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Decay rates for solutions of a system of wave equations with memory *

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Abstract

The purpose of this article is to study the asymptotic behavior of the solutions to a coupled system of wave equations having integral convolutions as memory terms. We prove that when the kernels of the convolutions decay exponentially, the first and second order energy of the solutions decay exponentially. Also we show that when the kernels decay polynomially, these energies decay polynomially.

1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary Γ. In this domain, we consider the initial boundary value problem

$$u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + \alpha(u-v) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds - \alpha(u-v) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$u = v = 0 \quad \text{on } \Gamma \times (0,\infty), \tag{1.3}$$

$$(u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad (u_t(0,x), v_t(0,x)) = (u_1(x), v_1(x)), \tag{1.4}$$

where u and v denote the transverse displacements of waves. Here, α a non-negative constant and g_i are positive functions satisfy

$$-c_0 g_i(t) \le g_i'(t) \le -c_1 g_i(t), \quad 0 \le g_i''(t) \le c_2 g_i(t) \quad \text{for } i = 1,2, \tag{1.5}$$

and for some positive constant c_j, $j = 0,1,2$. We also assume that

$$\beta_i := 1 - \int_0^\infty g_i(s)ds > 0, \quad \text{for } i = 1,2. \tag{1.6}$$

Dissipative coupled systems of the wave equations have been studied by several authors [1, 2, 3, 5, 6] whose results can be summarized as follows: Komornik

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and Rao [6] studied a linear system of two compactly coupled wave equations with boundary frictional damping in both equations. They show the existence, regularity and stability of the corresponding solutions. The stability results obtained in [6] were extended by Aassila [1] for a coupled system with weak frictional damping at the infinity. In another work, Aassila [2] removes the dissipation of the one equation and shows the strong asymptotic stability or the non uniform stability for some particular cases depending on the coupling constant. A similar coupled system with boundary frictional damping on only one of the equations was studied by Alabau [3]. He shows the polynomial decay of the corresponding strong solutions when the speed of wave propagation of the both equations are the same. Some others coupled systems with internal damping or with another coupling type can be found in [4, 5, 7, 9].

Our main result shows that the solution of system (1.1)-(1.4) decays uniformly in time, with rates depending on the rate of decay of the kernel of the convolutions. More precisely, the solution decays exponentially to zero provided g_i decays exponentially to zero. When g_i decays polynomially, we show that the corresponding solution also decays polynomially to zero with the same rate of decay. The method used here is based on the construction of suitable Lyapunov functionals, $\mathcal{L}$, satisfying

$$\frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t)^{1+\frac{1}{p}} + \frac{c_2}{(1+t)^{p+1}}.$$

for some positive constants c_1, c_2, γ and $p > 1$. The notation we use in this paper is standard and can be found in Lion's book [8]. In the sequel, c (some times $c_1, c_2, \dots$) denote various positive constants independent on t and on the initial data. The organization of this paper is as follows. In section 2 we establish a existence and regularity result. In section 3 we prove the uniform rate of exponential decay. Finally in section 4 we prove the uniform rate of polynomial decay.

2 Existence and Regularity

In this section we prove the existence and regularity of strong solutions of the coupled system of wave equations with memory. To simplify our analysis, we define the binary operator

$$g \,\Box\, \nabla u(t) = \int_0^t g(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 dx ds.$$

With this notation we have the following statement.

Lemma 2.1 *For $v \in C^1(0, T : H^1(\Omega))$,*

$$\begin{aligned}\int_\Omega \int_0^t g(t-s)\nabla v ds \cdot \nabla v_t dx &= -\frac{1}{2}g(t)\int_\Omega |\nabla v|^2 dx + \frac{1}{2} g' \,\Box\, \nabla v \\ &\quad -\frac{1}{2}\frac{d}{dt}\Big[g \,\Box\, \nabla v - \big(\int_0^t g(s)ds\big)\int_\Omega |\nabla v|^2 dx\Big].\end{aligned}$$

The proof of this lemma follows by differentiating the term $g \,\square\, \nabla v$. The first order energy of system (1.1)-(1.4) is

$$\begin{aligned}
E(t) := E(t;u,v) \quad &= \quad \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\big(1 - \int_0^t g_1(s)ds\big)\int_\Omega |\nabla u|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |v_t|^2 dx + \frac{1}{2}\big(1 - \int_0^t g_2(s)ds\big)\int_\Omega |\nabla v|^2 dx \\
&\quad + \frac{1}{2} g_1 \,\square\, \nabla u + \frac{1}{2} g_2 \,\square\, \nabla v + \frac{\alpha}{2}\int_\Omega |u - v|^2 dx.
\end{aligned}$$

The well-posedness of system (1.1)-(1.4) is given by the following theorem.

Theorem 2.2 *Assume* $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$ *and* $(u_1, v_1) \in (H_0^1(\Omega))^2$. *Then there exists only one strong solution* (u, v) *to (1.1)-(1.4) satisfying*

$$u, v \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}(0,T; H_0^1(\Omega)) \cap W^{2,\infty}(0,T; L^2(\Omega)).$$

