}, \ t \geq t_0, \tag{3.5}$$

$$m(t) \geq \max\Big\{ - \varphi(t) \cot \int_{s(t)}^{t} \varphi(s)ds; \varphi(t) \cot \int_{t}^{q(t)} \varphi(s)ds\Big\}, t > t_0. \tag{3.6}$$

Assume also that for every T there exists $\alpha > T$ such that

$$0 < \int_{t}^{\alpha} \varphi(s)ds < \frac{\pi}{2}, \ t \in (s(\alpha), \alpha), \quad 0 < \int_{\alpha}^{t} \varphi(s)ds < \pi, \ t \in (\alpha, \beta),$$

where β is defined by $\int_{\alpha}^{\beta} \varphi(s)ds = \pi$. If

$$a_k(t) \geq \tilde{a}_k(t), \quad k = 1, 2, \ t \geq t_0, \tag{3.7}$$

where $\tilde{a}_i$ are defined by (3.4), then all solutions of (1.1) are oscillatory.

On the one hand the following statement gives a new clear proof of the following well-known [33] fundamental oscillation criterion for autonomous equation with mixed deviations

$$x'(t) + a_1 x(t - \tau) + a_2 x(t + \sigma) = 0, \quad \tau > 0, \ \sigma > 0, \ a_k > 0, \ k = 1, 2. \tag{3.8}$$

On the other hand this statement demonstrates the sharpness of Theorem 3.2, which in particular case (an autonomous equation) enables one to obtain necessary and sufficient conditions of oscillation of all solutions of (1.1).

Corollary 3.5 *Suppose for the characteristic quasi-polynomial of (3.8)*

$$F(\lambda) := \lambda + a_1 \exp\{-\lambda\tau\} + a_2 \exp\{\lambda\sigma\}$$

the following condition holds

$$\{F(\lambda) > 0, \quad \textit{for all } \lambda \in (-\infty, \infty)\}. \tag{3.9}$$

Then all solutions of (3.8) are oscillatory.

(It is obvious that (3.9) is also necessary for oscillation of all solutions as well).

Proof. It is easy to see that the equation

$$F'(\lambda) := 1 - a_1\tau e^{-\lambda\tau} + a_2\sigma e^{\lambda\sigma} = 0 \tag{3.10}$$

has a unique root λ_0 and $\inf_{-\infty<\lambda<\infty} F(\lambda) = F(\lambda_0)$. Indeed, for every λ we have

$$\lim_{\lambda\to+\infty} F'(\lambda) = +\infty, \quad \lim_{\lambda\to-\infty} F'(\lambda) = -\infty, \quad F''(\lambda) > 0.$$

Equation (3.8) implies

$$\begin{gathered}\text{Cond. } (3.9) \iff \{F(\lambda_0) > 0\} \iff \sigma(\lambda_0 + a_1 e^{-\lambda_0\tau} + a_2 e^{\lambda_0\sigma}) > 0 \\ \iff \sigma\lambda_0 + \sigma a_1 e^{-\lambda_0\sigma} + (a_1\tau e^{-\lambda_0\tau} - 1) > 0 \iff a_1 > \frac{-\sigma\lambda_0 + 1}{\tau+\sigma} e^{\lambda_0\tau}.\end{gathered} \tag{3.11}$$

Similarly we have

$$\text{Cond. } (3.9) \iff \tau F(\lambda_0) > 0 \iff a_2 > \frac{-\tau\lambda_0 - 1}{\tau+\sigma} e^{-\lambda_0\sigma}. \tag{3.12}$$

Consider two possible cases for a_1. Assume first $a_1 > \frac{1}{\tau e}$. Then the equation

$$y'(t) + a_1 y(t-\tau) = 0$$

has no eventually positive solutions and therefore the inequality

$$y'(t) + a_1 y(t-\tau) \le 0$$

has no such solutions. Then (3.8) has no eventually positive solutions. Assume now that $a_1 \le \frac{1}{\tau e}$. Then

$$F'(-\frac{1}{\tau}) = 1 - a_1\tau e + a_2\sigma \exp\big\{-\frac{\sigma}{\tau}\big\} > 0.$$

