Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 32, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Existence of unstable manifolds for a certain class of delay differential equations *

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Abstract

We prove a theorem for unstable manifolds in a differential equation with a state-dependent delay. Although the equation cannot be formally linearized, we find an associated linear delay equation whose dynamics are qualitatively similar near the unstable manifold. Our proof relies upon estimates of the derivative of a trajectory on the unstable manifold near equilibrium.

1 Introduction

Various authors have studied the delay equation

$$\dot{x}(t) = f(x(t), x(t-r)), \quad r = r(x(t)) \tag{1.1}$$

over the past few years. Mackey [6] proposed that a special case of (1.1) could be used to model the spot price of an agricultural commodity, such as corn. Here, $r(x(t))$ represents the time delay between the production and delivery of the commodity and depends upon the price $x(t)$. Cooke and Huang [2] have examined some of the difficulties that arise when attempting to linearize (1.1) and have developed decay estimates for solutions near the origin. Mallet-Paret and Nussbaum [7] proved the existence of a sawtooth-shaped, slowly oscillating periodic solution to the equation

$$\varepsilon\dot{x}(t) = -x(t) + f(x(t-r)), \quad r = r(x(t)) \tag{1.1}$$

in the singular limit as ε goes to 0. Kuang and Smith [5], and Arino, Hadeler and Hbid [1] have proved the existence of periodic solutions for certain types of state-dependent delay equations, including (1.1).

When r is a constant, solutions to (1.1) are typically embedded in the space $C([-r,0])$, of continuous functions over the interval $[-r,0]$. If f is sufficiently well-behaved, it is possible to prove the existence and uniqueness of solutions and also the existence of a smooth unstable manifold relative to (1.1).

* *Mathematics Subject Classifications:* 34K40.
Key words: invariant manifolds, functional differential equations.

Submitted December 1, 1999. Published April 1, 2002.

When $r = r(x(t))$ is non-constant, we run into difficulty since solutions are not uniquely defined even locally in time. Unless we have a uniform bound for $|\dot{x}(t)|$, $x(t-r)$ and hence $f(x(t-r))$ will not be a Lipschitz function in the argument $x(t)$. In this paper, we develop an alternative phase space setting which allows us to control the derivative of $x(t)$; we then prove existence, uniqueness and unstable manifold results. Since the existence of an unstable manifold does not depend on the choice of phase-space, our approach provides a consistent way of looking at (1.1).

In section 2, we prove a straightforward existence result for solutions to (1.1), relative to the phase space $W^{1,\infty}$. In section 3, we show that (under appropriate technical conditions), the unstable manifold is a smooth graph whenever f is at least C^2. We will take advantage of the fact that trajectories on the unstable manifold are defined from time $-\infty$ to the present and are thus as smooth as possible.

2 Technical Preliminaries

Hale and Ladeira [3] have used the space $W^{1,\infty}$ to show that solutions to the equation

$$\dot{x}(t) = f(x(t), x(t-\rho)) \tag{2.1}$$

depend smoothly on the delay parameter ρ. In [3], ρ is neither time- nor state-dependent. However, it turns out that the space $W^{1,\infty}$ is useful in the case where the delay is state-dependent. We require the following definitions for which we use the notation in [3].

Suppose that E is a linear space equipped with the norms $|\cdot|$ and $N(\cdot)$; suppose further that we define the set $B_{R,N} = \{x \in E : N(x) \le R\}$ for any fixed $R > 0$. ($B_{R,N}$ is a closed ball of radius R centered at 0 relative to the $N(\cdot)$ norm.) Also suppose that, for any fixed $R > 0, (B_{R,N}, |\cdot|)$ is a complete metric space. Then we refer to E as a *quasi-Banach* space.

Let $W^{1,\infty}([-r^*, 0])$ be the linear space of absolutely continuous functions $\phi : [-r^*, 0] \to \mathbb{R}$ whose derivatives are essentially bounded.

Note that $W^{1,\infty}$ defines a quasi-Banach space when we set $N(\phi) = \|\phi\|^\infty = |\phi(-r^*)| + \operatorname{ess\,sup}_{\theta\in[-r^*,0]} |\dot{\phi}(\theta)|$ and

$$|\phi| = \|\phi\|^1 = |\phi(-r^*)| + \int_{-r^*}^{0} |\dot{\phi}(s)| ds.$$

For technical reasons, we also require the following definitions.

Let $W^{1,\infty}_{\alpha,0}([-r^*, \alpha]) = \{\phi \in W^{1,\infty}([-r^*, \alpha]) : \phi(s) = 0 \text{ for } s \in [-r^*, 0]\}$, where $W^{1,\infty}_{\alpha,0}$ is equipped with the norms $\|\phi\|^1_\alpha = \int_0^\alpha |\dot{\phi}(s)| ds$ and $\|\phi\|^\infty_\alpha = \operatorname{ess\,sup}_{s\in[0,\alpha]} |\dot{\phi}(s)|$.

Let

$$A(\alpha, \beta) = \{\phi \in W^{1,\infty}_{\alpha,0}([-r^*, \alpha]) : \|\phi\|^\infty_\alpha \le \beta\},$$

$$B(\alpha,\beta)=\{\phi\in W^{1,\infty}_{\alpha,0}([-r^*,\alpha]):\|\phi\|^1_\alpha\le\beta\}\,.$$

When $\alpha\le 1$, we note that $A(\alpha,\beta)\subset B(\alpha,\beta)$ since, for $\phi\in A(\alpha,\beta)$, $\operatorname{ess\,sup}_{f\in[0,\alpha]}|\dot\phi(s)|\le\beta$ implies that $\int_0^\alpha|\dot\phi(s)|ds\le\beta\int_0^\alpha ds=\beta\alpha\le\beta$. In general, we have the inclusion $A(\alpha,\beta)\subset B(\alpha,\alpha\beta)$. We may now consider the initial value problem

$$\begin{gathered}\dot x(t)=f(x(t),x(t-r))\,,\quad r=r(x(t))\,,\quad t>0\\ x(\theta)=\phi(\theta)\,,\quad \theta\in[-r^*,0],\phi(\cdot)\in W^{1,\infty}.\end{gathered}\tag{2.2}$$

For the rest of the paper, we will assume that (2.2) satisfies conditions (A1) and (A2) below.