Proof. Our starting point is to construct the Galerkin approximations u^m and v^m of the solution. Let

$$u^m(\cdot,t) = \sum_{j=1}^m h_{j,m}(t) w_j(\cdot), \quad v^m(\cdot,t) = \sum_{j=1}^m f_{j,m}(t) w_j(\cdot)$$

where the functions $f_{j,m}$ and $h_{j,m}$ are the solutions of the approximated systems

$$\begin{aligned}
&\int_\Omega u_{tt}^m w_j dx + \int_\Omega \nabla u^m \cdot \nabla w_j dx \\
&- \int_\Omega \int_0^t g_1(t-s)\nabla u^m(s) ds \cdot \nabla w_j dx + \alpha \int_\Omega (u^m - v^m) w_j dx \quad = \quad 0 \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega v_{tt}^m w_j dx + \int_\Omega \nabla v^m \cdot \nabla w_j dx \\
&- \int_\Omega \int_0^t g_2(t-s)\nabla v^m(s) ds \cdot \nabla w_j dx - \alpha \int_\Omega (u^m - v^m) w_j dx \quad = \quad 0 \qquad (2.2)
\end{aligned}$$

with the initial conditions $u^m(\cdot,0) = u_{0,m}$, $u_t^m(\cdot,0) = u_{1,m}$, $v^m(\cdot,0) = v_{0,m}$, and $v_t^m(\cdot,0) = v_{1,m}$, where

$$\begin{aligned}
u_{0,m} &= \sum_{j=1}^m \big\{\int_\Omega u_0 w_j dx\big\} w_j, \quad u_{1,m} = \sum_{j=1}^m \big\{\int_\Omega u_1 w_j dx\big\} w_j, \\
v_{0,m} &= \sum_{j=1}^m \big\{\int_\Omega v_0 w_j dx\big\} w_j, \quad v_{1,m} = \sum_{j=1}^m \big\{\int_\Omega v_1 w_j dx\big\} w_j.
\end{aligned}$$

The existence of the approximate solutions u^m and v^m are guaranteed by standard results on ordinary differential equations. Our next step is to show that the approximate solution remains bounded for any $m > 0$. To this end, let us multiply equation (2.1) by $h'_{j,m}$ and (2.2) by $f'_{j,m}$. Summing up the product result in j and using Lemma 2.1 we arrive at

$$\begin{aligned}\frac{d}{dt}E(t;u^m,v^m) &= -\frac{1}{2}g_1(t)\int_\Omega |\nabla u^m|^2dx - \frac{1}{2}g_2(t)\int_\Omega |\nabla v^m|^2dx \\ &\quad +\frac{1}{2}g'_1\,\square\,\nabla u^m + \frac{1}{2}g'_2\,\square\,\nabla v^m.\end{aligned}$$

Integrating from 0 to t the above relation follows that

$$E(t;u^m,v^m) \le E(0;u^m,v^m).$$

¿From our choice of $u_{0,m}$, $u_{1,m}$, $v_{0,m}$ and $v_{1,m}$ it follows that

$$E(t;u^m,v^m) \le c, \quad \forall t\in[0,T], \quad \forall m\in\mathbb{N}. \tag{2.3}$$

Next, we shall find an estimate for the second order energy. First, let us estimate the initial data $u^m_{tt}(0)$ and $v^m_{tt}(0)$ in the L^2-norm. Letting $t\to 0^+$ in the equations (2.1) and (2.2) and multiplying the result by $h''_{j,m}(0)$ and $f''_{j,m}(0)$, respectively, we obtain

$$\|u^m_{tt}(0)\|_2 + \|v^m_{tt}(0)\|_2 \le c, \quad \forall m\in\mathbb{N}. \tag{2.4}$$

Differentiating the equations (2.1) and (2.2) with respect to time, we obtain

$$\begin{aligned}&\int_\Omega u^m_{ttt}w_jdx + \int_\Omega \nabla u^m_t\cdot\nabla w_jdx + g_1(0)\int_\Omega \Delta u^m_0 w_jdx \\ &-\int_\Omega\int_0^t g'_1(t-s)\nabla u^m(s)ds\cdot\nabla w_jdx + \alpha\int_\Omega (u^m_t - v^m_t)w_jdx = 0 \quad (2.5)\\ &\int_\Omega v^m_{ttt}w_jdx + \int_\Omega \nabla v^m_t\cdot\nabla w_jdx + g_2(0)\int_\Omega \Delta v^m_0 w_jdx \\ &-\int_\Omega\int_0^t g'_2(t-s)\nabla v^m(s)ds\cdot\nabla w_jdx - \alpha\int_\Omega (u^m_t - v^m_t)w_jdx = 0. \quad (2.6)\end{aligned}$$

Multiplying equation (2.5) by $h''_{j,m}$ and (2.6) by $f''_{j,m}$ and using similar arguments as above,

$$E(t;u^m_t,v^m_t) \le c, \quad \forall t\in[0,T], \quad \forall m\in\mathbb{N}.$$

The rest of the proof is a matter of routine. ◊

3 Exponential Decay

In this section we study the asymptotic behavior of the solution of (1.1)-(1.4). The point of departure of this study is to establish the energy identities given in the next lemma.

Lemma 3.1 *Assume the initial data* $\{(u_0, v_0), (u_1, v_1)\} \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$*. Then the solution of (1.1)-(1.4) satisfies*

$$\begin{aligned} \frac{d}{dt}E(t;u,v) &= -\frac{1}{2}g_1(t)\int_\Omega |\nabla u|^2 dx - \frac{1}{2}g_2(t)\int_\Omega |\nabla v|^2 dx \\ &\quad + \frac{1}{2}g_1' \square \nabla u + \frac{1}{2}g_2' \square \nabla v, \\ \frac{d}{dt}E(t;u_t,v_t) &= -\frac{1}{2}g_1(t)\int_\Omega |\nabla u_t|^2 dx - \frac{1}{2}g_2(t)\int_\Omega |\nabla v_t|^2 dx + \frac{1}{2}g_1' \square \nabla u_t \\ &\quad + \frac{1}{2}g_2' \square \nabla v_t - g_1(t)\int_\Omega \Delta u_0 u_{tt} dx - g_2(t)\int_\Omega \Delta v_0 v_{tt} dx. \end{aligned}$$

Proof. Multiplying the equation (1.1) by u_t and applying Green's formula, we get

$$\begin{aligned} &\frac{1}{2}\Big\{\int_\Omega |u_t|^2 dx + \int_\Omega |\nabla u|^2 dx\Big\} \\ &- \int_\Omega \int_0^t g_1(t-s)\nabla u(s)ds \cdot \nabla u_t dx + \alpha \int_\Omega (u-v)u_t dx = 0. \end{aligned} \tag{3.1}$$