Since $F'(\lambda_0) = 0$ and $F''(\lambda) > 0$ for all λ, we obtain

$$\lambda_0 < -\frac{1}{\tau}. \tag{3.13}$$

Put $\varphi(t) := \nu, \quad m(t) := -\lambda_0$ in Theorem 3.2, where $\nu > 0$ is a sufficiently small number which will be chosen below. Then (3.13) implies

$$m(t) = -\lambda_0 > \frac{1}{\tau} > \frac{\nu}{\tan \nu\tau}.$$

Hence (3.3) holds. Thus, Theorem 3.2 implies that under the conditions

$$\begin{aligned} a_1 &> \frac{e^{\lambda_0 \tau}}{\sin \nu(\tau+\sigma)}\{-\lambda_0 \sin \nu\sigma + \nu \cos \nu\sigma\} := P_1(\nu) \\ a_2 &> \frac{e^{-\lambda_0 \sigma}}{\sin \nu(\tau+\sigma)}\{-\lambda_0 \sin \nu\tau - \nu \cos \nu\tau\} := P_2(\nu), \end{aligned} \tag{3.14}$$

all solutions of (3.8) are oscillatory.

It is easy to see that

$$\lim_{\nu\to 0} P_1(t) = e^{\lambda_0\tau}\frac{-\lambda_0\sigma + 1}{\tau+\sigma}, \quad \lim_{\nu\to 0} P_2(t) = e^{-\lambda_0\sigma}\frac{-\lambda_0\tau - 1}{\tau+\sigma}.$$

Inequalities (3.11) and (3.12) are sharp and so (3.14) holds for $\nu \in (0, \nu_0)$. Then Corollary 3.5 is proven.

Next we will obtain from Theorem 3.2 an explicit condition for oscillation not only in terms of pointwise estimations but in terms of *the integral average estimations* as well. To avoid unwieldy formulations we will omit the estimation for the length of sign-preserving intervals of solutions.

Theorem 3.6 *Let $r(t) \to \infty$ and assume that there exist functions $b_j(t)$, $j = 1, 2$, such that:*

$$a_j(t) \ge b_j(t) \ge 0, \ j = 1, 2, \ t \ge t_0; \tag{3.15}$$

the following limits exist and are finite:

$$B_{1j} := \lim_{t\to\infty} \int_{r(t)}^{t} b_j(s)ds, \quad B_{2j} := \lim_{t\to\infty} \int_{t}^{p(t)} b_j(s)ds, \quad j = 1, 2, \tag{3.16}$$

with

$$B_{11} + B_{22} > 0; \tag{3.17}$$

the following system has a positive solution $\{x_1; x_2\}$:

$$\begin{gathered} -(B_{11}B_{22} - B_{12}B_{21})x_1x_2 - B_{11}x_1 + B_{22}x_2 + 1 = 0 \\ \ln x_1 - B_{11}x_1 - B_{12}x_2 < 0 \\ \ln x_2 + B_{21}x_1 + B_{22}x_2 < 0\,. \end{gathered} \tag{3.18}$$

Then all solutions of (1.1) are oscillatory.

Proof. In view of the first inequality in (3.18) the system

$$\begin{aligned} (x_1B_{11}-1)\alpha_1 - x_2B_{12}\alpha_2 &= 0\\ -x_1B_{21}\alpha_1 + (x_2B_{22}+1)\alpha_2 &= 0\\ \alpha_1+\alpha_2 &= 1 \end{aligned}\tag{3.19}$$

has a solution $\{\alpha_1;\alpha_2\}$, $\alpha_j>0,\ j=1,2$ (we omit the details).