(A1) $r:W^{1,\infty}([-r^*,0])\to\mathbb{R}$ is smooth, with $\sup_{\phi\in W^{1,\infty}}|D_\phi r|\le c<\infty, D_\phi r$ denoting the Fréchet derivative of r with respect to ϕ. Also, $\alpha\le\inf_{\phi\in W^{1,\infty}}r(\phi)\le r^*<\infty$, for some $\alpha>0$ small.

(A2) $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz function. In particular, for

$$(\xi_1,\eta_1),\xi_2,\eta_2)\in\mathbb{R}\times\mathbb{R},|f(\xi_1,\eta_1)-f(\xi_2,\eta_2)|\le L|\xi_1-\xi_2|+M|\eta_1-\eta_2|,$$

which specifies the Lipschitz constants L and M.

To give precise smoothness results, it will sometimes be necessary to make the more stringent assumption

(A3) $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a C^k function, with $k\ge 2$, $f(0)=0$, and $f'(0)\ne 0$.

We shall proceed by reformulating the initial-value problem (2.2) as a fixed point equation in the phase space $W^{1,\infty}([-r^*,\alpha])$. First notice that $x(t)$ satisfies the system (2.2) over the interval $[-r^*,\alpha]$ if and only if $x(t)=\phi_0(t)+z(t)$, $\phi_0(t)=\phi(t)$ for $t\in[-r^*,0]$, $\phi_0(t)=\phi(0)$ for $t\in[0,\alpha]$, and $z(t)$ satisfies

$$z(t)=\begin{cases}0\,, & t\in[-r^*,0]\\ \int_0^t f(\phi(0)+z(s)\,,\,\phi(s-r(\phi(0)+z(s))))ds & t\in[0,\alpha].\end{cases}\tag{2.3}$$

Here we have made the observation that, since $\inf_\phi\,(r(\phi))>\alpha, \phi_0(s-r(x))=\phi(s-r(x))$ for any $x\in\mathbb{R}$ and $z(s-r(\phi(0)+z(s)))=0$. We can now define the operator $T:A(\alpha,\beta)\times B_{R,N}\times S\to A(\alpha,\beta)$, with the function $r\in S$ if and only if $r:W^{1,\infty}([-r^*,0])\to\mathbb{R}^+$, and r satisfies hypothesis (A1).

Solutions to (1.1) correspond to fixed points of the integral equation

$$T(z,\phi,r)(t)=\begin{cases}0\,, & t\in[-r^*,0]\\ \int_0^t f(\phi(0)+z(s),\phi(s-r(\phi(0)+z(s))) & \\ +z(s-r(\phi(0)+z(s))))ds, & t\in[0,\alpha].\end{cases}\tag{2.4}$$

defined on the interval $t\in[0,\alpha]$. If we can show that $z\in W^{1,\infty}_{\alpha,0}$ is a unique fixed point of the equation $z=T(z,\phi,r)$, it follows from the relation $x(t)=\phi_0(t)+z(t)$ that the map $x_t(\phi,\cdot)$ can be continued from $t=0$ to $t=\alpha$. We prove existence and uniqueness using the contraction mapping principle, dividing our proof into the following lemmas.

Lemma 2.1 *Suppose that (A1) and (A2) are satisfied. Then for any fixed* $R > 0$ *there exist* $\alpha, \beta > 0$ *such that* $T(\overline{B}(\alpha, \alpha\beta) \times B_R \times S) \subset \overline{B}(\alpha, \alpha\beta)$ *and also* $T(A(\alpha, \beta) \times B_R \times S) \subset A(\alpha, \beta)$. *Hence* T *is a self-mapping with respect to the sets* $A(\alpha, \beta)$ *and* $\overline{B}(\alpha, \alpha\beta)$.

Proof. Since the proofs are similar, we show only that $T(\overline{B}(\alpha, \alpha\beta) \times B_R \times S) \subset \overline{B}(\alpha, \alpha\beta)$. From (A2),

$$\begin{aligned}
\|T(z, \phi, r)(t)\|_\alpha^1 = &\int_0^\alpha |f(\phi(0) + z(s)\,, \quad \phi(s - r(z(s) + \phi(0))))|ds \\
\leq &\alpha\{ \sup_{S\in[0,\alpha]} M|\phi(0) + z(s)| + \sup_{S\in[0,\alpha]} N|\phi(s - r(z(s) + \phi(0)))|\} \\
\leq &\alpha\{M(R + \alpha\beta) + NR\};
\end{aligned}$$

thus, for $\alpha \leq \frac{1}{\beta}$ and $\beta \geq R(M + N) + 1$, $\alpha\{M(R + \alpha\beta) + NR\} \leq \alpha\beta$, and the proof is complete. $\square$

Note that $\alpha > 0$ can be made as small as necessary by choosing β large. Thus the assumption that $0 \leq \alpha < \inf_{x\in\mathbb{R}} r(x)$ is not overly restrictive.

Lemma 2.2 T *is a uniform contraction, with respect to the norms* $\|\cdot\|_\alpha^1$ *and* $\|\cdot\|_\alpha^\infty$, *over* $\overline{B}(\alpha, \alpha\beta)$.