Using Lemma 2.1 we obtain

$$\begin{aligned} &\int_\Omega \int_0^t g_1(t-s)\nabla u(s)ds \cdot \nabla u_t dx \\ &= -\frac{1}{2}g_1(t)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}g_1' \square \nabla u - \frac{1}{2}\frac{d}{dt}\Big[g_1 \square \nabla u - \big(\int_0^t g_1(s)ds\big)\int_\Omega |\nabla u|^2 dx\Big]. \end{aligned}$$

Substituting the above identity into (3.1) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_t|^2 dx + \big(1 - \int_0^t g_1(s)ds\big)\int_\Omega |\nabla u|^2 dx + g_1 \square \nabla u\Big\} + \alpha \int_\Omega (u-v)u_t dx \\ &= -\frac{1}{2}g_1(t)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}g_1' \square \nabla u. \end{aligned} \tag{3.2}$$

Similarly we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |v_t|^2 dx + \big(1 - \int_0^t g_2(s)ds\big)\int_\Omega |\nabla v|^2 dx + g_2 \square \nabla v\Big\} - \alpha \int_\Omega (u-v)v_t dx \\ &= -\frac{1}{2}g_2(t)\int_\Omega |\nabla v|^2 dx + \frac{1}{2}g_2' \square \nabla v. \end{aligned} \tag{3.3}$$

Summing (3.2) and (3.3) it follows the first identity of Lemma. Differentiating the equation (1.1) with respect to the time we get

$$u_{ttt} - \Delta u_t + g_1(0)\Delta u + \int_0^t g_1'(t-s)\Delta u(s)ds + \alpha(u_t - v_t) = 0. \tag{3.4}$$

Performing an integration by parts in the convolution term, we find that

$$u_{ttt} - \Delta u_t + \int_0^t g_1(t-s)\Delta u_t(s)ds + \alpha(u_t - v_t) = -g_1(t)\Delta u_0.$$

Taking $\varphi = u_t$ we may use the same reasoning as above we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{ \int_\Omega |u_{tt}|^2 dx + \big(1 - \int_0^t g_1(s)ds\big) \int_\Omega |\nabla u_t|^2 dx + g_1 \,\square\, \nabla u_t \Big\} \\
&+\alpha \int_\Omega (u_t - v_t) u_{tt} dx \\
&\quad = \quad -\frac{1}{2} g_1(t) \int_\Omega |\nabla u_t|^2 dx + \frac{1}{2} g_1' \,\square\, \nabla u_t - g_1(t) \int_\Omega \Delta u_0 u_{tt} dx.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{ \int_\Omega |v_{tt}|^2 dx + \big(1 - \int_0^t g_2(s)ds\big) \int_\Omega |\nabla v_t|^2 dx + g_2 \,\square\, \nabla v_t \Big\} \\
&-\alpha \int_\Omega (u_t - v_t) v_{tt} dx \\
&\quad = \quad -\frac{1}{2} g_2(t) \int_\Omega |\nabla v_t|^2 dx + \frac{1}{2} g_2' \,\square\, \nabla v_t - g_2(t) \int_\Omega \Delta v_0 v_{tt} dx.
\end{aligned}$$

Summing the two above identities it follows the conclusion of Lemma. ♢

To prove the exponential decay of the solutions, we define the following functionals:

$$\begin{aligned}
K(t; u, v) \quad = \quad & \frac{1}{2}\Big\{ \int_\Omega |u_{tt}|^2 dx + \int_\Omega |v_{tt}|^2 dx + \int_\Omega |\nabla u_t|^2 dx \\
& + \int_\Omega |\nabla v_t|^2 dx + 2 \int_\Omega \int_0^t f_1(t-s) \nabla u(s) ds \cdot \nabla u_t dx \\
& +2 \int_\Omega \int_0^t f_2(t-s) \nabla v(s) ds \cdot \nabla v_t dx \Big\},
\end{aligned}$$

$$I_1(t; u) = \int_\Omega u_{tt} u_t dx + \frac{1}{2} g_1' \,\square\, \nabla u - g_1(t) \int_\Omega |\nabla u|^2 dx,$$

$$I_2(t; v) = \int_\Omega v_{tt} v_t dx + \frac{1}{2} g_2' \,\square\, \nabla v - g_2(t) \int_\Omega |\nabla v|^2 dx.$$

where $f_i(t) = g_i(0) g_i(t) + g_i'(t)$, $i = 1, 2$.

Lemma 3.2 *Under the hypothesis of Lemma 3.1,*

$$
\begin{aligned}
&\frac{d}{dt}\big\{K(t;u,v)+\frac{2}{3}g_1(0)I_1(t;u)+\frac{2}{3}g_2(0)I_2(t;v)\big\}\\
&\le -\frac{g_1(0)}{3}\big\{\int_\Omega |u_{tt}|^2dx+\int_\Omega |\nabla u_t|^2dx\big\}-\frac{g_2(0)}{3}\big\{\int_\Omega |v_{tt}|^2dx+\int_\Omega |\nabla v_t|^2dx\big\}\\
&\quad+\big\{\frac{3(c_0g_1(0)+c_2)^2}{2}+\frac{c_0g_1(0)}{3}\big\}g_1(t)\int_\Omega |\nabla u|^2dx\\
&\quad+\big\{\frac{3(c_0g_1(0)+c_2)^2}{2}+\frac{c_0g_2(0)}{3}\big\}g_2(t)\int_\Omega |\nabla v|^2dx\\
&\quad+\big\{\frac{3(c_0g_1(0)+c_2)^2}{2g_1(0)}+\frac{c_2g_1(0)}{3}\big\}g_1\,\Box\,\nabla u\\
&\quad+\big\{\frac{3(c_0g_2(0)+c_2)^2}{2g_2(0)}+\frac{c_2g_2(0)}{3}\big\}g_2\,\Box\,\nabla v-\frac{2}{3}g_1(0)\alpha\int_\Omega (u_t-v_t)u_tdx\\
&\quad+\frac{2}{3}g_2(0)\alpha\int_\Omega (u_t-v_t)v_tdx.
\end{aligned}
$$