In Theorem 3.2, denote functions $m(t)>0$ and $\varphi(t)$ as follows:

$$\varphi(t) := \nu\alpha_1x_1q'(t)b_1[q(t)] - \nu\alpha_2x_2s'(t)b_2[s(t)],\tag{3.20}$$

$$m(t) := x_1q'(t)b_1[q(t)] + x_2s'(t)b_2[s(t)],\tag{3.21}$$

where $\nu>0$ is a sufficiently small number which will be chosen below. Note that $\varphi(t)$ defined by (3.20) is *not necessarily* nonnegative for all t. Nevertheless, from (3.20)-(3.21) we obtain

$$\begin{gathered} \lim_{t\to\infty}\int_t^{q(t)}\varphi(s)ds = \nu\alpha_1x_1B_{11} - \nu\alpha_2x_2B_{12} = \nu\alpha_1>0,\\ \lim_{t\to\infty}\int_{s(t)}^{t}\varphi(s)ds = \nu\alpha_1x_1B_{21} - \nu\alpha_2x_2B_{22} = \nu\alpha_2>0, \end{gathered}\tag{3.22}$$

$$\lim_{t\to\infty}\int_t^{q(t)}m(s)ds = x_1B_{11}+x_2B_{12},\quad \lim_{t\to\infty}\int_{s(t)}^{t}m(s)ds = x_1B_{21}+x_2B_{22}.\tag{3.23}$$

Furthermore, $\int^\infty\varphi(s)ds=\infty$ and (3.1), (3.2) hold for $t>T$, where T is sufficiently large.

The inequality

$$\nu\alpha_1x_1q'(t)b_1[q(t)] - \nu\alpha_2x_2s'(t)b_2[s(t)] \le \nu\alpha_1\left[x_1q'(t)b_1[q(t)] + x_2s'(t)b_2[s(t)]\right]$$

implies that for sufficiently large t

$$\begin{aligned} \varphi(t)\cot\int_t^{q(t)}\varphi(s)ds \le{}& \nu\alpha_1m(t)\cot\int_t^{q(t)}\varphi(s)ds\\ \le{}& \frac{\nu\alpha_1}{\tan\nu\alpha_1}\frac{\tan\nu\alpha_1}{\tan\int_t^{q(t)}\varphi(s)ds}\le m(t). \end{aligned}$$

Similarly, the inequality

$$\nu\alpha_1x_1q'(t)b_1[q(t)] - \nu\alpha_2x_2s'(t)b_2[s(t)] \ge -\nu\alpha_2\left[x_1q'(t)b_1[q(t)] + x_2s'(t)b_2[s(t)]\right]$$

implies

$$\begin{aligned} \varphi(t)\cot\int_{s(t)}^{t}\varphi(s)ds \ge{}& -\nu\alpha_2\cot\int_{s(t)}^{t}\varphi(s)ds\\ \ge{}& -\frac{\nu\alpha_2}{\tan\nu\alpha_2}\frac{\tan\nu\alpha_2}{\tan\int_{s(t)}^{t}\varphi(s)ds}\ge -m(t). \end{aligned}$$

Hence $m(t) \geq -\varphi(t)\cot\int_{s(t)}^{t}\varphi(s)ds$. Therefore (3.6) holds. Denote

$$D_1(t,\nu) := \csc\int_{s(t)}^{q(t)}\varphi(s)ds\exp\big\{-\int_t^{q(t)}m(s)ds\big\}\nu x_1,$$
$$D_2(t,\nu) := \csc\int_{s(t)}^{q(t)}\varphi(s)ds\exp\big\{\int_{s(t)}^{t}m(s)ds\big\}\nu x_2.$$

Then

$$\begin{aligned} G_1(\nu) &:= \lim_{t\to\infty} D_1(t,\nu) = \exp(-x_1B_{11} - x_2B_{12})\frac{\nu x_1}{\sin\nu},\\ G_2(\nu) &:= \lim_{t\to\infty} D_2(t,\nu) = \exp(x_1B_{21} + x_1B_{22})\frac{\nu x_2}{\sin\nu}. \end{aligned} \tag{3.24}$$