Proof. We need to show that, for $\alpha > 0$ sufficiently small and for any pair $z, w \in \overline{B}(\alpha, \alpha\beta)$, there exists an element $0 < c_0 < 1$ with $c_0 = c_0(\alpha)$ such that $\|T(z, \phi, r(z)) - T(w, \phi, r(w))\|_\alpha^1 \leq c_0\|z - w\|_\alpha^1$ for any fixed $\phi \in B_R$. To do this, we rewrite the quantity $\|T(z, \phi, r(z)) - T(w, \phi, r(w))\|_\alpha^1$ explicitly. We then obtain

$$\begin{aligned}
&\|T(z,\phi, r(z)) - T(w, \phi, r(w))\|_\alpha^1 \\
&= \int_0^\alpha |f(\phi(0) + z(s), \phi(s - r(\phi(0) + z(s)))) \\
&\quad - f(\phi(0) + w(s), \phi(s - r(\phi(0) + w(s))))|ds \\
&\leq \int_0^\alpha M|z(s) - w(s)| + N|\phi(s - r(\phi(s - r(\phi(0) + w(s)))|ds\,.
\end{aligned}$$

This last inequality follows from (A2). Now, since $z(0) = 0 = w(0)$, it follows that $\int_0^\alpha |z(s) - w(s)|ds \leq \alpha\int_0^\alpha |\dot{z}(s) - \dot{w}(s)|ds$ so it remains to estimate $\int_0^\alpha |\phi(s - r(\phi(0) + rz(s))) - \phi(s - r(\phi(0) + w(s)))|ds$. However, since $\phi \in B_R$, it follows that

$$\begin{aligned}
&|\phi(s - r(\phi(0) + z(s))) - \phi(s - r(\phi(0) + w(s)))| \\
&\quad \leq R|r(\phi(0) + z(s)) - r(\phi(0) + w(s))| \\
&\quad \leq Rc|z(s) - w(s)|\,,
\end{aligned}$$

where above inequality follows Assumption (A1). Then we obtain

$$\int_0^\alpha |\phi(s - r(\phi(0) + z(s))) - \phi(s - r(\phi(0) + w(s))|ds \\ \leq Rc\int_0^\alpha |z(s) - w(s)|ds \leq Rc\alpha\int_0^\alpha |\dot{z}(s) - \dot{w}(s)|ds.$$

Combining the above inequalities gives

$$\|T(z, \phi, r(z)) - T(w, \phi, r(w))\|_\alpha^1 \\ \leq c\int_0^\alpha |\dot{z}(s) - \dot{w}(s)|ds + \alpha Rc\int_0^\alpha |\dot{z}(s) - \dot{w}(s)|ds = \alpha(1 + Rc)\|z - w\|_\alpha^1.$$

Suppose that we fix any $\alpha < \frac{1}{1+Rc}$; it follows that, with respect to the constant $c_0 = \alpha(1 + Rc) < 1$, T is a contraction map on $\overline{B}(\alpha, \alpha\beta)$. The proof for the $\|\cdot\|_\alpha^\infty$ norm is similar and therefore omitted.

Lemma 2.3 *Consider $T : B(\alpha, \alpha\beta) \times B_R \times S \to B(\alpha, \alpha\beta)$ with $r \in S$ fixed. Then $T = T(r, z, \phi)$ is a Lipschitz continuous function in z and ϕ.*

Proof. Fix r and consider (z, ϕ), $(w, \psi) \in B(\alpha, \alpha\beta) \times B_R$; we need to show that $\|T(z, \phi) = T(w, \psi)\|_\alpha^1 \leq c\{\|z - w\|_\alpha^1 + \|\phi - \psi\|^1\}$ for some constant $0 \leq c = c(\alpha) < \infty$. Because of the triangle inequality, it suffices to show that $\|T(z, \phi) - T(w, \phi)\|_\alpha^1 + \|T(w, \phi) - T(w, \psi)|_\alpha^1 \leq c\{\|z - w\|_\alpha^1 + \|\phi - \psi\|^1\}$. From the proof of Lemma 2.2, we know that $\|T(z\phi) - T(w, \phi)\|_\alpha^1 \leq c\|z - w\|_\alpha^1$; thus, it remains to show that $\|T(w, \phi) - T(w, \psi)\|_\alpha^1 \leq c\{\|z - w\|_\alpha^1 + \|\phi - \psi\|^1\}$ for some $0 \leq c < \infty$. We write

$$\begin{aligned}\|T(w, \phi) - T(w, \psi)\|_\alpha^1 =&\int_0^\alpha \Big|f(\phi(0) + w(s)\,,\, \phi(s - r(\phi(0) + w(s)))) \\ &- f(\psi(0) + w(s)\,,\, \psi(s - r(\psi(0) + w(s))))\Big|ds \\ \leq& M\int_0^\alpha |\phi(0) - \psi(0)|ds + N\int_0^\alpha \Big|\phi(s - r(\phi(0) + w(s))) \\ &- \psi(s - r(\psi(0) + w(s)))\Big|ds.\end{aligned}$$

We know that

$$\begin{aligned}M\int_0^\alpha |\phi(0) - \psi(0)|ds =& M\alpha|\alpha(0) - \psi(0)| \\ =& M\alpha|\phi(-r^*) + \int_{-r^*}^0 \dot{\phi}(\theta)d\theta - \psi(-r^*) - \int_{-r^*}^0 \dot{\psi}(\theta)d\theta| \\ \leq& M\alpha\left\{|\phi(-r^*) - \psi(-r^*)| + \int_{-r^*}^0 |\dot{\phi}(\theta) - \dot{\psi}(\theta)|d\theta\right\} \\ =& M\alpha\|\phi - \psi\|_\alpha^1\end{aligned}$$