Proof. Substitution of the term Δu given by equation (1.1) into (3.4) yields

$$
\begin{aligned}
u_{ttt}-\Delta u_t+g_1(0)u_{tt}-g_1(0)\int_0^t g_1(t-s)\Delta u(s)ds&\\
-g_1(0)\alpha(u-v)+\int_0^t g_1'(t-s)\Delta u(s)ds+\alpha(u_t-v_t)&=0 \qquad (3.5)
\end{aligned}
$$

Multiplying the above equation by u_{tt} and integrating over Ω, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\big\{\int_\Omega |u_{tt}|^2dx+\frac{1}{2}\int_\Omega |\nabla u_t|^2dx\big\}+g_1(0)\int_\Omega |u_{tt}|^2dx&\\
+\int_\Omega\int_0^t f_1(t-s)\nabla u(s)ds\cdot\nabla u_{tt}dx&\\
-g_1(0)\alpha\int_\Omega (u-v)u_{tt}dx+\alpha\int_\Omega (u_t-v_t)u_{tt}dx&=0
\end{aligned}
$$

and since

$$
\begin{aligned}
&\int_\Omega\int_0^t f_1(t-s)\nabla u(s)ds\cdot\nabla u_{tt}dx\\
&=\frac{d}{dt}\big\{\int_\Omega\int_0^t f_1(t-s)\nabla u(s)ds\cdot\nabla u_tdx\big\}\\
&\quad-\int_\Omega\int_0^t f_1'(t-s)\nabla u(s)ds\cdot\nabla u_tdx-f_1(0)\int_\Omega \nabla u\cdot\nabla u_tdx
\end{aligned}
$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_{tt}|^2dx+\int_\Omega |\nabla u_t|^2dx+2\int_\Omega\int_0^t f_1(t-s)\nabla u(s)ds\cdot\nabla u_tdx\Big\}\\
&=-\,g_1(0)\int_\Omega |u_{tt}|^2dx-\int_\Omega\int_0^t f_1'(t-s)[\nabla u(s)-\nabla u(t)]ds\cdot\nabla u_tdx\\
&\quad-f_1(t)\int_\Omega \nabla u\cdot\nabla u_tdx+g_1(0)\alpha\int_\Omega (u-v)u_{tt}dx-\alpha\int_\Omega (u_t-v_t)u_{tt}dx.
\end{aligned}$$

Making use of the inequality

$$\begin{aligned}
&\Big|\int_\Omega\int_0^t g_i(t-s)[\nabla\phi(s)-\nabla\phi(t)]ds\cdot\nabla\varphi dx\Big|\\
&\leq\ \Big(\int_\Omega |\nabla\varphi|^2\Big)^{1/2}\Big(\int_0^t g_i(s)ds\Big)^{1/2}(g_i\,\Box\,\nabla\phi)^{1/2}
\end{aligned}$$

we conclude

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_{tt}|^2dx+\int_\Omega |\nabla u_t|^2dx+2\int_\Omega\int_0^t f_1(t-s)\nabla u(s)ds\cdot\nabla u_tdx\Big\}\\
&\leq-\,g_1(0)\int_\Omega |u_{tt}|^2dx+\frac{g_1(0)}{3}\int_\Omega |\nabla u_t|^2dx+\frac{3}{2}g_1(t)(g_1(0)+c_0)^2\int_\Omega |\nabla u|^2dx\\
&\quad+\frac{3}{2}\frac{(c_0g_1(0)+c_2)^2}{g_1(0)}g_1\,\Box\,\nabla u+g_1(0)\alpha\int_\Omega (u-v)u_{tt}dx-\alpha\int_\Omega (u_t-v_t)u_{tt}dx.
\end{aligned}\tag{3.6}$$

Similarly we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |v_{tt}|^2dx+\int_\Omega |\nabla v_t|^2dx+2\int_\Omega\int_0^t f_2(t-s)\nabla v(s)ds\cdot\nabla v_tdx\Big\}\\
&\leq-\,g_2(0)\int_\Omega |v_{tt}|^2dx+\frac{g_2(0)}{3}\int_\Omega |\nabla v_t|^2dx+\frac{3}{2}g_2(t)(g_2(0)+c_0)^2\int_\Omega |\nabla v|^2dx\\
&\quad+\frac{3}{2}\frac{(c_0g_2(0)+c_2)^2}{g_2(0)}g_2\,\Box\,\nabla v-g_2(0)\alpha\int_\Omega (u-v)v_{tt}dx+\alpha\int_\Omega (u_t-v_t)v_{tt}dx.
\end{aligned}\tag{3.7}$$

Summing the inequalities (3.6) and (3.7) we obtain

$$\begin{aligned}\frac{d}{dt}K(t;u,v) \leq\ & -g_1(0)\int_\Omega |u_{tt}|^2dx - g_2(0)\int_\Omega |v_{tt}|^2dx + \frac{g_1(0)}{3}\int_\Omega |\nabla u_t|^2dx \\ & +\frac{g_2(0)}{3}\int_\Omega |\nabla v_t|^2dx + \frac{3}{2}g_1(t)(g_1(0)+c_0)^2\int_\Omega |\nabla u|^2dx \\ & +\frac{3}{2}g_2(t)(g_2(0)+c_0)^2\int_\Omega |\nabla v|^2dx + \frac{3}{2}\frac{(c_0g_1(0)+c_2)^2}{g_1(0)}g_1\square\nabla u \\ & +\frac{3}{2}\frac{(c_0g_2(0)+c_2)^2}{g_2(0)}g_2\square\nabla v + g_1(0)\alpha\int_\Omega (u-v)u_{tt}dx \\ & -\alpha\int_\Omega (u_t-v_t)u_{tt}dx - g_2(0)\alpha\int_\Omega (u-v)v_{tt}dx \\ & +\alpha\int_\Omega (u_t-v_t)v_{tt}dx. \end{aligned} \tag{3.8}$$