Condition (3.18) implies $G_k(\nu) < 1$ $\nu\in(0,\nu_0)$, $k=1,2$. Hence

$$D_k(t,\nu) < 1, \quad \nu\in(0,\nu_0), \quad k=1,2, \quad t > T \gg 1$$

and therefore

$$\begin{aligned} q'(t)a_1[q(t)] &> D_1(t,\nu)q'(t))b_1[q(t)]\\ s'(t)a_2[q(t)] &> D_2(t,\nu)s'(t))b_2[s(t)]. \end{aligned} \tag{3.25}$$

for $\nu\in(0,\nu_0)$, $t>T$. The right-hand sides of (3.4) can be rewritten in the form $H_1(t,\nu) + D_1(t,\nu)q'(t))b_1[q(t)]$, and $H_2(t,\nu) + D_2(t,\nu)s'(t))b_2[s(t)]$, where

$$\begin{aligned} H_1(t,\nu) := \csc\int_{s(t)}^{q(t)}\varphi(s)ds\exp\big(-\int_t^{q(t)}m(s)ds\big)\\ \times\Big[m(t)\big\{\int_{s(t)}^{t}\varphi(s)ds - \nu\alpha_2\big\} + \varphi(t)\big\{\cos\int_{s(t)}^{t}\varphi(s)ds - 1\big\}\Big] \end{aligned}$$

and

$$\begin{aligned} H_2(t,\nu) := \csc\int_{s(t)}^{q(t)}\varphi(s)ds\exp\int_{s(t)}^{t}m(s)ds\\ \times\Big[m(t)\big\{\int_t^{q(t)}\varphi(s)ds - \nu\alpha_1\big\} - \varphi(t)\big\{cos\int_t^{q(t)}\varphi(s)ds - 1\big\}\Big]. \end{aligned}$$

It is easy to check that $\lim_{t\to\infty} H_i(t,\nu) = 0$, $i=1,2$, $\nu\in(0,\nu_0)$. Hence (3.7) holds and therefore all solutions of (1.1) are oscillatory.

Example Consider the equation

$$x'(t) + \frac{a_1}{t}x\left(\frac{t}{\mu}\right) + \frac{a_2}{t}x(t+\tau) = 0,\ t\geq t_0 > 0, \tag{3.26}$$

where $\mu>1$, $\tau>0$, $a_1,a_2>0$. Put $b_1(t) := a_1(t) = a_1/t$ and $b_2(t) := a_2(t) = a_2/t$ in Theorem 3.6. Then $B_{11} = a_1\ln\mu$, $B_{12} = a_2\ln\mu$, $B_{21} = B_{22} = 0$.

System (3.18) turns into the system

$$\begin{gathered}-a_1x_1\ln\mu+1=0\\ \ln x_1-a_1x_1\ln\mu-a_2x_2\ln\mu<0\\ \ln x_2<0\end{gathered}$$

which is equivalent to the system

$$\begin{gathered}x_1=\frac{1}{a_1\ln\mu}\\ -\ln[a_1\ln\mu]-1<x_2a_2\ln\mu\\ \ln x_2<0\end{gathered}$$

and this in turn is equivalent to the system

$$\begin{gathered}x_1=\frac{1}{a_1\ln\mu}\\ \frac{-\ln[a_1\ln\mu]-1}{a_2\ln\mu}<x_2<1\end{gathered}$$

The last system has a solution if and only if

$$\frac{-\ln[a_1\ln\mu]-1}{a_2\ln\mu}<1\Longleftrightarrow a_1\mu^{a_2}>\frac{1}{e\ln\mu}. \tag{3.27}$$

Thus, (3.27) is sufficient for oscillation of all solutions of (3.26). Note that (3.27) does not depend on τ.

Remark. It will be shown in Section 4 that if $a_1\mu^{a_2}<1/(e\ln\mu)$ then (3.26) has a non-oscillatory solution.

Example Consider the equation

$$x'(t)+\frac{a_1}{t}x\left(\frac{t}{\mu}\right)+\frac{a_2}{t^\beta}x(t+t^\alpha)=0,\ t\ge t_0>0, \tag{3.28}$$

where $a_1,a_2>0$, $\mu>1$, $0\le\alpha<1$, $0\le\beta<1$. In Theorem 3.6, Let

$$b_1(t):=a_1(t)=\frac{a_1}{t},\quad b_2(t):=\frac{A}{t}\le a_2(t)=\frac{a_2}{t^\beta},\quad t>T,$$

where T is sufficiently large, A is an arbitrarily large positive constant. Then $B_{11}=a_1\ln\mu$, $B_{12}=A\ln\mu$, $B_{21}=B_{22}=0$. One can repeat now all calculations in the previous example. Cond.(3.27) is $a_1\mu^A>\frac{1}{e\ln\mu}$ which holds for every $a_1>0$ if A is sufficiently large. Therefore, *all solutions of (3.28) are oscillatory for any* $\{a_1>0,\ a_2>0\}$.