Also we know that

$$
\begin{aligned}
&N\int_0^\alpha |\phi(s-r(\phi(0)+w(s)))-\psi(s-r(\psi(0)+w(s)))|ds\\
&\le N\Big\{\int_0^\alpha |\phi(s-r(\phi(0)+w(s)))-\phi(s-r(\psi(0)+w(s)))|ds\\
&\quad+\int_0^\alpha |\phi(s-r(\psi(0)+w(s)))-\psi(s-r(\psi(0)+w(s)))|ds\Big\}\\
&\le NR\alpha c|\psi(0)-\psi(0)|+N\int_{-r^*}^0 |\phi(\theta)-\psi(\theta)|d\theta\\
&\le NR\alpha c|\phi(0)-\psi(0)|+N\int_{-r^*}^0 [|\phi(-r^*)-\psi(-r^*)|+\int_{-r^*}^\theta |\dot\phi(\sigma)-\dot\psi(\sigma)|d\sigma]d\theta\\
&\le NR\alpha c|\phi(0)-\psi(0)|+Nr^*\|\phi-\psi\|^1\\
&\le N(R\alpha^2cM+r^*)\|\phi-\psi\|^1.
\end{aligned}
$$

The above estimates complete the proof. □

Using Lemmas 2.1, 2.2, and 2.3, with $\alpha > 0$ and sufficiently small, the contraction mapping theorem gives the following result.

Theorem 2.4 *Consider the initial-value problem (2.2) subject to assumptions (A1) and (A2), with $\phi(\cdot) \in B_R$ and $R > 0$ fixed. Then there exists a real number $\alpha = \alpha(R) > 0$, independent of ϕ, such that $x(t, \phi)$ exists and is unique on $[0, \alpha]$.*

It is also possible to establish a simple sufficient condition for global existence (i.e., existence on the interval $t \in [0, \infty)$), using the fact that equation (1.1) is autonomous. The theorem that follows implies that the only way in which a solution $x(t, \phi)$ can fail to exist is if it blows up in finite time.

Theorem 2.5 *Consider the initial-value problem (2.2) subject to assumptions (A1) and (A2), with $\phi(\cdot) \in B_R$ and $R > 0$ fixed. Further, suppose that the solution $x(t, \phi)$ is non-continuable on the interval $[0, b)$ for some $0 < b < \infty$. Then, for any fixed $c > 0$, there exists a $t \in [0, b)$ such that $|x(t, \phi)| > c$.*

Proof. We apply a contradiction argument. Suppose that for some fixed $\mu > 0$, $\sup_{t\in[0,b)}\sup_{\theta\in[-r^*,0]}|x(t+\theta)| < \mu$ and that $x(t, \phi)$ is not continuable on $[0, b)$. From assumption (A2), we know that, for $|\xi|, |\eta| < \mu$, $\sup_{\xi,\eta}|f(\xi, \eta)| \le (M + N)\mu < \infty$. Now we have assumed that $\|\phi(\cdot)\|^\infty < R < \infty$; hence, by definition, $\operatorname{ess\,sup}_{\theta\in[-r^*,0]} |\dot\phi(\theta)| \le R$ and $\|x_t(\phi, \cdot)\|^\infty \le \mu + \max(\mu, R)$ and for all $t \in [0, b)$, $x_t(\phi, \cdot) \in B_{\mu+\max(\mu,R)}$. Setting $R_c = \mu+\max(\mu, R)$, we know that, for all $t \in [0, b)$, there exists an $\alpha_0 = \alpha(R_0)$ such that $x(t, \phi)$ is continuable from t to $t + \alpha_0$. Thus we gain a contradiction if we choose $t = b - \frac{\alpha_0}{2}$, since $x(t, \phi)$ can now be continued beyond b.

3 An Unstable Manifold Theorem

In this section, we prove a theorem on local unstable manifolds for equation (1.1). Significantly, it will turn out that the unstable manifold forms a smooth graph, even though the vector field defined by (2.2) is not Fréchet differentiable in the initial conditions. We set up our problem in the following way.

Consider the following two equations, $\dot{x}(t) = f(x(t), x(t-r)), \quad r = r(x(t))$ and $\dot{x}(t) = f(x(t), x(t-r(0)))$ under the assumption that $f(0) = 0$. We do not claim that the second equation is a formal linearization of the first; however, we will show that it gives a good description of the local dynamics near the unstable manifold of $\dot{x}(t) = f(x(t), x(t-r)), \quad r = r(x(t))$.

Suppose that 0 is a hyperbolic equilibrium point of $\dot{x}(t) = f(x(t), x(t-r(0)))$; it follows that the set $\Lambda = \{\lambda \in \mathbb{C} : \lambda = -\alpha + f'(0)e^{-r(0)\lambda}, Re\lambda > 0\}$ is finite and that the space $W^{1,\infty}([-r^*, 0])$ can be decomposed as $W^{1,\infty} = U \oplus S$, as in Hale and Lunel [4]. Here we define $U = \{\phi \in W^{1,=\infty} : x_t(\phi, \cdot)$ exists and remains bounded (in the $\|\cdot\|^1$-norm) for all $t \leq 0\}$ and set $\phi \in U$ equal to ϕ^U. Similarly, we define $S = \{\phi \in W^{1,\infty} : x_t(\phi, \cdot)$ exists and remains bounded for all $t \geq 0\}$, where $x_t(\phi, \cdot)$ is a solution to the initial-value problem (2.2). The dimension of U is equal to the elements (including multiplicities) in Λ, and is finite. Associated with the sets U and S are the projections $\pi_U : W^{1,\infty} \to U$ and $\pi_S : W^{1,\infty} \to S$, with $\pi_U U = U$ and $\pi_S S = (I - \pi_U)S = S$.