On the other hand, multiplying equation (3.4) by u_t, integrating over Ω and using Lemma 2.1 we get

$$\begin{aligned}\frac{d}{dt}I_1(t;u) =\ & -\int_\Omega |\nabla u_t|^2dx - \frac{1}{2}g_1'(t)\int_\Omega |\nabla u|^2dx \\ & +\frac{1}{2}g_1''\square\nabla u - \alpha\int_\Omega (u_t-v_t)u_tdx.\end{aligned}$$

¿From hypothesis (1.5) we obtain

$$\begin{aligned}\frac{d}{dt}I_1(t;u) \leq\ & -\int_\Omega |\nabla u_t|^2dx + \frac{c_0}{2}g_1(t)\int_\Omega |\nabla u|^2dx \\ & +\frac{c_2}{2}g_1\square\nabla u - \alpha\int_\Omega (u_t-v_t)u_tdx.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\frac{d}{dt}I_2(t;v) \leq\ & -\int_\Omega |\nabla v_t|^2dx + \frac{c_0}{2}g_2(t)\int_\Omega |\nabla v|^2dx \\ & +\frac{c_2}{2}g_2\square\nabla v + \alpha\int_\Omega (u_t-v_t)v_tdx.\end{aligned}$$

Finally, going back to (3.8) and from the last two inequalities our conclusion follows. ◊

Let us introduce the functional

$$J(t;\varphi) = \int_\Omega \varphi_t\varphi dx.$$

Lemma 3.3 *Under hypothesis of the Lemma 3.1 we have*

$$\frac{d}{dt}J(t;u) \leq \alpha_0\int_\Omega |\nabla u_t|^2dx - \frac{\beta_1}{2}\int_\Omega |\nabla u|^2dx + \frac{1}{2\beta_1}\left(\int_0^t g_1(s)ds\right)g_1\square\nabla u,$$

$$\frac{d}{dt}J(t;v) \leq \alpha_0 \int_\Omega |\nabla v_t|^2 dx - \frac{\beta_2}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2\beta_2}\left(\int_0^t g_2(s)ds\right) g_2 \,\square\, \nabla v.$$

The proof of this Lemma is similar the proof of the Lemma 3.2, for this reason we omit it here.

Let us introduce the functionals

$$\begin{aligned}\mathcal{L}(t) &= N_1 E(t;u,v) + N_2 E(t;u_t,v_t) + K(t;u,v) + \frac{2g_1(0)}{3} I_1(t;u) \\ &\quad + \frac{2g_2(0)}{3} I_2(t;v) + \frac{g_1(0)}{12\alpha_0} J(t;u) + \frac{g_2(0)}{12\alpha_0} J(t;v),\end{aligned}$$

$$\begin{aligned}\mathcal{N}(t) &= \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx + \int_\Omega |\nabla u_t|^2 dx + \int_\Omega |\nabla v_t|^2 dx + \int_\Omega |u_{tt}|^2 dx \\ &\quad + \int_\Omega |v_{tt}|^2 dx + g_1 \,\square\, \nabla u + g_2 \,\square\, \nabla v + g_1 \,\square\, \nabla u_t + g_2 \,\square\, \nabla v_t.\end{aligned}$$

It is not difficult to see that there exist positive constants q_0 and q_1 for which

$$q_0 \mathcal{N}(t) \leq \mathcal{L}(t) \leq q_1 \mathcal{N}(t). \tag{3.9}$$

We will show later that the functional $\mathcal{L}$ satisfies the inequality of the following Lemma.

Lemma 3.4 *Let f be a real positive function of class C^1. If there exists positive constants γ_0, γ_1 and c_0 such that*

$$f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$f(t) \leq (f(0) + c)e^{-\gamma t}.$$

Proof. Suppose that $\gamma_0 < \gamma_1$ and define

$$F(t) := f(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$F'(t) = f'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 F(t).$$

Integrating from 0 to t we arrive to

$$F(t) \leq F(0)e^{-\gamma_0 t} \quad \Rightarrow \quad f(t) \leq \big(f(0) + \frac{c_0}{\gamma_1 - \gamma_0}\big)e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \geq \gamma_1$. In this conditions we get

$$f'(t) \leq -\gamma_1 f(t) + c_0 e^{-\gamma_1 t} \quad \Rightarrow \quad [e^{\gamma_1 t} f(t)]' \leq c_0.$$

Integrating from 0 to t we obtain

$$f(t) \leq (f(0) + c_0 t)e^{-\gamma_1 t}.$$

Since $t \leq (\gamma_1 - \epsilon)e^{(\gamma_1-\epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that

$$f(t) \leq [f(0) + c_0(\gamma_1 - \epsilon)]e^{-\epsilon t}.$$

This completes the proof. ♢

Now we shall show the main result of this section.

Theorem 3.5 *Let us suppose that the initial data $(u_0, v_0) \in [H_0^1(\Omega)]^2$ and $(u_1, v_1) \in [L^2(\Omega)]^2$ and that the kernel g_i satisfies the conditions (1.5) and (1.6). Then there exist positive constants k_1 and k_2 such that*

$$E(t; u, v) + E(t; u_t, v_t) \leq k_1(E(0; u, v) + E(0; u_t, v_t))e^{-k_2 t},$$

for all $t \geq 0$.

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in \left[H^2(\Omega) \cap H_0^1(\Omega)\right]^2$ and $(u_1, v_1) \in [H_0^1(\Omega)]^2$. Our conclusion follows by standard density arguments. From the Lemmas 3.1, 3.2 and 3.3 we get

$$\frac{d}{dt}\mathcal{L}(t) \leq -c_1\mathcal{N}(t) + c_2 R^2(t)$$

where $R(t) = g_1(t) + g_2(t)$. Using the exponential decay of g_1 and g_2 and Lemma 3.4 we conclude

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + c\}e^{-k_2 t}$$

for all $t \geq 0$. The conclusion of Theorem follows from (3.9). ♢

As a consequence of Theorem 3.5 we have that the first order energy also decays exponentially. We summarize this result in the following Corollary.