This result is rather unexpected. Indeed, for $a_2 = 0$ the condition $a_1 > \frac{1}{e \ln \mu}$ is necessary and sufficient for oscillation of all solutions for the "reduced" equations

$$y'(t) + \frac{a_1}{t} y\left(\frac{t}{\mu}\right) = 0.$$

For $a_1 = 0$ the "reduced" equation

$$z'(t) + \frac{a_2}{t^\beta} z(t + t^\alpha) = 0$$

has a non-oscillatory solution for every $a_2 > 0$, $0 \le \alpha < 1$, $0 \le \beta < 1$.

Remark In [36, Theorem 1] the same oscillation condition (3.27) for (3.26) is implied as in our Theorem 3.6. However, for (3.28) one can derive from [36] only condition (3.27) for oscillation of all solutions. Further, our condition $\{a_1 > 0,\ a_2 > 0\}$ covers all cases.

4 Non-oscillation conditions

In this section we apply some well known results [25, 28, 34] to the delay differential equation

$$x'(t) + \sum_{k=1}^{n} a_k(t) x[r_k(t)] = 0, \quad t \ge t_0, \tag{4.1}$$

and to the advanced differential equation

$$x'(t) - \sum_{k=1}^{n} b_k(t) x[p_k(t)] = 0, \quad t \ge t_0. \tag{4.2}$$

We assume for (4.1), (4.2) conditions (A1), and (A2) hold.

Lemma 4.1 *1.) Equation (4.1) has a non-oscillatory solution if and only if there exists a function $u(t) \ge 0$ and $t_1 \ge t_0$ such that*

$$u(t) \ge \sum_{k=1}^{m} a_k(t) \exp\Big\{ \int_{r_k(t)}^{t} u(s)ds \Big\}, \quad t \ge t_1.$$

2.) If $m = 1$ and $\lim_{t\to\infty} \sup \int_{r(t)}^{t} a(s)ds < 1/e$, then (4.1) has a non-oscillatory solution.

Lemma 4.2 *1.) Equation (4.2) has a non-oscillatory solution if and only if there exists a function $u(t) \ge 0$ and $t_1 \ge t_0$ such that*

$$u(t) \ge \sum_{k=1}^{m} b_k(t) \exp\Big\{ \int_{t}^{p_k(t)} u(s)ds \Big\}, \quad t \ge t_1.$$

2.) If $m = 1$ and $\lim_{t\to\infty} \sup \int_{t}^{p(t)} b(s)ds < 1/e$, then (4.2) has a non-oscillatory solution.

Theorem 4.3 *Suppose $a_1(t), a_2(t), r(t), p(t)$ are uniformly continuous on the interval $[t_0, \infty)$,*

$$\lim_{t\to\infty} \sup[t - r(t)] = r < \infty, \quad \lim_{t\to\infty} \sup[p(t) - t)] = p < \infty, \tag{4.3}$$

and there exists a non-oscillatory solution of the delay differential equation

$$y'(t) + a_1(t)y[r(t)] + a_2(t)y(t) = 0. \tag{4.4}$$

Then there exists a non-oscillatory solution of (1.1).