We now examine at the dynamics of the solution map $L : U \to U$ defined by $\dot{x}(t) = f(x(t), x(t-r(0)))$, where U is a finite-dimensional subspace of $W^{1,\infty}$. Since the origin is a hyperbolic equilibrium point, there exist constants $M_0, \beta_0 > 0$, independent of $\phi^U \in U$, such that, for $t \leq 0, \sup_{\theta\in[-r^*,0]}|T(t)\phi^U(\theta)| \leq M_0 e^{\beta_0 t}\sup_{\theta\in[-r^*,0]}|\phi^U(\theta)|$. Since all norms are equivalent in finite dimensions, we know that there exist constants $M, \beta > 0$, independent of ϕ^U, such that $\|T(t)\phi^U\|^1 \leq Me^{\beta t}\|\phi^U\|^1$. If we consider the restriction of g to U, the following smoothness result can be proved.

Lemma 3.1 *Suppose that, in (1.1) $\dot{x}(t) = -\alpha x(t) + f(x(t-r)), r = r(x(t)), r : \mathbb{R} \to \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ are C^2 functions. Then there exists a neighbourhood $N_\delta(0) \subset U$ such that the mapping $g : U \to \mathbb{R}$ is Fréchet differentiable at all points $x_t(\cdot) \in N_\delta(0)$.*

Proof. Suppose that we fix a point $x_t(\cdot) \in N_\delta(0)$. Since $f \in C^2$, from the chain rule it is sufficient to show that the function $g_1 : U \to \mathbb{R}$, defined by $g_1(x_t(\cdot)) = x(t - r(x(t)))$, is Fréchet differentiable. We now prove that, for $h_t(\cdot) \in U, D_\phi g_1(x_t(\cdot))h_t = -\frac{dr}{dx}\dot{x}(t - r(x(t)))h(t) + h(t - r(x(t)))$, where $D_\phi g_1$ is a linear operator since $x_t(\cdot)$ is fixed. We can write

$$
\begin{aligned}
&\big|g_1(x_t + h_t) - g_1(x_t) - D_\phi g_1(x_t)h_t\big| \\
&\quad = \big|x(t - r(x(t) + h(t))) + h(t - r(x(t) + h(t - r(x(t) + h(t))) \\
&\qquad - x(t - r(x(t))) + \frac{dr}{dx}\dot{x}(t - r(x(t)))h(t) - h(t - r(x(t)))\big|
\end{aligned}
$$

$$
\begin{aligned}
=&\big|x(t-r(x(t))-\frac{dr}{dx}h(t)+o(h))+h(t-r(x(t))-\frac{dr}{dx}h(t)+o(h))\\
&-x(t-r(x(t)))+\frac{dr}{dx}\dot{x}(t-r(x(t)))h(t)-h(t-r(x(t)))\big|\\
=&\big|\dot{x}(t-r(x(t))\big(-\frac{dr}{dx}h(t)+o(h)\big)+\dot{h}(t-r(x(t)))\big(-\frac{dr}{dx}h(t)+o(h)\big)\\
&+x(t-r(x(t)))+h(t-r(x(t)))-x(t-r(x(t)))\\
&+\frac{dr}{dx}\dot{x}(t-r(x(t)))h(t)-h(t-r(x(t)))\big|\\
=&|-\frac{dr}{dx}\dot{h}(t-r(x(t)))h(t)+o(h)|.
\end{aligned}
$$

The proof will be complete if we can show that $-\frac{dr}{dx}\dot{h}(t-r(x(t)))h(t)$ is $o(h)$. In particular, we need to show that, for any sequence $\{h_n\}$, with each $h_n \in U, \lim_{n\to\infty}\|h_n\|^1 = 0$ implies that $\lim_{n\to\infty}\sup_{\theta\in[-r^*,0]}|\dot{h}_n(\theta)| = 0$. We proceed by contradiction and suppose that there exists an $\varepsilon > 0$ such that, for all $N \in \mathbb{Z}^+$ and some $n \geq N, \sup_{\theta\in[-r^*,0]}|\dot{h}_n(\theta)| > \varepsilon$. From the fundamental theorem of calculus, it follows that $\sup_{\theta\in[-r^*,o]}|h_n(\theta)| \leq \|h_n\|^1$. Also, it is obvious that $\lim_{n\to\infty}\sup_{\theta\in[-r^*,0]}|h_n(\theta-r^*)| = 0$ whenever $\lim_{n\to\infty}\sup_{\theta\in[-r^*,0]}|h_n(\theta)|$, since $h_n(\cdot) \in U$. Next, we substitute $h_n(\cdot - r^*)$ into $\dot{x}(t) = x(t-r(0))$, thus defining $\dot{h}_n(\cdot)$ over the interval $[-r^*,0]$. We now find that, for all $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\sup|\dot{h}_n(\cdot)| < \varepsilon$ whenever $\|h_n\|^1 < \delta$. Since this property holds for all $n \geq N = N(\varepsilon)$, we arrive at a contradiction and the proof is complete. ☐

Remark. We may regard Lemma 3.1 from the following perspective. Although, relative to the background space $W^{1,\infty}([-r^*,0])$, the $\|\cdot\|^\infty$-norm is not equivalent to the $\|\cdot\|^1$-norm, relative to U, the $\|\cdot\|^\infty$- and $\|\cdot\|^1$-norms are equivalent. This property guarantees the Fréchet differentiability of g when restricted to U.