Corollary 3.6 *Under the hypotheses of Theorem 3.5, we have that there exist positive constants c and k_1 such that*

$$E(t; u, v) \leq cE(0; u, v)e^{-k_2 t}, \quad \forall t \geq 0.$$

4 Polynomial rate of decay

In this section we assume that the kernel g_i decays polynomially to zero as time goes to infinity. That is, instead of hypothesis (1.5) we consider

$$-c_0 g_i^{1+\frac{1}{p}}(t) \leq g_i'(t) \leq -c_1 g_i^{1+\frac{1}{p}}(t), \quad 0 \leq g_i''(t) \leq c_2 g_i^{1+\frac{1}{p}}(t), \tag{4.1}$$

$$\alpha_i := \int_0^\infty g_i^{1-\frac{1}{p}}(s)ds < \infty \tag{4.2}$$

for some $p > 1$ and for $i = 1, 2$. The following lemmas will play an important role in the sequel.

Lemma 4.1 *Suppose that g and h are continuous functions satisfying $g \in L^{1+\frac{1}{q}}(0,\infty)\cap L^1(0,\infty)$ and $g^r \in L^1(0,\infty)$ for some $0 \le r < 1$. Then*

$$\begin{aligned}&\int_0^t |g(t-s)h(s)|ds\\ &\le \Big\{\int_0^t |g(t-s)|^{1+\frac{1-r}{q}}|h(s)|ds\Big\}^{\frac{q}{q+1}}\Big\{\int_0^t |g(t-s)|^r|h(s)|ds\Big\}^{\frac{1}{q+1}}.\end{aligned}$$

Proof. Without loss of generality we can suppose that $g, h \ge 0$. Note that for any fixed t we have

$$\int_0^t g(t-s)h(s)ds = \lim_{\|\Delta s_i\|\to 0}\sum_{i=1}^m g(t-s_i)h(s_i)\Delta s_i.$$

Letting $I_m^r := \sum_{j=1}^m g^r(t-s_j)h(s_j)\Delta s_j$, we may write

$$\sum_{i=1}^m g(t-s_i)h(s_i)\Delta s_i = \sum_{i=1}^m \varphi_i\theta_i,$$

where

$$\varphi_i = (g^{1-r}(t-s_i)I_m^r),\quad \theta_i = \big(\frac{g^r(t-s_i)h(s_i)\Delta s_i}{I_m^r}\big).$$

Since the function $F(z) := |z|^{1+\frac{1}{q}}$ is convex, it follows that

$$F\big(\sum_{i=1}^m g(t-s_i)h(s_i)\Delta s_i\big) = F\big(\sum_{i=1}^m \varphi_i\theta_i\big) \le \sum_{i=1}^m \theta_i F(\varphi_i),$$

so, we have

$$\Big\{\sum_{i=1}^m g(t-s_i)h(s_i)\Delta s_i\Big\}^{1+\frac{1}{q}} \le |I_m^r|^{\frac{1}{q}}\sum_{i=1}^m g^{1+\frac{1-r}{q}}(t-s_i)h(s_i)\Delta s_i. \tag{4.3}$$

In view of

$$\lim_{\|\Delta s_i\|\to 0} I_m^r = \int_0^t g^r(t-s)h(s)ds,$$

letting $\|\Delta s_i\| \to 0$ in (4.3), we get

$$\Big\{\int_0^t g(t-s)h(s)ds\Big\}^{1+\frac{1}{q}} \le \Big\{\int_0^t g^r(t-s)h(s)ds\Big\}^{1+q}\Big\{\int_0^t g^{1+\frac{1-r}{q}}(t-s)h(s)ds\Big\},$$

from which our result follows. ♢

Lemma 4.2 *Let $w \in C(0,T;H_0^1(\Omega))$ and g be a continuous function satisfying hypotheses (4.1)-(4.2). Then for $0 < r < 1$ we have*

$$g\,\Box\,\nabla w \le 2\Big\{\int_0^t g^r ds\|w\|_{C(0,T;H_0^1)}\Big\}^{\frac{1}{1+(1-r)p}}\{g^{1+\frac{1}{p}}\,\Box\,\nabla w\}^{\frac{(1-r)p}{1+(1-r)p}},$$

while for $r = 0$ we get

$$g \,\square\, \nabla w \le 2\Big\{ \int_0^t \|w(s)\|_{H_0^1}^2 ds + t\|w(t)\|_{H_0^1}^2 \Big\}^{\frac{1}{p+1}} \{g^{1+\frac{1}{p}} \,\square\, \nabla w\}^{\frac{p}{1+p}}.$$

Proof. ¿From the hypotheses on w and Lemma 4.1 we get

$$\begin{aligned} g \,\square\, \nabla w &= \int_0^t g(t-s)h(s)ds \\ &\le \Big\{ \int_0^t g^r(t-s)h(s)ds \Big\}^{\frac{1}{1+p(1-r)}} \Big\{ \int_0^t g^{1+\frac{1}{p}}(t-s)h(s)ds \Big\}^{\frac{(1-r)p}{1+p(1-r)}} \\ &\le \big\{ g^r \,\square\, \nabla w \big\}^{\frac{1}{1+p(1-r)}} \big\{ g^{1+\frac{1}{p}} \,\square\, \nabla w \big\}^{\frac{(1-r)p}{1+p(1-r)}} \end{aligned} \tag{4.4}$$

where

$$h(s) = \int_\Omega |\nabla w(t) - \nabla w(s)|^2 ds.$$

For $0 < r < 1$ we have

$$g^r \,\square\, \nabla w = \int_\Omega \int_0^t g^r(t-s)|\nabla w(t) - \nabla w(s)|^2 ds dx \le 4 \int_0^t g^r(s)ds \|w\|_{C(0,T;H_0^1)}^2,$$

from which the first inequality of Lemma 4.2 follows. To prove the last part, let us take $r = 0$ in Lemma 4.1 to get

$$\begin{aligned} 1 \,\square\, \nabla w &= \int_\Omega \int_0^t |\nabla w(t) - \nabla w(s)|^2 ds dx \\ &\le 2t\|w(t)\|_{H_0^1}^2 + 2\int_0^t \|w(s)\|_{H_0^1}^2 ds. \end{aligned}$$