Proof. Lemma 4.1 implies the existence of a function $u_0(t) \geq 0$ and $t_1 \geq t_0$ such that

$$u_0(t) \geq a_1(t) \exp\Big\{\int_{r(t)}^{t} u_0(s)ds\Big\} + a_2(t), \quad t \geq t_1. \tag{4.5}$$

Consider the space $\mathbf{C}[t_0, \infty)$ of all continuous and bounded functions on $[t_0, \infty)$ with supremum norm $\|\cdot\|$ and consider the operator

$$(Au)(t) := a_1(t) \exp\Big\{\int_{r(t)}^{t} u(s)ds\Big\} + a_2(t) \exp\Big\{-\int_{t}^{p(t)} u(s)ds\Big\}.$$

Let $S = \{u \in \mathbf{C} : 0 \leq u(t) \leq u_0(t)\}$. Equation (4.5) implies $0 \leq (Au)(t) \leq u_0(t)$. For $u \in S$, denote the integral operators

$$(Hu)(t) := \int_{r(t)}^{t} u(s)ds, \quad (Ru)(t) := \int_{t}^{p(t)} u(s)ds.$$

Inequalities (4.3) imply

$$|(Hu)(t)| \leq r\|u_0\|, \quad |(Ru)(t)| \leq p\|u_0\|.$$

Hence the sets HS and RS are bounded in the space $\mathbf{C}[t_0, \infty)$. For $u \in S$, we obtain

$$\begin{aligned} |(Hu)(t_2) - (Hu)(t_1)| &\leq \Big|\int_{r(t_1)}^{r(t_2)} u(s)ds\Big| + \Big|\int_{t_1}^{t_2} u(s)ds\Big| \\ &\leq \|u_0\|(|r(t_2) - r(t_1)| + |t_2 - t_1|) \end{aligned}$$

and

$$|(Ru)(t_2) - (Ru)(t_1)| \leq \|u_0\|(|p(t_2) - p(t_1)| + |t_2 - t_1|).$$

Hence the families HS and RS are equicontinuous. Then the sets HS and RS are compact. Therefore, the set AS is also compact.

Schauder's Fix Point Theorem implies that there exists a solution $u \in S$ of the equation $u = Au$. Therefore, the function $x(t) = x(t_1)\exp\{-\int_{t_1}^{t} u(s)ds\}$, $t \geq t_1$, is a positive solution of (1.1).

Corollary 4.4 *Suppose* $a_1(t)$, $a_2(t)$, $r(t)$, *and* $p(t)$ *are uniformly continuous on* $[t_0, \infty)$, *(4.3) holds and*

$$\limsup_{t\to\infty} \int_{r(t)}^{t} a_1(s) \exp\Big\{ \int_{r(s)}^{s} a_2(\tau)d\tau \Big\} ds < \frac{1}{e}. \tag{4.6}$$

Then (1.1) has a non-oscillatory solution.

Proof. Substituting $y(t)$ by $z(t)\exp\{-\int_{t_0}^{t} a_2(s)ds\}$ in (4.4), we obtain

$$\dot{z}(t) + a_1(t)\exp\Big\{ \int_{r(t)}^{t} a_2(s)ds \Big\} z(r(t)) = 0.$$

Lemma 4.2 and (4.6) imply that (4.4) and therefore (1.1) has a non-oscillatory solution.

Remark. Corollary 4.4 implies that if

$$\frac{-\ln[a_1 \ln\mu] - 1}{a_2 \ln\mu} < 1 \Longleftrightarrow a_1\mu^{a_2} < \frac{1}{e\ln\mu}$$

then (3.26) has a non-oscillatory solution. This implies that conditions of Theorems 3.6 and 4.3 are sharp for oscillation and non-oscillation of (1.1).

Remark. Corollary 4.4 improves some results in [26].

Corollary 4.5 *Suppose* $a_1(t)$, $a_2(t)$, $r(t)$, *and* $p(t)$ *are uniformly continuous on* $[t_0, \infty)$, *(4.3) holds and*

$$\int_{t_0}^{\infty} a_1(s)ds < \infty. \tag{4.7}$$

Then (1.1) has a non-oscillatory solution.

Proof. Condition (4.3) implies

$$\begin{aligned} &\limsup_{t\to\infty} \int_{r(t)}^{t} a_1(s) \exp\Big\{ \int_{r(s)}^{s} a_2(\tau)d\tau \Big\} ds \\ &\qquad \le e^{r\|a_2\|} \limsup_{t\to\infty} \int_{r(t)}^{t} a_1(s)ds = 0 < \frac{1}{e}. \end{aligned}$$

Hence (1.1) has a non-oscillatory solution.