Trajectories on the unstable manifold appear as solutions to an integral equation associated with $\dot{x}(t) = f(x(t), x(t-r(0)))$. In order to specify the integral equation, we require some additional notation. Suppose that Φ is a basis for U and Ψ is a basis for U^T, with $(\Phi,\Psi) = 1$, so that $\pi_U\phi = \Phi(\Psi,\phi)$. We then define $X(\cdot)$ to be the fundamental matrix solution of $\dot{x}(t) = f(x(t), x(t-r(0)))$,

$$K(t,s)(\theta) = \int_0^s X(t+\theta-\tau)d\tau\,, \quad X_0^u = \Phi\Psi(0)\,, \quad K(t,s)^u = \int_0^s T(t-\tau)X_0^u d\tau,$$

and

$$K(t,s)^s = \pi_s K(t,s) = K(t,s) - \Phi(\Psi, K(t,s)).$$

A function $x_t^*(\phi,\cdot) \in W^{1,\infty}([-r^*,0])$ satisfying equation $\dot{x}(t) = f(x(t), x(t-r(0)))$ must also satisfy the variation-of-constants formula (3.1) $x_t(\phi,\cdot) = T(t)\phi + \int_0^t d[K(t,\tau)]g(x_\tau)$ (for a full treatment of (3.1), see [4]). The following lemma, which is proved in [4], allows us to characterize the unstable set of $\dot{x}(t) =$

$f(x(t), x(t-r(0)))$. For convenience, we shall set $W^u(0) = \{\phi \in W^{1,\infty} : x_t(\cdot,\phi)$ exists for all $t \le 0$ and $\lim_{t\to-\infty} x_t(\cdot,\phi) = 0\}$, and $W^s(0) = \{\phi \in W^{1,\infty} : x_t(\cdot,\phi)$ exists for all $t \ge 0$ and $\lim_{t\to+\infty} x_t(\cdot,\phi) = 0\}$.

Lemma 3.2 *Suppose that $x^*(t,\phi)$ is a solution of (1.1) that is defined and bounded for all $t \le 0$. Then $x^*_t(\cdot,\phi) \in W^{1,\infty}([-r^*,0])$ satisfies the integral equation*

$$x_t(\cdot) = T(t)\phi^u + \int_0^t T(t-\tau)X_0^u f(x_\tau)d\tau + \int_{-\infty}^t d[K(t,\tau)^s]f(x_\tau). \tag{3.2}$$

The proof of this theorem can be found in [4]. We can now prove our main theorem. We remark that if $f \in C^k$ in the theorem below, $k \ge 2$, then $W^u_{\rm loc}(0)$ will be a C^{k-1}-manifold.

Theorem 3.3 *Suppose that 0 is a hyperbolic equilibrium point of equation (1.1) and $f \in C^2$. Then there exists a neighborhood $N_\delta(0)$ of 0 in $W^{1,\infty}([-r^*,0])$ and a map $\pi : U \cap N_\delta(0) \longrightarrow W^u(0)$ such that $(\cdot,\pi(\cdot))$ defines a smooth graph.*

Proof. We start by proving the existence of the unstable manifold $W^u(0)$. In Lemma 3.1, we proved that $g : W^{1,\infty}([-r^*,0]) \to \mathbb{R}$ is Lipschitz continuous. Also, after writing

$$\begin{aligned}\dot{x}(t) &= -\alpha x(t) + f(x(t-r))\\ &= -\alpha x(t) + f'(0)x(t-r(0)) + (f(x(t-r)) - f'(0)x(t-r(0)))\\ &= -\alpha x(t) + f'(0)x(t-r(0)) + f_1(x(t-r)),\end{aligned}$$

with $r = r(x(t))$, we know that there exists a monotone increasing, continuous function $\eta(r) : [0,\infty) \to [0,\infty)$ with $\eta(0) = 0$ such that, for any pair $\phi, \psi \in T(r^*)W^{1,\infty}([-r^*,0])$ with $\|\phi\|^1, \|\psi\|^1 \le \sigma, |f_1(\phi) - f_1(\psi)| \le \eta(\sigma)\|\phi-\psi\|^1$. We now apply the contraction mapping principle to equation (3.2) for $\|\phi\|^1$ sufficiently small. Since $\|T(t)\phi^u\|^1 \le Me^{\beta t}\|\phi^u\|^1$ for $t \le 0$, we know that there exists a constant $C_0 > 0$ such that

$$\|T|_u x_t\|^1 \le C_0(e^{\beta t}\|\phi^u\|^1 + \eta(\delta)\int_t^0 e^{\beta(t-\tau)}\|x_\tau\|^1 d\tau + \eta(\delta)\int_{-\infty}^t e^{-\beta(t-\tau)}\|x_t\|^1 d\tau),$$

where

$$T|_u x_t = T(t)\phi^u + \int_0^t T(t-\tau)X_0^u f(x_\tau)d\tau + \int_{-\infty}^t d[K(t,\tau)^s]f(x_t)$$

and $\|\phi\|^1 < \frac{\delta}{2C_0} << 1$. We may view $T|_U$ as a self-mapping over the set

$$S(\phi,\delta) = \big\{x_t : (-\infty,0] \to W^{1,\infty}([-r^*,0]),\ \pi_u x_0 = \phi^u \in U,\ \sup_{t\in[-\infty,0)} \|x_t\|^1 \le \delta\big\}.$$

It follows that $S(\phi,\delta)$ is closed and bounded with respect to the norm $\|x_t\| = \sup_{t\in(-\infty,0]}\|x_t\|^1$. Now, if δ is chosen so that $\eta(\delta) < \frac{\beta}{4C_0}$, then an application of Gronwall's inequality gives the estimate $\|T|_U x_t\|^1 < \delta(\frac{1}{2}+\frac{2C_0}{\beta}\eta(\delta)) < \delta$, and $T|_U = T(\phi^u)$ defines a contraction map. Thus, $T|_U$ has a unique fixed point, which we call $x_t^*(\phi^U,\cdot)$. $x_t^*(\phi^U,\cdot)$ is an absolutely continuous function.