Substitution of the above inequality into (4.4) yields the second inequality. The proof is now complete. ♢

¿From the above Lemma, for $0 < r < 1$, we get

$$g \,\square\, \nabla w \le c_0 (g^{1+\frac{1}{p}} \,\square\, \nabla w)^{\frac{(1-r)p}{1+p(1-r)}}, \tag{4.5}$$

Lemma 4.3 *Let f be a non-negative C^1 function satisfying*

$$f'(t) \le -k_0[f(t)]^{1+\frac{1}{p}} + \frac{k_1}{(1+t)^{p+1}},$$

for some positive constants k_0, k_1 and $p > 1$. Then there exists a positive constant c_1 such that

$$f(t) \le c_1 \frac{pf(0) + 2k_1}{(1+t)^p}.$$

Proof. Let $h(t) := \frac{2k_1}{p(1+t)^p}$ and $F(t) := f(t) + h(t)$. Then

$$\begin{aligned} F'(t) &= f'(t) - \frac{2k_1}{(1+t)^{p+1}} \\ &\leq -k_0[f(t)]^{1+\frac{1}{p}} - \frac{k_1}{(1+t)^{p+1}} \\ &\leq -k_0\Big\{[f(t)]^{1+\frac{1}{p}} + \frac{p^{1+\frac{1}{p}}}{2k_0k_1^{\frac{1}{p}}}[h(t)]^{1+\frac{1}{p}}\Big\}. \end{aligned}$$

¿From which it follows that there exists a positive constant c_1 such that

$$F'(t) \leq -c_1\{[f(t)]^{1+\frac{1}{p}} + [h(t)]^{1+\frac{1}{p}}\} \leq -c_1[F(t)]^{1+\frac{1}{p}},$$

which gives the required inequality. ♢

Theorem 4.4 *Assume that* $(u_0, v_0) \in [H_0^1(\Omega)]^2$, *that* $(u_1, v_1) \in [L^2(\Omega)]^2$, *and that (1.6), (4.1), (4.2) hold. Then any solution* (u, v) *of system (1.1)-(1.4) satisfies*

$$E(t; u, v) + E(t; u_t, v_t) \leq c\big\{E(0; u, v) + E(0; u_t, v_t)\big\}(1+t)^{-p}$$

for $p > 1$.

Proof. We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big)^2$ and $(u_1, v_1) \in [H_0^1(\Omega)]^2$. Our conclusion will follow by standard density arguments. From Lemma 3.1 and hypotheses (4.1) and (4.2), we get

$$\begin{aligned} \frac{d}{dt}E(t; u, v) &\leq -\frac{1}{2}g_1(t)\int_\Omega |\nabla u|^2 dx - g_2(t)\int_\Omega |\nabla v|^2 dx \\ &\quad -\frac{c_1}{2}g_1^{1+\frac{1}{p}} \square \nabla u - \frac{c_1}{2}g_2^{1+\frac{1}{p}} \square \nabla v, \qquad (4.6) \\ \frac{d}{dt}E(t; u_t, v_t) &\leq -\frac{1}{2}g_1(t)\int_\Omega |\nabla u_t|^2 dx - g_2(t)\int_\Omega |\nabla v_t|^2 dx \\ &\quad -\frac{c_1}{2}g_1^{1+\frac{1}{p}} \square \nabla u_t - \frac{c_1}{2}g_2^{1+\frac{1}{p}} \square \nabla v_t \\ &\quad -g_1(t)\int_\Omega \Delta u_0 u_{tt} dx - g_2(t)\int_\Omega \Delta v_0 v_{tt} dx, \qquad (4.7) \end{aligned}$$

for some $c_1 > 0$. Since the inequalities

$$\begin{aligned} &\Big|\int_\Omega\int_0^t f_i(t-s)[\nabla\varphi(s) - \nabla\varphi(t)]ds \cdot \nabla\varphi_t\Big| \\ &\leq (c_0 g_i(0) + c_2)\Big\{\epsilon_i\int_\Omega |\nabla\varphi_t|^2 dx + \frac{(g_i(0))^{1/p}}{4\epsilon_i}\Big(\int_0^t g_i(s)ds\Big)g_i^{1+\frac{1}{p}} \square \nabla\varphi\Big\}, \qquad (4.8) \end{aligned}$$

$$\begin{aligned}&\Big|f_i(t)\int_\Omega \nabla\varphi\cdot\nabla\varphi_t dx\Big|\\ &\quad\le g_i(t)\{c_0[g_i(0)]^{1/p}+g_i(0)\}\Big\{\lambda_i\int_\Omega|\nabla\varphi_t|^2dx+\frac{1}{4\lambda_i}\int_\Omega|\nabla\varphi|^2dx\Big\}\end{aligned}\tag{4.9}$$

hold for any $\epsilon_i > 0$ and $\lambda_i > 0$ $(i = 1, 2)$. It follows of Lemma 3.2 that