Theorem 4.6 *Suppose* $\int_{t_0}^{\infty} a_1(s)ds = \infty$ *and* x *is a non-oscillatory solution of (1.1). Then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose $x(t) > 0$, $t \geq t_1$ and $r(t) \geq t_1,\ t \geq t_2$. Then $\dot{x}(t) \leq 0,\ t \geq t_2$. Denote $u(t) = -\frac{\dot{x}(t)}{x(t)}$, $t \geq t_2$. Then $u(t) \geq 0$, $t \geq t_2$. After substituting

$$x(t) = x(t_2) \exp\big\{ -\int_{t_2}^{t} u(s)ds \big\}, \quad t \geq t_2, \tag{4.8}$$

into (1.1) we have

$$u(t) = a_1(t) \exp\big\{ \int_{r(t)}^{t} u(s)ds \big\} + a_2(t) \exp\big\{ -\int_{t}^{p(t)} u(s)ds \big\}, \quad t \geq t_2. \tag{4.9}$$

Hence $u(t) \geq a_1(t)$ and therefore $\int_{t_0}^{\infty} u(s)ds = \infty$. Equality (4.8) implies, that $\lim_{t\to\infty} x(t) = 0$.

Consider now the mixed differential equation

$$x'(t) - a_1(t)x[r(t)] - a_2(t)x[p(t)] = 0,\ t \geq t_0. \tag{4.10}$$

Theorem 4.7 *Suppose $a_1(t)$, $a_2(t)$, $r(t)$, and $p(t)$ are uniformly continuous on $[t_0, \infty)$, (4.3) holds and*

$$y'(t) - a_1(t)y(t) - a_2(t)y[p(t)] = 0,\ t \geq t_0, \tag{4.11}$$

has a non-oscillatory solution. Then (4.10) has a non-oscillatory solution.

Proof. In the space $\mathbf{C}[t_0, \infty)$, consider the operator

$$(Bu)(t) := a_1(t) \exp\big\{ -\int_{r(t)}^{t} u(s)ds \big\} + a_2(t) \exp\big\{ \int_{t}^{p(t)} u(s)ds \big\}.$$

Lemma 4.2 implies that there exists a nonnegative solution $u_0(t)$ of the inequality

$$u(t) \geq a_1(t) + a_2(t) \exp\big\{ \int_{t}^{p(t)} u(s)ds \big\}, \quad t \geq t_1.$$

Let $S = \{u : 0 \leq u(t) \leq u_0(t)\}$. As in the proof of Theorem 4.3, we obtain $BS \subset S$, and the set BS is a compact. Therefore, the equation $u = Bu$ has a nonnegative solution u. Hence a function

$$x(t) = x(t_1) \exp\big\{ \int_{t_1}^{t} u(s)ds \big\}, \quad t \geq t_0,$$

is a positive solution of (4.10).

Corollary 4.8 . *Suppose $a_1(t), a_2(t), r(t), p(t)$ are uniformly continuous on $[t_0, \infty)$, (4.3) holds and*

$$\limsup_{t\to\infty} \int_{t}^{p(t)} a_2(s) \exp\big\{ \int_{s}^{p(s)} a_1(\tau)d\tau \big\}ds < \frac{1}{e}.$$

Then (4.10) has a non-oscillatory solution.

The proof of this corollary is similar to the proof of Corollary 4.4.

Corollary 4.9 *Suppose $a_1(t)$, $a_2(t)$, $r(t)$, and $p(t)$ are uniformly continuous on $[t_0, \infty)$, (4.3) holds and $\int_{t_0}^{\infty} a_2(s)ds < \infty$. Then (4.10) has a non-oscillatory solution.*

Theorem 4.10 *Suppose $\int_{t_0}^{\infty} a_2(s)ds = \infty$ and x is a non-oscillatory solution of (4.10). Then $\lim_{t\to\infty} x(t) = \infty$.*

The proof of this theorem is similar to the proof of Theorem 4.6.

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