We now show that $x_t^*(\phi^u,\cdot)$ is Lipschitz in ϕ^u. In particular, for any pair $\phi^u,\psi^u \in U$ with $\|\phi^u\|^1, \|\psi^u\|^1$ sufficiently small, we find a constant $C_1 > 0$, independent of ϕ^u,ψ^u, such that $\|x_t^*(\phi^u,\cdot) - x_t^*(\psi^u,\cdot)\|^1 \le C_1\|\phi^u - \psi^u\|^1$ for $t \le 0$. But now

$$\begin{aligned} x_t^*(\phi^u,\cdot) &- x_t^*(\psi^u,\cdot) \\ =&T(t)(\phi^u-\psi^u) + \int_0^t T(t-\tau)X_0^u(f(x_\tau^*(\phi^u,\cdot)) - f(x_\tau^*(\psi^u,\cdot))d\tau \\ &+ \int_{-\infty}^t d[K(t,\tau)^s](f(x_\tau^*(\phi^u,\cdot)) - f(x_\tau^*(\psi^u,\cdot)) \end{aligned}$$

and hence

$$\begin{aligned} \|x_t^*(\phi^u,\cdot) &- x_t^*(\psi^u,\cdot)\|^1 \\ \le& C_0\Big(e^{\beta t}\|\phi^u-\psi^u\|^1 + \eta(\delta)\int_t^0 e^{\beta(t-\tau)}\|x_\tau^*(\phi^u,\cdot) - x_\tau^*(\psi^u,\cdot)\|^1 d\tau \\ &+ \eta(\delta)\int_{-\infty}^t e^{-\beta(t-\tau)}\|x_\tau^*(\phi^u,\cdot) - x_\tau^*(\psi^u,\cdot)\|d\tau\Big). \end{aligned}$$

Our proof relies upon the estimate

$$\|x_t^*(\phi^u,\cdot) - x_t^*(\psi^u,\cdot)\|^1 \le Ce^{\beta_0 t}\|\phi^u - \psi^u\|^1,$$

which is valid for some $\beta_0, C > 0$ and for all $t \le 0$ and $\|\phi^u\|^1, \|\psi^u\|^1 \le \frac{\delta}{2C_0}$. In this case, $x_t^*(\phi^u,\cdot)$ must be Lipschitz in ϕ^u with respect to the constant C. We verify the estimate by first defining the weighted norm $\|x_t(\phi^u,\cdot)\|^{1,\beta_0} = \sup_{-\infty<t\le 0} e^{-\beta_0 t}\|x_t(\phi^u,\cdot)\|^1$ on $S(\phi,\delta)$. If $\beta_0 \in (\beta-1,\beta), \beta_0 > 0$, it immediately follows that $\|x_t(\phi^u,\cdot)\|^{1,\beta_0} \le \delta$. Also,

$$\begin{aligned} &e^{-\beta_0 t}\|x_t(\phi^u,\cdot) - x_t(\psi^u,\cdot)\|^1 \\ &\le C_0\Big[e^{(\beta-\beta_0)t}\|\phi^u-\psi^u\|^1 + \eta(\delta)\int_t^0 e^{(\beta-\beta_0)(t-\tau)}e^{-\beta_0\tau}\|x_\tau^*(\phi^u,\cdot) - x_\tau^*(\psi^u,\cdot)\|^1 d\tau \\ &\quad + \eta(\delta)\int_{-\infty}^t e^{-(\beta+\beta_0)(t-\tau)}\|x_\tau^*(\phi^u,\cdot) - x_\tau^*(\psi^u,\cdot)\|^1 d\tau\Big] \\ &\le C_0\Big[e^{(\beta-\beta_0)t}\|\phi^u-\psi^u\|^1 + \eta(\delta)\|x_t^*(\phi^u,\cdot) - x_t^*(\phi^u,\cdot)\|^{1,\beta_0}\int_t^0 e^{(\beta-\beta_0)(t-\tau)}d\tau \\ &\quad + \eta(\delta)\|x_t^*(\phi^u,\cdot) - x_t^*(\psi^u,\cdot)\|^{1,\beta_0}\int_{-\infty}^t e^{-(\beta+\beta_0)(t-\tau)}d\tau\Big] \\ &\le C_0\Big[e^{(\beta-\beta_0)t}\|\phi^u-\psi^u\|^1 + \eta(\delta)\big(\frac{1}{\beta-\beta_0} - 1\big)\|x_t^*(\phi^u,\cdot) - x_t^*(\psi^u,\cdot)\|^{1,\beta_0} \end{aligned}$$

$$+\eta(\delta)\frac{1}{\beta+\beta_0}\|x_t^*(\phi^u,\cdot)-x_t^*(\psi^u,\cdot)\|^{1,\beta_0}\Big]$$
$$\leq C_0\|\phi^u-\psi^u\|^1+C_0\eta(\delta)\Big[\frac{1}{\beta-\beta_0}-1+\frac{1}{\beta+\beta_0}\Big]\|x_t^*(\phi^u,\cdot)-x_t^*(\psi^u,\cdot)\|^{1,\beta_0}.$$

But now $\|x_t(\phi_t(\phi^u,\cdot)-x_t(\phi^u,\cdot)\|^1\leq Ce^{\beta_0 t}\|\phi^u-\psi^u\|^1$, where

$$C=\frac{C_0}{1-C_0\eta(\delta)\big[\frac{1}{\beta-\beta_0}-1+\frac{1}{\beta+\beta_0}\big]}\,,$$

and thus $W^u(0,N_\delta(0))$ defines a Lipschitz graph.