$$\begin{aligned}&\frac{d}{dt}\{K(t;u,v)+\frac{2}{3}g_1(0)I_1(t;u)+\frac{2}{3}g_2(0)I_2(t;v)\}\\ &\le\; -\frac{g_1(0)}{6}\{\int_\Omega|u_{tt}|^2dx+\int_\Omega|\nabla u_t|^2dx\}-\frac{g_2(0)}{6}\{\int_\Omega|v_{tt}|^2dx+\int_\Omega|\nabla v_t|^2dx\}\\ &\quad+\{\frac{3}{2}(g_1(0)+[g_1(0)]^{\frac{1}{p}}c_0)^2+\frac{c_0}{3}g_1^{1+\frac{1}{p}}(0)\}g_1(t)\int_\Omega|\nabla u|^2dx\\ &\quad+\{\frac{3}{2}(g_2(0)+[g_2(0)]^{\frac{1}{p}}c_0)^2+\frac{c_0}{3}g_2^{1+\frac{1}{p}}(0)\}g_2(t)\int_\Omega|\nabla v|^2dx\\ &\quad+\Big\{\frac{3(c_0g_1(0)+c_2)^2}{2g_1^{1+\frac{1}{p}}(0)}+\frac{c_2}{3}g_1(0)\Big\}g_1^{1+\frac{1}{p}}\Box\nabla u\\ &\quad+\Big\{\frac{3(c_0g_2(0)+c_2)^2}{2g_2^{1+\frac{1}{p}}(0)}+\frac{c_2}{3}g_2(0)\Big\}g_2^{1+\frac{1}{p}}\Box\nabla v\end{aligned}\tag{4.10}$$

for

$$\epsilon_i=\frac{g_i(0)}{6(c_0g_i(0)+c_2)};\quad \lambda_i=\frac{1}{6(g_i(0)+[g_i(0)]^{\frac{1}{p}}c_0)}.$$

Note that

$$\begin{aligned}&\int_\Omega\int_0^t g_i(t-s)\nabla\varphi(s)ds\cdot\nabla\varphi dx\\ &=\;\int_\Omega\int_0^t g_i(t-s)[\nabla\varphi(s)-\nabla\varphi(t)]ds\cdot\nabla\varphi dx+\int_0^t g_i(s)ds\int_\Omega|\nabla\varphi|^2dx\\ &\le\;\frac{1}{4\epsilon_i}g_i^{1+\frac{1}{p}}\Box\nabla\varphi+(\epsilon_i\alpha_i+\int_0^t g(s)ds)\int_\Omega|\nabla\varphi|^2dx.\end{aligned}\tag{4.11}$$

On taking $\epsilon_i = \frac{\beta_i}{2\alpha_i}$ in (4.11) and using Lemma 3.3 it follows that

$$\frac{d}{dt}J(t;u)\;\le\;2\alpha_0\int_\Omega|\nabla u_t|^2dx-\frac{\beta_1}{2}\int_\Omega|\nabla u|^2dx+\frac{\alpha_1}{2\beta_1}g_1^{1+\frac{1}{p}}\Box\nabla u,\tag{4.12}$$

$$\frac{d}{dt}J(t;v)\;\le\;2\alpha_0\int_\Omega|\nabla v_t|^2dx-\frac{\beta_2}{2}\int_\Omega|\nabla v|^2dx+\frac{\alpha_2}{2\beta_2}g_2^{1+\frac{1}{p}}\Box\nabla v.\tag{4.13}$$

¿From (4.6)-(4.13) we find that

$$\frac{d}{dt}\mathcal{L}(t)\le cR(t)-k_0[M(t)+S(t)]$$

where

$$M(t) = \int_\Omega |u_{tt}|^2 dx + \int_\Omega |v_{tt}|^2 dx + \int_\Omega |\nabla u_t|^2 dx + \int_\Omega |\nabla v_t|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx,$$

$$S(t) = g_1^{1+\frac{1}{p}} \Box \nabla u + g_2^{1+\frac{1}{p}} \Box \nabla v + g_1^{1+\frac{1}{p}} \Box \nabla u_t + g_2^{1+\frac{1}{p}} \Box \nabla v_t.$$

Since the energy is bounded, Lemma 4.2 implies

$$M(t) \geq cM(t)^{\frac{1+p(1-r)}{p(1-r)}},$$
$$S(t) \geq c\big\{g_1 \Box \nabla u + g_2 \Box \nabla v + g_1 \Box \nabla u_t + g_2 \Box \nabla v_t\big\}^{\frac{1+p(1-r)}{p(1-r)}}.$$

It is not difficult to see that we can take N_1, N_2 large enough such that $\mathcal{L}$ satisfies

$$c\{E(t; u, v) + E(t; u_t, v_t)\} \leq \mathcal{L}(t) \leq c_1\{M(t) + S(t)\}^{\frac{p(1-r)}{1+p(1-r)}}. \tag{4.14}$$

¿From which it follows that

$$\frac{d}{dt}\mathcal{L}(t) \leq cR(t) - c_2\mathcal{L}(t)^{\frac{1+p(1-r)}{p(1-r)}}.$$

Using Lemma 4.3 we obtain

$$\mathcal{L}(t) \leq c\{\mathcal{L}(0) + c_2\}\frac{1}{(1+t)^{p(1-r)}}.$$

¿From which it follows that the energies decay to zero uniformly. Using Lemma 4.2 for $r = 0$ we get

$$M(t) \geq cM(t)^{\frac{1+p}{p}},$$
$$S(t) \geq c\{g_1 \Box \nabla u + g_2 \Box \nabla v + g_1 \Box \nabla u_t + g_2 \Box \nabla v_t\}^{\frac{1+p}{p}}.$$

Repeating the same reasoning as above, we get

$$\mathcal{L}(t) \leq c\{\mathcal{L}(0) + c_2\}\frac{1}{(1+t)^p}.$$

¿From which our result follows. The proof is now complete. ♢

Finally, as a consequence of Theorem 4.4 we conclude that the first order energy also decays polynomially.

Corollary 4.5 *Under the hypotheses of Theorem 4.4, there exists a positive constant c such that for $p > 1$,*

$$E(t; u, v) \leq c\{E(0; u, v)\}(1+t)^{-p}.$$

Remark: Using the same ideas for proving Theorems 3.5 and 4.4 it is possible to obtain the same results for a similar coupled system of the plate equations.

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