It remains to prove that $W^u(0,N_\delta(0))$ defines a smooth graph over the domain $U\cap N_\delta(0)$. From Lemma 3.1, we know that $D_\phi g$ is continuous, so that the proof follows from the estimate

$$\begin{aligned}
&\|x_t^*(\psi^u+h,\cdot)-x_t^*(\psi^u,\cdot)-x_t^*(\phi^u+h,\cdot)+x_t^*(\phi^u,\cdot)\|^1\\
&=\|T(t)(\psi^u+h-\psi^u-\phi^u-h+\phi^u)\\
&\quad+\int_t^0 T(t-\tau)X_0^u\,[f((x_\tau^*(\psi^u+h,\cdot))-f(x_\tau^*(\psi^u,\cdot))]\,d\tau\\
&\quad-\int_t^0 T(t-\tau)X_0^u\,[f(x_\tau^*(\phi^u+h,\cdot))-f(x_\tau^*(\psi^u,\cdot)]\,d\tau\\
&\quad+\int_{-\infty}^t d[K(t,\tau)^s][f(x_\tau^*(\psi^u+h,\cdot))-f(x_\tau^*(\psi^u,\cdot))]\\
&\quad-\int_{-\infty}^t d[K(t,\tau)^s][f(x_\tau^*(\phi^u+h,\cdot))-f(x_\tau^*(\phi^u,\cdot))]\|^1\\
&=\|\int_t^0 T(t-\tau)X_0^u D_\phi f(x_\tau^*(\psi^u,\cdot))h_\tau d\tau-\int_t^0 T(t-\tau)X_0^u D_\phi f(x_\tau^*(\phi^u,\cdot))h_\tau d\tau\\
&\leq\|\int_t^0 T(t-\tau)X_0^u(D_\phi f(x_\tau^*(\phi^u,\cdot))-D_\phi f(x_\tau^*(\phi^u,\cdot)))h_\tau d\tau\|^1\\
&\quad+\|\int_{-\infty}^t\frac{d[K(t,\tau)^s]}{d\tau}(D_\phi f(x_\tau^*(\psi^u,\cdot))-D_\phi f(x_\tau^*(\phi^u,\cdot)))h_\tau d\tau\|^1+o(h)\\
&\leq C_1\|\int_t^0 e^{\beta(t-\tau)}(D_\phi f(x_\tau^*(\psi^u,\cdot))-D_\phi f(x_\tau^*(\phi^u,\cdot)))h_\tau d\tau\|^1\\
&\quad+C_2\|\int_{-\infty}^t e^{-\beta(t-\tau)}(D_\phi f(x_\tau^*(\psi^u,\cdot))-D_\phi f(x_\tau^*(\phi^u,\cdot)))h_\tau d\tau\|^1+o(h)\\
&\leq C_1\|\sup_{t\leq 0}\|(D_\phi f(x_t^*(\psi^u,\cdot))-D_\phi f(x_t^*(\phi^u,\cdot))h_t\|^1\int_t^0 e^{\beta(t-\tau)}d\tau\|^1\\
&\quad+C_2\|\sup_{t\leq 0}\|(D_\phi f(x_t^*(\psi^u,\cdot))-D_\phi f(x_t^*(\phi^u,\cdot)))h_t\|^1\int_{-\infty}^t e^{-\beta(t-\tau)}d\tau\|^1+o(h)\\
&=o(h),
\end{aligned}$$

since the Fréchet derivative $D_\phi f$ is continuous. Thus $x_t^*(\phi^u,\cdot)$ varies smoothly in ϕ^u, and $W^u(0, N_\delta(0))$ defines a smooth graph. The proof of Theorem 3.3 is now complete. □

The set $W^u(0, N_\delta(0))$ defines a smooth graph over U. Thus, a typical solution $x_t(\phi, \cdot)$ will have a saddle structure near the origin. More precisely, $x_t(\phi, \cdot)$ will approach 0 along a path nearby $W^s(0, N_\delta(0))$ before asymptotically tending to the smooth set $W^u(0, N_\delta(0))$. In this sense any solution which does not decay to 0 becomes more well-behaved as t increases. Here we have defined $W^s(0) = \{\phi \in W^{1,\infty}([-r^*, 0]) : \lim_{t\to\infty} x_t(\phi, \cdot) = 0\}$, but have made no claims about the structure or smoothness properties of $W^s(0)$. In general, it is to be expected that the dynamics on $W^s(0)$ will be considerably more complicated than the dynamics on $W^u(0)$.

Acknowledgements This paper forms part of the author's Ph.D thesis (applied math, Brown University, 1998). The author would like to thank the reviewer for his helpful comments, his advisor John Mallet-Paret, and Jan Garbarek.

References

[1] O. Arino, K. P. Hadeler and M. L. Hbid, , *Existence of periodic solutions for delay differential equations with state dependent delay.* J. Diff. Eqns. 144 (1998), 263-301.

[2] K. L. Cooke and W. Huang, *On the problem of linearization for state-dependent delay differential equations.* Proc. Amer. Math. Soc. 124 (1996), 1417-1426.

[3] J. K. Hale and L. A. C. Ladeira, *Differentiability with respect to delays.* J. Diff. Eqns. 92 (1991), 14-26.

[4] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, 1993.

[5] Y. Kuang and H. Smith, *Slowly Oscillating Periodic Solutions of Autonomous State-dependent Delay Equations.* Nonlinear Analysis, T.M.A. 19 (1992), 855-872.

[6] M. C. Mackey, *Commodity price fluctuations: price-dependent delays and nonlinearities as explanatory factors.* J. Econ. Th. 48 (1989), 497-509.

[7] J. Mallet-Paret and R. D. Nussbaum, *Boundary layer phenomena for differential equations with state-dependent time lags: I.* Arch. Rat. Mech. Anal. 120 (1992), 99-146.

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